SOFIC MEAN DIMENSION

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ABSTRACT. We introduce mean dimensions for continuous actions of countable sofic groups on compact metrizable spaces. These generalize the Gromov-Lindenstrauss-Weiss mean dimensions for actions of countable amenable groups, and are useful for distinguishing continuous actions of countable sofic groups with infinite entropy.

1. INTRODUCTION

Mean dimension was introduced by Gromov [Gro99b] about a decade ago, as an analogue of dimension for dynamical systems, and was studied systematically by Lindenstrauss and Weiss [LW] for continuous actions of countable amenable groups on compact metrizable spaces. Among many beautiful results they obtained, it is especially notable that they used mean dimension to show that there exits a minimal action of $\mathbb{Z}$ on some compact metrizable space which can not be equivariantly embedded into $[0,1]^{\mathbb{Z}}$ equipped with the shift $\mathbb{Z}$-action. Mean dimension is further explored in [Coo, CK, Gut, Kri06, Kri09, Lin].

The notion of sofic groups was also introduced by Gromov [Gro99a] around the same time. The class of sofic groups include all discrete amenable groups and residually finite groups, and it is still an open question whether every group is sofic. For some nice exposition on sofic groups, see [CC, ES05, ES06, Pes, Wei]. Using the idea of counting sofic approximations, in [Bow10] Bowen defined entropy for measure-preserving actions of countable sofic groups on probability measure spaces, when there exists a countable generating partition with finite Shannon entropy. Together with David Kerr, in [KL11, KL13] we extended Bowen's measure entropy to all measure-preserving actions of countable sofic groups on standard probability measure spaces, and defined topological entropy for continuous actions of countable sofic groups on compact metrizable spaces. The sofic measure entropy and sofic topological entropy are related by the variational principle [KL11]. Furthermore, the sofic entropies coincide with the classical entropies when the group is amenable [Bow12, KL13].

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The goal of this article is to extend mean dimension to continuous actions of countable sofic groups G on compact metrizable spaces X. In order to define sofic mean dimension, we use some approximate actions of G on finite sets as models, and replace X by certain spaces of approximately G-equivariant maps from the finite sets to X, which appeared first in the definition of sofic topological entropy [KL13]. A novelty here is that we replace open covers of X by certain open covers on these map spaces.

Lindenstrauss and Weiss studied two kinds of mean dimensions for actions of countable amenable groups in [LW], one is topological, as the analogue of the covering dimension, and the other is metric, as the analogue of the lower box dimension. We define sofic mean topological dimension and establish some basic properties in Section 2, and show that it coincides with the Gromov-Lindenstrauss-Weiss mean topological dimension when the group is amenable in Section 3. Similarly, we discuss sofic metric mean dimension in Sections 4 and 5. It is shown in Section 6 that sofic mean topological dimension is always bounded above by sofic metric mean dimension. We calculate sofic mean dimensions for some Bernoulli shifts and show that every non-trivial factor of the shift action of G on $[0,1]^G$ has positive sofic mean dimensions in Section 7. In the last section, we show that actions with small-boundary property have zero or $-\infty$ sofic mean topological dimension.

To round up this section, we fix some notation.

Definition 1.1. For $d \in \mathbb{N}$, we write $[d]$ for the set $\{1, \dots, d\}$ and $\mathrm{Sym}(d)$ for the permutation group of $[d]$. A countable group G is called *sofic* if there is a *sofic approximation sequence* $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i=1}^{\infty}$ for G, namely the following three conditions are satisfied:

(1) for any $s, t \in G$, one has $\lim_{i\to\infty} \frac{|\{a\in[d_i]:\sigma_i(s)\sigma_i(t)(a)=\sigma_i(st)(a)\}|}{d_i} = 1$;
(2) for any distinct $s, t \in G$, one has $\lim_{i\to\infty} \frac{|\{a\in[d_i]:\sigma_i(s)(a)=\sigma_i(t)(a)\}|}{d_i} = 0$;
(3) $\lim_{i\to\infty} d_i = +\infty$.

For a map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$, we write $\sigma(s)(a)$ as $\sigma_s(a)$ or sa, when there is no confusion. We say that σ is a *good enough sofic approximation* for G if for some large finite subset F of G which will be clear from the context, one has $\frac{|\{a\in[d]:\sigma(s)\sigma(t)(a)=\sigma(st)(a)\}|}{d}$ very close to 1 for all $s, t \in F$ and $\frac{|\{a\in[d]:\sigma(s)(a)=\sigma(t)(a)\}|}{d}$ very close to 0 for all distinct $s, t \in F$.

Throughout this paper, G will be a countable sofic group with identity element e_G, and we fix a sofic approximation sequence Σ for G.

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2. Sofic Mean Topological Dimension

In this section we define the sofic mean topological dimension and establish some basic properties.

We start with recalling the definitions of covering dimension of compact metrizable spaces and mean topological dimension for actions of countable amenable groups. For a compact space Y and two finite open covers $\mathcal{U}$ and $\mathcal{V}$ of Y, we say that $\mathcal{V}$ *refines* $\mathcal{U}$, and write $\mathcal{V} \succ \mathcal{U}$, if every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$.

Definition 2.1. Let Y be a compact space and $\mathcal{U}$ a finite open cover of Y. We denote

$$\operatorname{ord}(\mathcal{U}) = \max_{y \in Y} \sum_{U \in \mathcal{U}} 1_U(y) - 1, \text{ and } \mathcal{D}(\mathcal{U}) = \min_{\mathcal{V} \succ \mathcal{U}} \operatorname{ord}(\mathcal{V}),$$

where $\mathcal{V}$ ranges over finite open covers of Y refining $\mathcal{U}$.

For a compact metrizable space X, its *(covering) dimension* $\dim(X)$ is defined as $\sup_{\mathcal{U}} \mathcal{D}(\mathcal{U})$ for $\mathcal{U}$ ranging over finite open covers of X.

Definition 2.2. A countable group G is called *amenable* if for any finite subset K of G and any $\varepsilon > 0$ there exists a finite subset F of G with $|KF \setminus F| < \varepsilon |F|$. Equivalently, G has a *left Følner sequence* $\{F_n\}_{n \in \mathbb{N}}$, i.e. each F_n is a nonempty finite subset of G and $\frac{|sF_n \setminus F_n|}{|F_n|} \to 0$ as $n \to 0$ for every $s \in G$.

Let a countable amenable group G act continuously on a compact metrizable space X. Let $\mathcal{U}$ be a finite open cover of X. For a nonempty finite subset F of G, we set $\mathcal{U}^F = \bigvee_{s \in F} s^{-1}\mathcal{U}$. The function $F \mapsto \mathcal{D}(\mathcal{U}^F)$ defined on the set of nonempty finite subsets of G satisfies the conditions of the Ornstein-Weiss lemma [OW] [LW, Theorem 6.1], thus $\frac{\mathcal{D}(\mathcal{U}^F)}{|F|}$ converges to some real number, denoted by $\operatorname{mdim}(\mathcal{U})$, as F becomes more and more left invariant. That is, for any $\varepsilon > 0$, there exist a nonempty finite subset K of G and $\delta > 0$ such that $|\frac{\mathcal{D}(\mathcal{U}^F)}{|F|} - \operatorname{mdim}(\mathcal{U})| < \varepsilon$ for every nonempty finite subset F of G satisfying $|KF \setminus F| < \delta |F|$. In terms of any left Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ of G, one has

$$\operatorname{mdim}(\mathcal{U}) = \lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U}^{F_n})}{|F_n|}.$$

The *mean topological dimension* of X [LW, page 13] is defined as

$$\operatorname{mdim}(X) = \sup_{\mathcal{U}} \operatorname{mdim}(\mathcal{U}),$$

where $\mathcal{U}$ ranges over finite open covers of X.

Throughout the rest of this section, we fix a countable sofic group G and a sofic approximation sequence $\Sigma = \{\sigma_i : G \to \operatorname{Sym}(d_i)\}_{i=1}^{\infty}$ for G, as defined in Section 1. Let α be a continuous action of G on a compact metrizable space X.

Let ρ be a continuous pseudometric on X. For a given $d \in \mathbb{N}$, we define on the set of all maps from $[d]$ to X the pseudometrics

$$\rho_2(\varphi, \psi) = \left(\frac{1}{d}\sum_{a\in[d]}(\rho(\varphi(a), \psi(a)))^2\right)^{1/2},$$
$$\rho_\infty(\varphi, \psi) = \max_{a\in[d]} \rho(\varphi(a), \psi(a)).$$

Definition 2.3. Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We define $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be the set of all maps $\varphi : [d] \to X$ such that $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$ for all $s \in F$. We consider $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be a topological space with the topology inherited from X^d.

The space $\mathrm{Map}(\rho, F, \delta, \sigma)$ appeared first in [KL13, Section 2], and was used to define the topological entropy of the action α. Eventually we shall take σ to be σ_i for large i. Then the condition (1) in the definition of Σ says that σ is approximately a group homomorphism of G into $\mathrm{Sym}(d)$, and therefore we can think of σ as an approximate action of G on $[d]$. The space $\mathrm{Map}(\rho, F, \delta, \sigma)$ is the set of approximately G-equivariant maps from $[d]$ into X.

For a finite open cover $\mathcal{U}$ of X, we denote by $\mathcal{U}^d$ the finite open cover of $X^{[d]}$ consisting of $U_1 \times U_2 \times \cdots \times U_d$ for $U_1, \ldots, U_d \in \mathcal{U}$. Note that $\mathrm{Map}(\rho, F, \delta, \sigma)$ is a closed subset of $X^{[d]}$. Consider the restriction $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)} = \mathcal{U}^d \cap \mathrm{Map}(\rho, F, \delta, \sigma)$ of $\mathcal{U}^d$ to $\mathrm{Map}(\rho, F, \delta, \sigma)$. Denote $\mathcal{D}(\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)})$ by $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)$.

The set $[d]$ is the analogue of an approximately left invariant finite subset H of G in the amenable group case, $\mathrm{Map}(\rho, F, \delta, \sigma)$ is the analogue of the subset $\{(sx)_{s\in H} : x \in X\}$ of X^H which can be identified with X naturally (this will be made clear in the proof of Theorem 3.1 below), $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$ is then the analogue of $\mathcal{U}^H$, and $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)$ is the analogue of $\mathcal{D}(\mathcal{U}^H)$.

Definition 2.4. Let ρ be a compatible metric on X. Let F be a nonempty finite subset of G and $\delta > 0$. For a finite open cover $\mathcal{U}$ of X we define

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) = \overline{\lim_{i\to\infty}} \frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma_i)}{d_i},$$
$$\mathcal{D}_\Sigma(\mathcal{U}, \rho, F) = \inf_{\delta>0} \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta),$$
$$\mathcal{D}_\Sigma(\mathcal{U}, \rho) = \inf_{F} \mathcal{D}_\Sigma(\mathcal{U}, \rho, F),$$

where F in the third line ranges over the nonempty finite subsets of G. If $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) = -\infty$. We define the *sofic mean topological dimension* of α as

$$\mathrm{mdim}_\Sigma(X, \rho) = \sup_{\mathcal{U}} \mathcal{D}_\Sigma(\mathcal{U}, \rho)$$

for $\mathcal{U}$ ranging over finite open covers of X. As shown by Lemma 2.9 below, the quantities $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F), \mathcal{D}_\Sigma(\mathcal{U}, \rho)$ and $\mathrm{mdim}_\Sigma(X, \rho)$ do not depend on the choice of

ρ, and we shall write them as $\mathcal{D}_\Sigma(\mathcal{U}, F), \mathcal{D}_\Sigma(\mathcal{U})$ and $\mathrm{mdim}_\Sigma(X)$ respectively. In particular, $\mathrm{mdim}_\Sigma(\cdot)$ is an invariant of topological dynamical systems.

Remark 2.5. Note that $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta)$ decreases when δ decreases and F increases. Thus in the definitions of $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F)$ and $\mathcal{D}_\Sigma(\mathcal{U}, \rho)$ one can also replace $\inf_{\delta>0}$ and $\inf_F$ by $\lim_{\delta\to 0}$ and $\lim_{F\to\infty}$ respectively, where $F_1 \leq F_2$ means $F_1 \subseteq F_2$. If we partially order the set of all such (F, δ) by $(F, \delta) \geq (F', \delta)$ when $F \supseteq F'$ and $\delta \leq \delta'$, then

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho) = \lim_{(F,\delta)\to\infty} \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta).$$

Remark 2.6. From Definition 2.4 and [KL13, Proposition 2.4] one gets that the following conditions are equivalent:

(1) $\mathrm{mdim}_\Sigma(X) \geq 0$;
(2) For any finite subset F of G, any $\delta > 0$, and any $N \in \mathbb{N}$, there is some $i \geq N$ such that $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is nonempty;
(3) The sofic topological entropy $h_\Sigma(X) \geq 0$.

Also, by the variational principle [KL11, Theorem 6.1], these conditions imply the following

(4) X has a G-invariant Borel probability measure.

Every non-amenable countable group has a continuous affine action on a compact metrizable convex set (in some locally convex Hausdorff topological vector space) admitting no fixed point, equivalently, admitting no invariant Borel probability measure [CC, Corollary 4.10.2]. Thus every non-amenable sofic group has a continuous action on some compact metrizable space with mean topological dimension $-\infty$. However, we do not know whether there is any such an example with a unique invariant Borel probability measure.

Remark 2.7. The definitions of $\mathrm{mdim}_\Sigma(X), \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$ in the sequel and $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X)$ depend on the choice of the sofic approximation sequence Σ. However, we do not know any examples for which different choices of Σ lead to different values of these invariants, though Lewis Bowen [Bow09] showed that the sofic measure entropy of the trivial action of $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 2$ (more generally groups with property (τ)) on the two-point set equipped with the uniform distribution does depend on the choice of Σ.

We need the following simple observation several times.

Lemma 2.8. *Let ρ be a continuous pseudometric on X, F a nonempty finite subset of G, $\delta > 0$, and σ a map from G to* $\mathrm{Sym}(d)$ *for some $d \in \mathbb{N}$. For any $\varphi \in$* $\mathrm{Map}(\rho, F, \delta, \sigma)$ *and $s \in F$, setting $\mathcal{W} = \{a \in [d] : \rho(s\varphi(a), \varphi(sa)) \leq \sqrt{\delta}\}$, one has $|\mathcal{W}| \geq (1 - \delta)d$.*

Proof. This follows from

$$\delta^2 \geq (\rho_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s))^2 \geq \frac{1}{d}|[d] \setminus \mathcal{W}|\delta = (1 - \frac{|\mathcal{W}|}{d})\delta.$$

□

Lemma 2.9. *Let ρ and ρ' be compatible metrics on X. For any nonempty finite subset F of G and any finite open cover $\mathcal{U}$ of X, one has $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F) = \mathcal{D}_\Sigma(\mathcal{U}, \rho', F)$.*

Proof. By symmetry it suffices to show $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F)$. Let $\delta > 0$. Take $\delta' > 0$ be a small positive number which we shall determine in a moment. We claim that for any map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ one has $\mathrm{Map}(\rho, F, \delta', \sigma) \subseteq \mathrm{Map}(\rho', F, \delta, \sigma)$. Let $\varphi \in \mathrm{Map}(\rho, F, \delta', \sigma)$. For each $s \in F$, set

$$\mathcal{W}_s = \{a \in [d] : \rho(s\varphi(a), \varphi(sa)) \leq \sqrt{\delta'}\}.$$

By Lemma 2.8 one has $|\mathcal{W}_s| \geq (1 - \delta')d$. Taking δ' small enough, we may assume that for any $x, y \in X$ with $\rho(x, y) \leq \sqrt{\delta'}$, one has $\rho'(x, y) \leq \delta/2$. Then one has

$$\begin{aligned}(\rho_2'(\alpha_s \circ \varphi, \varphi \circ \sigma_s))^2 &\leq \frac{|\mathcal{W}_s|}{d} \cdot \frac{\delta^2}{4} + (1 - \frac{|\mathcal{W}_s|}{d})(\mathrm{diam}(X, \rho'))^2 \\ &\leq \frac{\delta^2}{4} + \delta'(\mathrm{diam}(X, \rho'))^2 \leq \delta^2,\end{aligned}$$

granted that δ' is small enough. Therefore $\varphi \in \mathrm{Map}(\rho', F, \delta, \sigma)$. This proves the claim.

Since $\mathrm{Map}(\rho, F, \delta', \sigma) \subseteq \mathrm{Map}(\rho', F, \delta, \sigma)$, clearly $\mathcal{D}(\mathcal{U}, \rho, F, \delta', \sigma) \leq \mathcal{D}(\mathcal{U}, \rho', F, \delta, \sigma)$. Thus $\mathcal{D}(\mathcal{U}, \rho, F) \leq \mathcal{D}(\mathcal{U}, \rho, F, \delta') \leq \mathcal{D}(\mathcal{U}, \rho', F, \delta)$. Letting $\delta \to 0$, we get $\mathcal{D}(\mathcal{U}, \rho, F) \leq \mathcal{D}(\mathcal{U}, \rho', F)$ as desired. □

We need the following lemma several times.

Lemma 2.10. *Let α^X and α^Y be continuous actions of G on compact metrizable spaces X and Y respectively, and $\pi : X \to Y$ be an equivariant continuous map. Let ρ^X and ρ^Y be compatible metrics on X and Y respectively. Let F be a nonempty finite subset of G and $\delta > 0$. Then there exists $\delta' > 0$ such that for every map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ and every $\varphi \in \mathrm{Map}(\rho^X, F, \delta', \sigma)$, one has $\pi \circ \varphi \in \mathrm{Map}(\rho^Y, F, \delta, \sigma)$.*

Proof. Since π is continuous and X is compact, we can find $\delta' > 0$ small enough such that $|F|\delta'(\mathrm{diam}(Y, \rho^Y))^2 \leq \delta^2/2$ and for any $x, x' \in X$ with $\rho^X(x, x') \leq \sqrt{\delta'}$ one has $\rho^Y(\pi(x), \pi(x')) \leq \delta/2$.

Denote by $\mathcal{W}$ the set of all $a \in [d]$ satisfying $\rho^X(s\varphi(a), \varphi(sa)) \leq \sqrt{\delta'}$. By Lemma 2.8 one has $|\mathcal{W}| \geq (1 - |F|\delta')d$. For each $a \in \mathcal{W}$, by the choice of δ' one has

$$\rho^Y(s\pi(\varphi(a)), \pi(\varphi(sa)) = \rho^Y(\pi(s\varphi(a)), \pi(\varphi(sa)) \leq \delta/2.$$

Thus

$$\begin{aligned}(\rho_2^Y(\alpha_s^Y\circ\pi\circ\varphi,\pi\circ\varphi\circ\sigma_s))^2 &\le \frac{|\mathcal{W}|}{d}(\frac{\delta}{2})^2+\frac{|[d]\setminus\mathcal{W}|}{d}(\mathrm{diam}(Y,\rho^Y))^2\\ &\le \delta^2/4+|F|\delta'(\mathrm{diam}(Y,\rho^Y))^2\le\delta^2/4+\delta^2/2<\delta^2,\end{aligned}$$

and hence $\pi\circ\varphi\in\mathrm{Map}(\rho^Y,F,\delta,\sigma)$. $\square$

Lindenstrauss and Weiss established the next two propositions in the case G is amenable [LW, page 5, Proposition 2.8].

Proposition 2.11. *Let G act continuously on a compact metrizable space X. Let Y be a closed G-invariant subset of X. Then* $\mathrm{mdim}_\Sigma(Y)\le\mathrm{mdim}_\Sigma(X)$.

Proof. Let ρ be a compatible metric on X. Then ρ restricts to a compatible metric ρ' on Y.

Let $\mathcal{U}$ be a finite open cover of Y. Then we can find a finite open cover $\mathcal{V}$ of X such that $\mathcal{U}$ is the restriction of $\mathcal{V}$ to Y. Note that $\mathrm{Map}(\rho',F,\delta,\sigma)\subseteq\mathrm{Map}(\rho,F,\delta,\sigma)$ for any nonempty finite subset F of G, any $\delta>0$, and any map σ from G to $\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$. Furthermore, the restriction of $\mathcal{V}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$ on $\mathrm{Map}(\rho',F,\delta,\sigma)$ is exactly $\mathcal{U}^d|_{\mathrm{Map}(\rho',F,\delta,\sigma)}$. Thus $\mathcal{D}(\mathcal{U},\rho',F,\delta,\sigma)\le\mathcal{D}(\mathcal{V},\rho,F,\delta,\sigma)$. It follows that $\mathrm{mdim}_\Sigma(\mathcal{U})\le\mathrm{mdim}_\Sigma(\mathcal{V})\le\mathrm{mdim}_\Sigma(X)$. Since $\mathcal{U}$ is an arbitrary finite open cover of Y, we get $\mathrm{mdim}_\Sigma(Y)\le\mathrm{mdim}_\Sigma(X)$. $\square$

Proposition 2.12. *Let G act continuously on a compact metrizable space X_n for each $1\le n<R$, where $R\in\mathbb{N}\cup\{\infty\}$. Consider the product action of G on $X:=\prod_{1\le n<R}X_n$. Then* $\mathrm{mdim}_\Sigma(X)\le\sum_{1\le n<R}\mathrm{mdim}_\Sigma(X_n)$.

Proof. Let ρ and $\rho^{(n)}$ be compatible metrics on X and X_n respectively. Denote by π_n the projection of X onto X_n. Let $\mathcal{U}$ be a finite open cover of X. Then there are an $N\in\mathbb{N}$ with $N<R$ and a finite open cover $\mathcal{V}_n$ of X_n for all $1\le n\le N$ such that

$$\mathcal{V}:=\bigvee_{n=1}^N\pi_n^{-1}(\mathcal{V}_n)\succ\mathcal{U}.$$

Let F be a nonempty finite subset of G and $\delta>0$. By Lemma 2.10 we can find $\delta'>0$ such that for any map σ from G to $\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$ and any $\varphi\in\mathrm{Map}(\rho,F,\delta',\sigma)$ one has $\pi_n\circ\varphi\in\mathrm{Map}(\rho^{(n)},F,\delta,\sigma)$ for all $1\le n\le N$. It follows that we have a continuous map $\Phi_n:\mathrm{Map}(\rho,F,\delta',\sigma)\to\mathrm{Map}(\rho^{(n)},F,\delta,\sigma)$ sending φ to $\pi_n\circ\varphi$ for each $1\le n\le N$. Note that

$$\mathcal{V}^d|_{\mathrm{Map}(\rho,F,\delta',\sigma)}=\bigvee_{n=1}^N\Phi_n^{-1}(\mathcal{V}_n^d|_{\mathrm{Map}(\rho^{(n)},F,\delta,\sigma)}).$$

For any finite open covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of a compact metrizable space Y one has $\mathcal{D}(\mathcal{U}_1 \vee \mathcal{U}_2) \leq \mathcal{D}(\mathcal{U}_1) + \mathcal{D}(\mathcal{U}_2)$ [LW, Corollary 2.5]. Thus

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta', \sigma) \leq \mathcal{D}(\mathcal{V}, \rho, F, \delta', \sigma) \leq \sum_{n=1}^{N} \mathcal{D}(\mathcal{V}_n, \rho^{(n)}, F, \delta, \sigma),$$

and hence $\mathcal{D}_\Sigma(\mathcal{U}, \rho) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta') \leq \sum_{n=1}^{N} \mathcal{D}_\Sigma(\mathcal{V}_n, \rho^{(n)}, F, \delta)$. Since F and δ are arbitrary, we get

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho) \leq \sum_{n=1}^{N} \mathcal{D}_\Sigma(\mathcal{V}_n, \rho^{(n)}) \leq \sum_{n=1}^{N} \mathrm{mdim}_\Sigma(X_n) \leq \sum_{1 \leq n < R} \mathrm{mdim}_\Sigma(X_n).$$

Therefore $\mathrm{mdim}_\Sigma(X) \leq \sum_{1 \leq n < R} \mathrm{mdim}_\Sigma(X_n)$ as desired. $\square$

If a property P for continuous G-actions on compact Hausdorff spaces is preserved by products, subsystems, and isomorphisms, and the trivial action of G on the one-point set $\bullet$ has property P, then for any continuous G-action on a compact Hausdorff space X, there is a largest factor Y of X with property P [GM, Proposition 2.9.1]. We prove a similar fact for the category of actions on compact metrizable spaces, the proof of which is implicit in the proof of [Lin, Proposition 6.12].

Lemma 2.13. *Let Γ be a topological group. Let P be a property for continuous Γ-actions on compact metrizable spaces. Suppose that P is preserved by countable products, subsystems, and isomorphisms, and that the trivial action of Γ on the one-point set $\bullet$ has property P. Then any continuous Γ-action on a compact metrizable space X has a largest factor Y with property P, i.e. for any factor Z of X with property P there is a unique (Γ-equivariant continuous surjective) map $Y \to Z$ making the following diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

commute.

Proof. For each factor Z of X with factor map $\pi_Z : X \to Z$, denote by R_Z the closed subset $\{(x, y) \in X^2 : \pi_Z(x) = \pi_Z(y)\}$ of X^2. Denote by R the set $\bigcap_Z R_Z$ for Z ranging over factors of X with property P. Since X^2 is compact metrizable, it has a countable base. Thus every subset W of X^2 with the topology inherited from X^2 is a Lindelöf space in the sense that every open cover of W has a countable subcover [Kel, page 49]. Taking $W = X^2 \setminus R$ and considering the open cover of $X^2 \setminus R$ consisting of $X^2 \setminus R_Z$ for all factors Z of X with property P, we find factors $Z_1, Z_2, \ldots$ of X with property P such that $\bigcap_{n=1}^{\infty} R_{Z_n} = R$. Consider the map $\pi : X \to \prod_{n=1}^{\infty} Z_n$ sending x to $(\pi_{Z_n}(x))_{n=1}^{\infty}$. Then $Y := \pi(X)$ is a closed Γ-invariant subset of $\prod_{n=1}^{\infty} Z_n$ and is a factor of X. Furthermore, $R_Y = R$. By the assumption on P and Z_n, we see that the Γ-action on Y has property P. Since $R_Z \supseteq R = R_Y$ for every factor Z of X with property P, clearly Y is the largest factor of X with property P. $\square$

By Remark 2.6 and Lemma 2.10 if G acts continuously on a compact metrizable space X with $\mathrm{mdim}_\Sigma(X) \geq 0$, then $\mathrm{mdim}_\Sigma(Y) \geq 0$ for every factor Y of X. From Lemma 2.13 and Propositions 2.11 and 2.12, taking property P to be having sofic mean topological dimension at most 0 and observing that $\mathrm{mdim}_\Sigma(\bullet) = 0$ for the trivial action of G on the one-point set $\bullet$, we obtain the following result, which was established by Lindenstrauss for countable amenable groups [Lin, Proposition 6.12].

Proposition 2.14. *Let G act continuously on a compact metrizable space X with $\mathrm{mdim}_\Sigma(X) \geq 0$. Then X has a largest factor Y satisfying $\mathrm{mdim}_\Sigma(Y) = 0$.*

3. Sofic Mean Topological Dimension for Amenable Groups

In this section we show that the sofic mean topological dimension extends the mean topological dimension for actions of countably infinite amenable groups:

Theorem 3.1. *Let a countably infinite (discrete) amenable group G act continuously on a compact metrizable space X. Let Σ be a sofic approximation sequence of G. Then*

$$\mathrm{mdim}_\Sigma(X) = \mathrm{mdim}(X).$$

Theorem 3.1 follows directly from Lemmas 3.5 and 3.7 below.

We need the following Rokhlin lemma several times. Though for ergodic measure-preserving actions of $\mathbb{Z}$ one needs only one Rokhlin tower, for actions of general countable amenable groups one needs several Rokhlin towers. Here one should think of $[d]$ as equipped with the uniform distribution. The assumption about σ says that it is approximately a free action of G on $[d]$ (preserving the uniform distribution), while the conclusion says that $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ are the bases for Rokhlin towers and that for each $k = 1, \ldots, \ell$, $\{\sigma(s)\mathcal{C}_k\}_{s \in F_k}$ is a Rokhlin tower.

Lemma 3.2. *[KL13, Lemma 4.6] Let G be a countable amenable group. Let $0 \leq \tau < 1$, $0 < \eta < 1$, $\delta > 0$, and K be a nonempty finite subset of G. Then there are an $\ell \in \mathbb{N}$, nonempty finite subsets $F_1, \ldots, F_\ell$ of G with $|KF_k \setminus F_k| < \delta|F_k|$ and $|F_kK \setminus F_k| < \delta|F_k|$ for all $k = 1, \ldots, \ell$, a finite set $F \subseteq G$ containing e_G, and an $\eta' > 0$ such that, for every $d \in \mathbb{N}$, every map $\sigma : G \to \mathrm{Sym}(d)$ for which there is a set $\mathcal{B} \subseteq [d]$ satisfying $|\mathcal{B}| \geq (1-\eta')d$ and*

$$\sigma_{st}(a) = \sigma_s\sigma_t(a), \sigma_s(a) \neq \sigma_{s'}(a), \sigma_{e_G}(a) = a$$

for all $a \in \mathcal{B}$ and $s, t, s' \in F$ with $s \neq s'$, and every set $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1-\tau)d$, there exist $\mathcal{C}_1, \ldots, \mathcal{C}_\ell \subseteq \mathcal{W}$ such that

1. *for every $k = 1, \ldots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times \mathcal{C}_k$ to $\sigma(F_k)\mathcal{C}_k$ is bijective,*

(2) *the sets* $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$ *are pairwise disjoint and* $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1-\tau-\eta)d$.

Remark 3.3. Let G be a finite group. Note that the only nonempty finite subset F of G satisfying $|GF \setminus F| < \frac{1}{|G|}|F|$ is G. From Lemma 3.2 one deduces the following: Let $0 \leq \tau < 1$ and $0 < \eta < 1$, then there is an $\eta' > 0$ such that, for every $d \in \mathbb{N}$, every map $\sigma : G \to \mathrm{Sym}(d)$ for which there is $\mathcal{B} \subseteq [d]$ satisfying $|\mathcal{B}| \geq (1-\eta')d$ and

$$\sigma_{st}(a) = \sigma_s\sigma_t(a), \sigma_s(a) \neq \sigma_{s'}(a), \sigma_{e_G}(a) = a$$

for all $a \in \mathcal{B}$ and $s, t, s' \in G$ with $s \neq s'$, and every set $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1-\tau)d$, there exists $\mathcal{C} \subseteq \mathcal{W}$ such that the map $(s, c) \mapsto \sigma_s(c)$ from $G \times \mathcal{C}$ to $\sigma(G)\mathcal{C}$ is bijective and $|\sigma(G)\mathcal{C}| \geq (1-\tau-\eta)d$.

Combined with Lemma 3.2, the following lemma tells us how to construct elements in $\mathrm{Map}(\rho, F, \delta, \sigma)$.

Lemma 3.4. *Let* α *be a continuous action of a countable group* G *on a compact metrizable space* X. *Let* ρ *be a continuous pseudometric on* X. *Let* $\delta, \delta' > 0$ *with* $\sqrt{\delta'}\mathrm{diam}(X, \rho) < \delta/2$. *Let* $\ell \in \mathbb{N}$ *and* $F, F_1, \dots, F_\ell$ *be nonempty finite subsets of* G *with* $|FF_k \setminus F_k| < \delta'|F_k|$ *for all* $k = 1, \dots, \ell$. *Let* σ *be a map* $G \to \mathrm{Sym}(d)$ *for some* $d \in \mathbb{N}$. *Denote by* $\mathcal{W}$ *the set of elements* a *in* $[d]$ *satisfying* $\sigma_t\sigma_s(a) = \sigma_{ts}(a)$ *for all* $t \in F$ *and* $s \in \bigcup_{k=1}^{\ell} F_k$. *Suppose that there are* $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$ *satisfying the following:*

(1) *for every* $k = 1, \dots, \ell$, *the map* $(s, c) \mapsto \sigma_s(c)$ *from* $F_k \times \mathcal{C}_k$ *to* $\sigma(F_k)\mathcal{C}_k$ *is bijective,*

(2) *the sets* $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$ *are pairwise disjoint and* $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1-\delta')d$.

For any $h = (h_k)_{k=1}^{\ell} \in \prod_{k=1}^{\ell} X^{\mathcal{C}_k}$, *if* $\varphi : [d] \to X$ *satisfies*

$$\varphi(sc) = s(h_k(c))$$

for all $k \in \{1, \dots, \ell\}, c \in \mathcal{C}_k$, *and* $s \in F_k$, *then* $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$.

Proof. Note that if $t \in F$, $k \in \{1, \dots, \ell\}$, $s \in F_k$, $c \in \mathcal{C}_k$, and $ts \in F_k$, then $\sigma_t\sigma_s(c) = \sigma_{ts}(c)$, and hence $\alpha_t \circ \varphi(sc) = \varphi \circ \sigma_t(sc)$. For every $t \in F$, one has

$$\begin{aligned}
(\rho_2(\alpha_t \circ \varphi, \varphi \circ \sigma_t))^2 &\leq \frac{d - |\bigcup_{k=1}^{\ell} \sigma(F_k \cap t^{-1}F_k)\mathcal{C}_k|}{d}(\mathrm{diam}(X, \rho))^2 \\
&= \frac{d - |\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| + |\bigcup_{k=1}^{\ell} \sigma(F_k \setminus t^{-1}F_k)\mathcal{C}_k|}{d}(\mathrm{diam}(X, \rho))^2 \\
&\leq \frac{\delta' d + \delta'|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k|}{d}(\mathrm{diam}(X, \rho))^2 \\
&\leq 2\delta'(\mathrm{diam}(X, \rho))^2 < \delta^2.
\end{aligned}$$

Thus $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$. ∎

We show first $\mathrm{mdim}_\Sigma(X) \geq \mathrm{mdim}(X)$ for infinite G. This amounts to show that $\mathrm{Map}(\rho, F, \delta, \sigma)$ is large enough. When $[d] = H$ for some approximately left invariant finite subset H of G and $\sigma_s \in \mathrm{Sym}(d)$ is essentially the left multiplication by s for all $s \in F$, one can embed X into $\mathrm{Map}(\rho, F, \delta, \sigma)$ sending x to $(sx)_{s\in H}$. In general, $[d]$ may fail to be of the form H, but Lemma 3.2 tells us $[d]$ is roughly the disjoint union of some such H_j for $j \in J$. Then we do such embedding on each H_j and hence embed X^J into $\mathrm{Map}(\rho, F, \delta, \sigma)$.

Lemma 3.5. *Let a countably infinite amenable group G act continuously on a compact metrizable space X. Then for any finite open cover $\mathcal{U}$ of X we have $\mathcal{D}_\Sigma(\mathcal{U}) \geq \mathrm{mdim}(\mathcal{U})$. In particular, $\mathrm{mdim}_\Sigma(X) \geq \mathrm{mdim}(X)$.*

Proof. It suffices to show that $\mathcal{D}_\Sigma(\mathcal{U}) \geq \mathrm{mdim}(\mathcal{U}) - 2\theta$ for every $\theta > 0$.

Fix a compatible metric ρ on X. Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Now it suffices to show that if σ is a good enough sofic approximation then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \geq \mathrm{mdim}(\mathcal{U}) - 2\theta.$$

Take a finite subset K of G containing F and $\varepsilon > 0$ such that for any nonempty finite subset F' of G with $|KF' \setminus F'| < \varepsilon|F'|$ one has

$$\frac{\mathcal{D}(\mathcal{U}^{F'})}{|F'|} \geq \mathrm{mdim}(\mathcal{U}) - \theta.$$

Take $0 < \delta' < 1$ to be small such that $(\mathrm{mdim}(\mathcal{U}) - \theta)(1 - \delta') \geq \mathrm{mdim}(\mathcal{U}) - 2\theta$ and $\sqrt{\delta'}\mathrm{diam}(X, \rho) < \delta/2$. By Lemma 3.2 there are an $\ell \in \mathbb{N}$ and nonempty finite subsets $F_1, \dots, F_\ell$ of G satisfying $|KF_k \setminus F_k| < \min(\varepsilon, \delta')|F_k|$ for all $k = 1, \dots, \ell$ such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G and every $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1 - \delta'/2)d$ there exist $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$ satisfying the following:

(1) for every $k = 1, \dots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times \mathcal{C}_k$ to $\sigma(F_k)\mathcal{C}_k$ is bijective,
(2) the sets $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$ are pairwise disjoint and $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1 - \delta')d$.

Let $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ be a good enough sofic approximation for G such that $|\mathcal{W}| \geq (1 - \delta'/2)d$ for

$$\mathcal{W} := \{a \in [d] : \sigma_t\sigma_s(a) = \sigma_{ts}(a) \text{ for all } t \in F, s \in \bigcup_{k=1}^{\ell} F_k\}.$$

Then we have $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ as above.

Since G is infinite, there exist maps $\psi_k : \mathcal{C}_k \to G$ for $k = 1, \dots, \ell$ such that the map Ψ from $\bigsqcup_{k=1}^{\ell} F_k \times \mathcal{C}_k$ to G sending $(s, c) \in F_k \times \mathcal{C}_k$ to $s\psi_k(c)$ is injective.

Denote by $\tilde{F}$ the range of Ψ. Note that $|K\tilde{F} \setminus \tilde{F}| < \varepsilon|\tilde{F}|$ because for every k, $|KF_k \setminus F_k| < \varepsilon|F_k|$. Thus

$$\frac{\mathcal{D}(\mathcal{U}^{\tilde{F}})}{|\tilde{F}|} \geq \mathrm{mdim}(\mathcal{U}) - \theta.$$

Pick $x_0 \in X$. For each $x \in X$ define a map $\varphi_x : [d] \to X$ by $\varphi_x(a) = x_0$ for all $a \in [d] \setminus \bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k$, and

$$\varphi_x(sc) = s\psi_k(c)x$$

for all $k \in \{1, \dots, \ell\}$, $c \in \mathcal{C}_k$, and $s \in F_k$. By Lemma 3.4 one has $\varphi_x \in \mathrm{Map}(\rho, F, \delta, \sigma)$.

Note that the map Φ from X to $\mathrm{Map}(\rho, F, \delta, \sigma)$ sending x to φ_x is continuous, and $\Phi^{-1}(\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}) = \mathcal{U}^{\tilde{F}}$. Thus $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \geq \mathcal{D}(\mathcal{U}^{\tilde{F}})$. Therefore

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \geq \frac{\mathcal{D}(\mathcal{U}^{\tilde{F}})}{|\tilde{F}|} \cdot \frac{|\tilde{F}|}{d} \geq (\mathrm{mdim}(\mathcal{U}) - \theta)(1 - \delta') \geq \mathrm{mdim}(\mathcal{U}) - 2\theta,$$

as desired. □

Let $\mathcal{U}$ be a finite open cover of a compact metrizable space X. A continuous map f from X into another compact metrizable space Y is said to be $\mathcal{U}$-*compatible* if for each $y \in Y$, the set $f^{-1}(y)$ is contained in some $U \in \mathcal{U}$ [LW, Definition 2.2 and Proposition 2.3]. We need the following fact:

Lemma 3.6. *[LW, Proposition 2.4] Let $\mathcal{U}$ be a finite open cover of a compact metrizable space X, and $k \geq 0$. Then $\mathcal{D}(\mathcal{U}) \leq k$ if and only if there is a $\mathcal{U}$-compatible continuous map $f : X \to Y$ for some compact metrizable space Y with dimension k.*

Next we show $\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}(X)$. We first use Lemma 3.2 to decompose $[d]$ into the disjoint union of some approximately left invariant finite subsets $\{H_j\}_{j\in J}$ of G and a small portion $[d] \setminus \bigcup_j H_j$. Take a continuous $\mathcal{U}^{H_j}$-compatible map $X \to Y_j$ with $\dim(Y_j) \leq \mathcal{D}(\mathcal{U}^{H_j})$ for each $j \in J$. Anticipating each element of $\mathrm{Map}(\rho, F, \delta, \sigma)$ being essentially of the form $(sx_j)_{s\in H_j}$ with some $x_j \in X$ on H_j for each $j \in J$, we map $\mathrm{Map}(\rho, F, \delta, \sigma)$ to $\prod_j Y_j$. To take care of the coordinates on $[d] \setminus \bigcup_j H_j$, we also take a continuous $\mathcal{U}$-compatible map $X \to Z$ with $\dim(Z) \leq \mathcal{D}(\mathcal{U})$, and map $\mathrm{Map}(\rho, F, \delta, \sigma)$ to $\prod_{a\in[d]\setminus\bigcup_j H_j} Z$. These two maps combined together are $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$-compatible on nice elements which are almost of the form $(sx_j)_{s\in H_j}$ with some $x_j \in X$ on H_j for each $j \in J$. The points of $\mathrm{Map}(\rho, F, \delta, \sigma)$ not so nice can be bad only at a small portion of $[d]$. Then we use some auxiliary map h to shrink X to one point at all the good places, making the bad parts to live in a small-dimensional space (relative to d).

Lemma 3.7. *Let a countable amenable group G act continuously on a compact metrizable space X. Then $\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}(X)$.*

Proof. Fix a compatible metric ρ on X. Let $\mathcal{U}$ be a finite open cover of X. It suffices to show that $\mathcal{D}_\Sigma(\mathcal{U}) \leq \mathrm{mdim}(X)$.

Take a finite open cover $\mathcal{V}$ of X such that for every $V \in \mathcal{V}$, one has $\overline{V} \subseteq U$ for some $U \in \mathcal{U}$. Then it suffices to show $\mathcal{D}_\Sigma(\mathcal{U}) \leq \mathrm{mdim}(\mathcal{V}) + 3\theta$ for every $\theta > 0$. We can find $\eta > 0$ such that for every $V \in \mathcal{V}$, one has $B(V, \eta) = \{x \in X : \rho(x, V) < \eta\} \subseteq U$ for some $U \in \mathcal{U}$.

Take a nonempty finite subset K of G and $\varepsilon > 0$ such that for any nonempty finite subset F' of G with $|KF' \setminus F'| < \varepsilon|F'|$ one has

$$\frac{\mathcal{D}(\mathcal{V}^{F'})}{|F'|} \leq \mathrm{mdim}(\mathcal{V}) + \theta.$$

Take $\tau > 0$ with $\tau\mathcal{D}(\mathcal{U}) \leq \theta$. By Lemma 3.2 there are an $\ell \in \mathbb{N}$ and nonempty finite subsets $F_1, \dots, F_\ell$ of G satisfying $|KF_k \setminus F_k| < \varepsilon|F_k|$ for all $k = 1, \dots, \ell$ such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G and every $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1 - \tau/2)d$ there exist $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$ satisfying the following:

(1) for every $k = 1, \dots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times \mathcal{C}_k$ to $\sigma(F_k)\mathcal{C}_k$ is bijective,

(2) the sets $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$ are pairwise disjoint and $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1 - \tau)d$.

Set $F = \bigcup_{k=1}^{\ell} F_k^{-1}$. Take $\kappa > 0$ such that for any $x, y \in X$ with $\rho(x, y) \leq \kappa$ one has $\rho(s^{-1}x, s^{-1}y) < \eta$ for all $s \in F$. Take $\delta > 0$ with $\delta^{1/2} < \kappa$ and $\delta|\mathcal{U}||F| \leq \theta$.

Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Now it suffices to show that if σ is a good enough sofic approximation then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \leq \mathrm{mdim}(\mathcal{V}) + 3\theta.$$

Denote by $\mathcal{W}$ the subset of $[d]$ consisting of a satisfying $\sigma_s\sigma_{s^{-1}}(a) = \sigma_{e_G}(a) = a$ for all $s \in F$. Assuming that σ is a good enough sofic approximation, we have $|\mathcal{W}| \geq (1 - \tau/2)d$ and can find $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ as above. Set $\mathcal{Z} = [d] \setminus \bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k$. Then $|\mathcal{Z}| \leq \tau d$.

For every $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$, by Lemma 2.8 the set Λ_φ of all $a \in [d]$ satisfying

$$\rho(\varphi(sa), s\varphi(a)) \leq \delta^{1/2}$$

for all $s \in F$ has cardinality at least $(1 - |F|\delta)d$.

Take a partition of unity $\{\zeta_U\}_{U \in \mathcal{U}}$ for X subordinate to $\mathcal{U}$. That is, each ζ_U is a continuous function $X \to [0, 1]$ with support contained in U, and

$$\sum_{U \in \mathcal{U}} \zeta_U = 1.$$

Define a continuous map $\overrightarrow{\xi} : X \to [0,1]^{\mathcal{U}}$ by $\overrightarrow{\xi}(x)_U = \zeta_U(x)$ for $x \in X$ and $U \in \mathcal{U}$. Consider the continuous map $\overrightarrow{h} : \mathrm{Map}(\rho, F, \delta, \sigma) \to ([0,1]^{\mathcal{U}})^{[d]}$ defined by

$$\overrightarrow{h}(\varphi)_a = \overrightarrow{\xi}(\varphi(a)) \max(\max_{s \in F} \rho(s\varphi(a), \varphi(sa)) - \kappa, 0)$$

for $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ and $a \in [d]$. Denote by ν the point of $[0,1]^{\mathcal{U}}$ having all coordinates 0. Set X_0 to be the subset of $([0,1]^{\mathcal{U}})^{[d]}$ consisting of elements whose coordinates are equal to ν at at least $(1 - |F|\delta)d$ elements of $[d]$. For each $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$, note that $\overrightarrow{h}(\varphi)_a = \nu$ for all $a \in \Lambda_\varphi$ by our choice of δ and hence $\overrightarrow{h}(\varphi) \in X_0$. Thus we may think of $\overrightarrow{h}$ as a map from $\mathrm{Map}(\rho, F, \delta, \sigma)$ into X_0. Since the union of finitely many closed subsets of dimension at most m has dimension at most m [HW, page 30 and Theorem V.8] and $\delta > 0$ was chosen such that $|\mathcal{U}||F|\delta \leq \theta$, we get $\dim(X_0) \leq |\mathcal{U}||F|\delta d \leq \theta d$.

For each $1 \leq k \leq \ell$, by Lemma 3.6 we can find a compact metrizable space Y_k with $\dim(Y_k) \leq \mathcal{D}(\mathcal{V}^{F_k})$ and a $\mathcal{V}^{F_k}$-compatible continuous map $f_k : X \to Y_k$.

By Lemma 3.6 we can find a compact metrizable space Z with $\dim(Z) \leq \mathcal{D}(\mathcal{U})$ and a $\mathcal{U}$-compatible continuous map $g : X \to Z$.

Now define a continuous map $\Psi : \mathrm{Map}(\rho, F, \delta, \sigma) \to X_0 \times (\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k) \times (\prod_{a \in \mathcal{Z}} Z)$ as follows. For $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$, the coordinate of $\Psi(\varphi)$ in X_0 is $\overrightarrow{h}(\varphi)$, in Y_k for $1 \leq k \leq \ell$ and $c \in \mathcal{C}_k$ is $f_k(\varphi(c))$, in Z for $a \in \mathcal{Z}$ is $g(\varphi(a))$. We claim that Ψ is $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$-compatible. Let $w \in X_0 \times (\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k) \times (\prod_{a \in \mathcal{Z}} Z)$. We need to show that for each $a \in [d]$ there is some $U \in \mathcal{U}$ depending only on w and a such that $\varphi(a) \in U$ for every $\varphi \in \Psi^{-1}(w)$. We write the coordinates of w in X_0, $\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k$, and $\prod_{a \in \mathcal{Z}} Z$ as w^1, w^2, and w^3 respectively.

For each $a \in \mathcal{Z}$, since g is $\mathcal{U}$-compatible, one has $g^{-1}(w^3_a) \subseteq U_{w^3_a}$ for some $U_{w^3_a} \in \mathcal{U}$. Then $\varphi(a) \in U_{w^3_a}$ for every $\varphi \in \Psi^{-1}(w)$ and $a \in \mathcal{Z}$.

For every $1 \leq k \leq \ell$ and $c \in \mathcal{C}_k$, since f_k is $\mathcal{V}^{F_k}$-compatible, one has $f_k^{-1}(w^2_{k,c}) \subseteq \bigcap_{s^{-1} \in F_k} sV_{k,c,s}$ for some $V_{k,c,s} \in \mathcal{V}$ for every $s^{-1} \in F_k$. By the choice of η, $B(V_{k,c,s}, \eta)$ is contained in some $U_{k,c,s} \in \mathcal{U}$. For every $a \in [d] \setminus \mathcal{Z}$, we distinguish the two cases $w^1_a \neq \nu$ and $w^1_a = \nu$. If $w^1_a \neq \nu$, then $(w^1_a)_U \neq 0$ for some $U \in \mathcal{U}$, and then for $\varphi \in \Psi^{-1}(w)$ one has $\zeta_U(\varphi(a)) > 0$ and hence $\varphi(a) \in U$. Suppose that $w^1_a = \nu$. Say, $a = \sigma(s^{-1})c$ for some $1 \leq k \leq \ell$, $s^{-1} \in F_k$ and $c \in \mathcal{C}_k$. Let $\varphi \in \Psi^{-1}(w)$. Since $c \in \mathcal{C}_k \subseteq \mathcal{W}$ and $s \in F$, one has $sa = \sigma_s \sigma_{s^{-1}}(c) = c$. As $\{\zeta_U\}_{U \in \mathcal{U}}$ is a partition of unity of X, $\overrightarrow{\xi}(\varphi(a)) \neq \nu$. But $\overrightarrow{h}(\varphi)_a = w^1_a = \nu$. Thus $\max_{s' \in F} \rho(s'\varphi(a), \varphi(s'a)) \leq \kappa$. In particular, one has $\rho(s\varphi(a), \varphi(c)) = \rho(s\varphi(a), \varphi(sa)) \leq \kappa$. From our choice of κ, one gets $\rho(\varphi(a), s^{-1}\varphi(c)) < \eta$. Since $f_k(\varphi(c)) = w^2_{k,c}$, we have $\varphi(c) \in sV_{k,c,s}$ and hence $s^{-1}\varphi(c) \in V_{k,c,s}$. Therefore $\varphi(a) \in U_{k,c,s}$. This proves the claim.

From Lemma 3.6 we get

$$\mathcal{D}(\mathcal{U},\rho,F,\delta,\sigma)\leq \dim\big(X_0\times(\prod_{k=1}^{\ell}\prod_{c\in\mathcal{C}_k}Y_k)\times(\prod_{a\in\mathcal{Z}}Z)\big).$$

Since the dimension of the product of two compact metrizable spaces is at most the sum of the dimensions of the factors [HW, page 33 and Theorem V.8], we have

$$\begin{aligned}\dim\big(X_0\times(\prod_{k=1}^{\ell}\prod_{c\in\mathcal{C}_k}Y_k)\times(\prod_{a\in\mathcal{Z}}Z)\big) &\leq \dim(X_0)+\sum_{k=1}^{\ell}|\mathcal{C}_k|\dim(Y_k)+|\mathcal{Z}|\dim(Z)\\ &\leq \theta d+\sum_{k=1}^{\ell}|\mathcal{C}_k|\mathcal{D}(\mathcal{V}^{F_k})+|\mathcal{Z}|\mathcal{D}(\mathcal{U})\\ &\leq \theta d+\sum_{k=1}^{\ell}|\mathcal{C}_k||F_k|(\mathrm{mdim}(\mathcal{V})+\theta)+\tau d\mathcal{D}(\mathcal{U})\\ &\leq \theta d+d(\mathrm{mdim}(\mathcal{V})+\theta)+\theta d\\ &= d(\mathrm{mdim}(\mathcal{V})+3\theta).\end{aligned}$$

Therefore $\mathcal{D}(\mathcal{U},\rho,F,\delta,\sigma)\leq d(\mathrm{mdim}(\mathcal{V})+3\theta)$ as desired. $\square$

Remark 3.8. Theorem 3.1 fails when G is finite. Indeed, when a finite group G acts continuously on a compact metrizable space X, one has $\mathrm{mdim}(X)=\frac{1}{|G|}\dim(X)$. There are compact metrizable finite-dimensional spaces X satisfying $\dim(X^2)<2\dim(X)$ (see [Bol]). For such X, Lemma 3.9 below implies that $\mathrm{mdim}_\Sigma(X)<\mathrm{mdim}(X)$.

If X is a compact metrizable space with finite dimension, then for any $n,m\in\mathbb{N}$ one has $\dim(X^n\times X^m)\leq\dim(X^n)+\dim(X^m)$ [HW, page 33 and Theorem V.8] and hence $\frac{\dim(X^n)}{n}\to\inf_{m\in\mathbb{N}}\frac{\dim(X^m)}{m}$ as $n\to\infty$.

Lemma 3.9. *Let a finite group G act continuously on a compact metrizable finite-dimensional space X. Then* $\mathrm{mdim}_\Sigma(X)\leq\frac{1}{|G|}\inf_{m\in\mathbb{N}}\frac{\dim(X^m)}{m}$.

Proof. The proof is similar to that of Lemma 3.7. Fix a compatible metric ρ on X. Set $\lambda=\frac{1}{|G|}\inf_{m\in\mathbb{N}}\frac{\dim(X^m)}{m}$. Let $\mathcal{U}$ be a finite open cover of X and $\theta>0$. It suffices to show that $\mathcal{D}_\Sigma(\mathcal{U})\leq\lambda+3\theta$.

We can find $\eta>0$ such that for every $y\in X$, one has $\{x\in X:\rho(x,y)<\eta\}\subseteq U$ for some $U\in\mathcal{U}$.

Take $M>0$ such that $\frac{\dim(X^m)}{m}\leq(\lambda+\theta)|G|$ for all $m\geq M$.

Take $\tau>0$ with $\tau\dim(X)\leq\theta$. By Remark 3.3 for every map $\sigma:G\to\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$ which is a good enough sofic approximation for G and every $\mathcal{W}\subseteq[d]$ with $|\mathcal{W}|\geq(1-\tau/2)d$ there exists $\mathcal{C}\subseteq\mathcal{W}$ such that the map $(s,c)\mapsto\sigma_s(c)$ from $G\times\mathcal{C}$ to $\sigma(G)\mathcal{C}$ is bijective and $|\sigma(G)\mathcal{C}|\geq(1-\tau)d$.

Take $\kappa > 0$ such that for any $x, y \in X$ with $\rho(x, y) \le \kappa$ one has $\rho(sx, sy) < \eta$ for all $s \in G$. Take $\delta > 0$ with $\delta^{1/2} < \kappa$ and $\delta|\mathcal{U}||G| \le \theta$.

Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Now it suffices to show that if σ is a good enough sofic approximation and d is sufficiently large then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma)}{d} \le \lambda + 3\theta.$$

Denote by $\mathcal{W}$ the subset of $[d]$ consisting of a satisfying $\sigma_s\sigma_{s^{-1}}(a) = \sigma_{e_G}(a) = a$ for all $s \in G$. Assuming that σ is a good enough sofic approximation, we have $|\mathcal{W}| \ge (1 - \tau/2)d$ and can find $\mathcal{C}$ as above. Set $\mathcal{Z} = [d] \setminus \sigma(G)\mathcal{C}$. Then $|\mathcal{Z}| \le \tau d$.

For every $\varphi \in \mathrm{Map}(\rho, G, \delta, \sigma)$, by Lemma 2.8 the set Λ_φ of all $a \in [d]$ satisfying

$$\rho(\varphi(sa), s\varphi(a)) \le \delta^{1/2}$$

for all $s \in G$ has cardinality at least $(1 - |G|\delta)d$.

Take a partition of unity $\{\zeta_U\}_{U\in\mathcal{U}}$ for X subordinate to $\mathcal{U}$. That is, each ζ_U is a continuous function $X \to [0, 1]$ with support contained in U, and

$$\sum_{U\in\mathcal{U}} \zeta_U = 1.$$

Define a continuous map $\overrightarrow{\xi} : X \to [0, 1]^{\mathcal{U}}$ by $\overrightarrow{\xi}(x)_U = \zeta_U(x)$ for $x \in X$ and $U \in \mathcal{U}$. Consider the continuous map $\overrightarrow{h} : \mathrm{Map}(\rho, G, \delta, \sigma) \to ([0, 1]^{\mathcal{U}})^{[d]}$ defined by

$$\overrightarrow{h}(\varphi)_a = \overrightarrow{\xi}(\varphi(a)) \max(\max_{s\in G} \rho(s\varphi(a), \varphi(sa)) - \kappa, 0)$$

for $\varphi \in \mathrm{Map}(\rho, G, \delta, \sigma)$ and $a \in [d]$. Denote by ν the point of $[0, 1]^{\mathcal{U}}$ having all coordinates 0. Set X_0 to be the subset of $([0, 1]^{\mathcal{U}})^{[d]}$ consisting of elements whose coordinates are equal to ν at at least $(1 - |G|\delta)d$ elements of $[d]$. For each $\varphi \in \mathrm{Map}(\rho, G, \delta, \sigma)$, note that $\overrightarrow{h}(\varphi)_a = \nu$ for all $a \in \Lambda_\varphi$ by our choice of δ and hence $\overrightarrow{h}(\varphi) \in X_0$. Thus we may think of h as a map from $\mathrm{Map}(\rho, G, \delta, \sigma)$ into X_0. Since the union of finitely many closed subsets of dimension at most m has dimension at most m [HW, page 30] and $\delta > 0$ was chosen such that $|\mathcal{U}||G|\delta \le \theta$, we get $\dim(X_0) \le |\mathcal{U}||G|\delta d \le \theta d$.

Now define a continuous map $\Psi : \mathrm{Map}(\rho, G, \delta, \sigma) \to X_0 \times (\prod_{c\in\mathcal{C}} X) \times (\prod_{a\in\mathcal{Z}} X)$ as follows. For $\varphi \in \mathrm{Map}(\rho, G, \delta, \sigma)$, the coordinate of $\Psi(\varphi)$ in X_0 is $\overrightarrow{h}(\varphi)$, in X for $c \in \mathcal{C}$ is $\varphi(c)$, in X for $a \in \mathcal{Z}$ is $\varphi(a)$. We claim that Ψ is $\mathcal{U}^d|_{\mathrm{Map}(\rho,G,\delta,\sigma)}$-compatible. Let $w \in X_0 \times (\prod_{c\in\mathcal{C}} X) \times (\prod_{a\in\mathcal{Z}} X)$. We need to show that for each $a \in [d]$ there is some $U \in \mathcal{U}$ depending only on w and a such that $\varphi(a) \in U$ for every $\varphi \in \Psi^{-1}(w)$. We write the coordinates of w in X_0, $\prod_{c\in\mathcal{C}} X$, and $\prod_{a\in\mathcal{Z}} X$ as w^1, w^2, and w^3 respectively.

For each $a \in \mathcal{Z}$, one has $w^3_a \in U_{w^3_a}$ for some $U_{w^3_a} \in \mathcal{U}$. Then $\varphi(a) \in U_{w^3_a}$ for every $\varphi \in \Psi^{-1}(w)$ and $a \in \mathcal{Z}$.

For every $a \in [d] \setminus \mathcal{Z}$, we distinguish the two cases $w_a^1 \neq \nu$ and $w_a^1 = \nu$. If $w_a^1 \neq \nu$, then $(w_a^1)_U \neq 0$ for some $U \in \mathcal{U}$, and then for $\varphi \in \Psi^{-1}(w)$ one has $\zeta_U(\varphi(a)) > 0$ and hence $\varphi(a) \in U$. Suppose that $w_a^1 = \nu$. Say, $a = \sigma(s^{-1})c$ for some $s^{-1} \in G$ and $c \in \mathcal{C}$. Then $\{x \in X : \rho(x, s^{-1}\varphi(c)) < \eta\} \subseteq U$ for some $U \in \mathcal{U}$. Let $\varphi \in \Psi^{-1}(w)$. Since $c \in \mathcal{C} \subseteq \mathcal{W}$, one has $sa = \sigma_s\sigma_{s^{-1}}(c) = c$. As $\{\zeta_U\}_{U\in\mathcal{U}}$ is a partition of unity of X, $\overrightarrow{\xi}(\varphi(a)) \neq \nu$. But $h(\varphi)_a = w_a^1 = \nu$. Thus $\max_{s'\in G} \rho(s'\varphi(a), \varphi(s'a)) \leq \kappa$. In particular, one has $\rho(s\varphi(a), \varphi(c)) = \rho(s\varphi(a), \varphi(sa)) \leq \kappa$. From our choice of κ, one gets $\rho(\varphi(a), s^{-1}\varphi(c)) < \eta$. Thus $\varphi(a) \in U$. This proves the claim.

From Lemma 3.6 we get

$$\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma) \leq \dim\big(X_0 \times (\prod_{c\in\mathcal{C}} X) \times (\prod_{a\in\mathcal{Z}} X)\big).$$

Taking d to be sufficiently large, we have $|\mathcal{C}| \geq M$ and hence $\dim(X^{|\mathcal{C}|}) \leq |\mathcal{C}||G|(\lambda+\theta)$. Since the dimension of the product of two compact metrizable spaces is at most the sum of the dimensions of the factors [HW, page 33 and Theorem V.8], we have

$$\begin{aligned}\dim\big(X_0 \times (\prod_{c\in\mathcal{C}} X) \times (\prod_{a\in\mathcal{Z}} X)\big) &\leq \dim(X_0) + \dim(X^{|\mathcal{C}|}) + |\mathcal{Z}|\dim(X)\\ &\leq \theta d + |\mathcal{C}||G|(\lambda+\theta) + \tau d\dim(X)\\ &\leq \theta d + d(\lambda+\theta) + \theta d\\ &= d(\lambda + 3\theta).\end{aligned}$$

Therefore $\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma) \leq d(\lambda + 3\theta)$ as desired. □

4. Sofic Metric Mean Dimension

In this section we define the sofic metric mean dimension and establish some basic properties for it.

We start with recalling the definitions of the lower box dimension for a compact metric space and the metric mean dimension for actions of countable amenable groups.

For a pseudometric space (Y, ρ) and $\varepsilon > 0$ we say a subset Z of Y is (ρ, ε)-*separated* if $\rho(y, z) \geq \varepsilon$ for all distinct $y, z \in Z$. Denote by $N_\varepsilon(Y, \rho)$ the maximal cardinality of (ρ, ε)-separated subsets of Y.

The *lower box dimension* of a compact metric space (Y, ρ) is defined as

$$\underline{\dim}_B(Y, \rho) := \varliminf_{\varepsilon\to 0} \frac{\log N_\varepsilon(Y, \rho)}{|\log \varepsilon|}.$$

Let a countable (discrete) amenable group G act continuously on a compact metrizable space X. Let ρ be a continuous pesudometric on X. For a finite open cover $\mathcal{U}$ of X, we define the mesh of $\mathcal{U}$ under ρ by

$$\mathrm{mesh}(\mathcal{U}, \rho) = \max_{U\in\mathcal{U}} \mathrm{diam}(U, \rho).$$

For a nonempty finite subset F of G, we define a pseudometric ρ_F on X by

$$\rho_F(x,y) = \max_{s\in F} \rho(sx, sy)$$

for $x, y \in X$. The function $F \mapsto \log \min_{\mathrm{mesh}(\mathcal{U},\rho_F)<\varepsilon} |\mathcal{U}|$ defined on the set of nonempty finite subsets of G satisfies the conditions of the Ornstein-Weiss lemma [OW] [LW, Theorem 6.1], thus $\min_{\mathrm{mesh}(\mathcal{U},\rho_F)<\varepsilon} \frac{\log|\mathcal{U}|}{|F|}$ converges to some real number, denoted by $S(X,\varepsilon,\rho)$, as F becomes more and more left invariant. The *metric mean dimension of X with respect to ρ* [LW, page 13] is defined as

$$\mathrm{mdim_M}(X,\rho) = \varliminf_{\varepsilon\to 0} \frac{S(X,\varepsilon,\rho)}{|\log\varepsilon|}.$$

For any nonempty finite subset F of G and $\varepsilon > 0$, it is easy to check that

$$\min_{\mathrm{mesh}(\mathcal{U},\rho_F)<\varepsilon} |\mathcal{U}| \geq N_\varepsilon(X,\rho_F) \geq \min_{\mathrm{mesh}(\mathcal{U},\rho_F)<2\varepsilon} |\mathcal{U}|. \tag{1}$$

As discussed on page 14 of [LW], using (1) one can also write $\mathrm{mdim_M}(X,\rho)$ in a way similar to $\underline{\dim}_B(Y,\rho)$:

$$\mathrm{mdim_M}(X,\rho) = \varliminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \varlimsup_{n\to\infty} \frac{\log N_\varepsilon(X,\rho_{F_n})}{|F_n|}$$

for any left Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G (see Definition 2.2). Also, define

$$\mathrm{mdim_M}(X) = \inf_\rho \mathrm{mdim_M}(X,\rho)$$

for ρ ranging over compatible metrics on X.

In the rest of this section we fix a countable sofic group G and a sofic approximation sequence $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i=1}^\infty$ for G. We also fix a continuous action α of G on a compact metrizable space X.

As we discussed in Section 2, when defining sofic invariants, we replace X by $\mathrm{Map}(\rho,F,\delta,\sigma)$. We also replace (ρ_F,ε)-separated subsets of X by $(\rho_\infty,\varepsilon)$-separated subsets of $\mathrm{Map}(\rho,F,\delta,\sigma)$. Though we are mainly interested in the sofic metric mean dimension with respect to a metric, sometimes it can be calculated using certain continuous pseudometric as Lemma 4.4 and Section 7 indicate. Thus we start various definitions for continuous pseudometrics.

Definition 4.1. Let F be a nonempty finite subset of G and $\delta > 0$. For $\varepsilon > 0$ and ρ a continuous pseudometric on X we define

$$\begin{aligned}
h^\varepsilon_{\Sigma,\infty}(\rho,F,\delta) &= \varlimsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}(\rho,F,\delta,\sigma_i),\rho_\infty),\\
h^\varepsilon_{\Sigma,\infty}(\rho,F) &= \inf_{\delta>0} h^\varepsilon_{\Sigma,\infty}(\rho,F,\delta),\\
h^\varepsilon_{\Sigma,\infty}(\rho) &= \inf_F h^\varepsilon_{\Sigma,\infty}(\rho,F),
\end{aligned}$$

where F in the third line ranges over the nonempty finite subsets of G. If $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h^{\varepsilon}_{\Sigma,\infty}(\rho, F, \delta) = -\infty$. We define the *sofic metric mean dimension of α with respect to ρ* as

$$\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) = \varliminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} h^{\varepsilon}_{\Sigma,\infty}(\rho).$$

We also define

$$\mathrm{mdim}_{\Sigma,\mathrm{M}}(X) = \inf_{\rho} \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho),$$

for ρ ranging over compatible metrics on X. Note that $\mathrm{mdim}_{\Sigma,\mathrm{M}}(\cdot, \cdot)$ is an invariant of metric dynamical systems, while $\mathrm{mdim}_{\Sigma,\mathrm{M}}(\cdot)$ is an invariant of topological dynamical system.

Remark 4.2. Note that $h^{\varepsilon}_{\Sigma,\infty}(\rho, F, \delta)$ decreases when δ decreases and F increases. Thus in the definitions of $h^{\varepsilon}_{\Sigma,\infty}(\rho, F)$ and $h^{\varepsilon}_{\Sigma,\infty}(\rho)$ one can also replace $\inf_{\delta>0}$ and $\inf_F$ by $\lim_{\delta\to 0}$ and $\lim_{F\to\infty}$ respectively, where $F_1 \leq F_2$ means $F_1 \subseteq F_2$.

The *sofic topological entropy* $h_\Sigma(X)$ of α was defined in [KL11, Definition 4.6]. It was shown in Proposition 2.4 of [KL13] that

$$h_\Sigma(X) = \lim_{\varepsilon\to 0} h^{\varepsilon}_{\Sigma,\infty}(\rho)$$

for every compatible metric ρ on X. Thus we have

Proposition 4.3. *If $h_\Sigma(X) < +\infty$, then $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) \leq 0$ for every compatible metric ρ on X.*

The amenable group case of Proposition 4.3 was observed by Lindenstrauss and Weiss [LW, page 14].

We say that a continuous pseudometric ρ on X is *dynamically generating* [Li, Sect. 4] if for any distinct points $x, y \in X$ one has $\rho(sx, sy) > 0$ for some $s \in G$. The next lemma says that for any dynamically generating continuous pseudometric ρ, one can construct a compatible metric ρ' on X such that $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) = \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho')$. This is the analogue of the Kolmogorov-Sinai theorem that for measure-preserving actions of $\mathbb{Z}$ the entropy of any generating finite partition is equal to the entropy of the action. In some examples X is naturally a closed invariant subset of Y^G equipped with the shift action of G for some compact metrizable space Y which is much simpler than X. In such case one can take a compatible metric ρ on Y and think of it as a dynamically generating continuous pseudometric on X via the coordinate map $X \to Y$ at e_G. Then the calculation using ρ is much simpler than that using ρ' (see for example Section 7 and [BL, KL11, KL13]).

Lemma 4.4. *Let ρ be a dynamically generating continuous pseudometric on X. Enumerate the elements of G as $s_1, s_2, \ldots$. Define ρ' by $\rho'(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho(s_n x, s_n y)$*

for all $x, y \in X$. Then ρ' is a compatible metric on X. Furthermore, if $e_G = s_m$, then for any $\varepsilon > 0$ one has

$$h^{4\varepsilon}_{\Sigma,\infty}(\rho') \le h^{\varepsilon}_{\Sigma,\infty}(\rho) \le h^{\varepsilon/2^m}_{\Sigma,\infty}(\rho').$$

In particular, $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X,\rho) = \mathrm{mdim}_{\Sigma,\mathrm{M}}(X,\rho')$.

Proof. Clearly ρ' is a continuous pseudometric on X. Since ρ is dynamically generating, ρ' separates the points of X. Thus ρ' is a compatible metric on X. Let $\varepsilon > 0$.

We show first $h^{\varepsilon}_{\Sigma,\infty}(\rho) \le h^{\varepsilon/2^m}_{\Sigma,\infty}(\rho')$. Let F be a finite subset of G containing e_G and $\delta > 0$. Take $k \in \mathbb{N}$ with $2^{-k}\mathrm{diam}(X,\rho) < \delta/2$. Set $F' = \bigcup_{n=1}^k s_n F$ and take $1 > \delta' > 0$ to be small which we shall fix in a moment.

Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G. We claim that $\mathrm{Map}(\rho, F', \delta', \sigma) \subseteq \mathrm{Map}(\rho', F, \delta, \sigma)$. Let $\varphi \in \mathrm{Map}(\rho, F', \delta', \sigma)$. By Lemma 2.8 one has

$$|\mathcal{W}| \ge (1 - \delta'|F'|)d,$$

for

$$\mathcal{W} := \{a \in [d] : \max_{s\in F'} \rho(\varphi\circ\sigma_s(a), \alpha_s\circ\varphi(a)) \le \sqrt{\delta'}\}.$$

Set $\mathcal{R} = \mathcal{W} \cap \bigcap_{t\in F} \sigma_t^{-1}(\mathcal{W})$. Then $|\mathcal{R}| \ge (1 - \delta'|F'|(1+|F|))d$. Also set

$$\mathcal{Q} = \{a \in [d] : \sigma_{s_n}\circ\sigma_t(a) = \sigma_{s_n t}(a) \text{ for all } 1 \le n \le k \text{ and } t \in F\}.$$

For any $a \in \mathcal{R}\cap\mathcal{Q}$ and $t \in F$, since $a, \sigma_t(a) \in \mathcal{W}$ and $s_n, s_n t \in F'$ for all $1 \le n \le k$, we have

$$\begin{aligned}
&\rho'(\varphi\circ\sigma_t(a), \alpha_t\circ\varphi(a)) \\
\le\ & 2^{-k}\mathrm{diam}(X,\rho) + \sum_{n=1}^k \frac{1}{2^n}\rho(\alpha_{s_n}\circ\varphi\circ\sigma_t(a), \alpha_{s_n}\circ\alpha_t\circ\varphi(a)) \\
\le\ & \delta/2 + \sum_{n=1}^k \frac{1}{2^n}\Big(\rho(\alpha_{s_n}\circ\varphi\circ\sigma_t(a), \varphi\circ\sigma_{s_n}\circ\sigma_t(a)) + \rho(\varphi\circ\sigma_{s_n t}(a), \alpha_{s_n t}\circ\varphi(a))\Big) \\
\le\ & \delta/2 + \sum_{n=1}^k \frac{1}{2^n}\cdot 2\sqrt{\delta'} \le \delta/2 + 2\sqrt{\delta'}.
\end{aligned}$$

When σ is a good enough sofic approximation for G, one has $|\mathcal{Q}| \ge (1-\delta'|F'|)d$ and hence for any $t \in F$,

$$\begin{aligned}
(\rho'_2(\varphi\circ\sigma_t, \alpha_t\circ\varphi))^2 &\le \frac{1}{d}(|\mathcal{R}\cap\mathcal{Q}|(\delta/2+2\sqrt{\delta'})^2 + (d - |\mathcal{R}\cap\mathcal{Q}|)(\mathrm{diam}(X,\rho'))^2) \\
&\le (\delta/2+2\sqrt{\delta'})^2 + \delta'|F'|(2+|F|)(\mathrm{diam}(X,\rho'))^2 < \delta^2,
\end{aligned}$$

when δ' is small enough independent of σ and φ. Therefore $\varphi \in \mathrm{Map}(\rho', F, \delta, \sigma)$. This proves the claim.

Note that $\frac{1}{2^m}\rho_\infty \leq \rho'_\infty$ on $\mathrm{Map}(\rho, F', \delta', \sigma)$. Thus

$$N_\varepsilon(\mathrm{Map}(\rho, F', \delta', \sigma), \rho_\infty) \leq N_{\varepsilon/2^m}(\mathrm{Map}(\rho, F', \delta', \sigma), \rho'_\infty) \leq N_{\varepsilon/2^m}(\mathrm{Map}(\rho', F, \delta, \sigma), \rho'_\infty),$$

when σ is a good enough sofic approximation for G. It follows that $h^\varepsilon_{\Sigma,\infty}(\rho, F', \delta') \leq h^{\varepsilon/2^m}_{\Sigma,\infty}(\rho', F, \delta)$, and hence $h^\varepsilon_{\Sigma,\infty}(\rho) \leq h^{\varepsilon/2^m}_{\Sigma,\infty}(\rho')$ as desired.

Next we show $h^{4\varepsilon}_{\Sigma,\infty}(\rho') \leq h^\varepsilon_{\Sigma,\infty}(\rho)$. It suffices to show $h^{4\varepsilon}_{\Sigma,\infty}(\rho') \leq h^\varepsilon_{\Sigma,\infty}(\rho) + \theta$ for every $\theta > 0$. Take $k \in \mathbb{N}$ with $2^{-k}\mathrm{diam}(X, \rho) < \varepsilon/2$.

Let F be a finite subset of G containing $\{s_1, \dots, s_k\}$ and $\delta > 0$ be sufficiently small which we shall specify in a moment. Set $\delta' = \delta/2^m$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some sufficiently large $d \in \mathbb{N}$. Note that $\frac{1}{2^m}\rho_2(\varphi, \psi) \leq \rho'_2(\varphi, \psi)$ for all maps $\varphi, \psi : [d] \to X$. Thus $\mathrm{Map}(\rho, F, \delta, \sigma) \supseteq \mathrm{Map}(\rho', F, \delta', \sigma)$.

Let $\mathscr{E}$ be a $(\rho'_\infty, 4\varepsilon)$-separated subset of $\mathrm{Map}(\rho', F, \delta', \sigma)$ with $|\mathscr{E}| = N_{4\varepsilon}(\mathrm{Map}(\rho', F, \delta', \sigma), \rho'_\infty)$. For each $\varphi \in \mathscr{E}$ denote by $\mathcal{W}_\varphi$ the set of $a \in [d]$ satisfying $\rho(\alpha_s \circ \varphi(a), \varphi \circ \sigma_s(a)) \leq \sqrt{\delta}$ for all $s \in F$. By Lemma 2.8 one has $|\mathcal{W}_\varphi| \geq (1 - |F|\delta)d$. Take δ to be small enough so that $|F|\delta < 1/2$. Stirling's approximation says that $\frac{t!}{(2\pi t)^{1/2}(t/e)^t} \to 1$ as $t \to +\infty$. The number of subsets of $[d]$ of cardinality at most $|F|\delta d$ is equal to $\sum_{j=0}^{\lfloor |F|\delta d \rfloor}\binom{d}{j}$, which is at most $|F|\delta d\binom{d}{|F|\delta d}$ (here we use $|F|\delta < 1/2$), which by Stirling's approximation is less than $\exp(\beta d)$ for some $\beta > 0$ depending on δ and $|F|$ but not on d when d is sufficiently large with $\beta \to 0$ as $\delta \to 0$ for a fixed $|F|$. Take δ to be small enough such that $\beta < \theta/2$. Then, when d is sufficiently large, we can find a subset $\mathscr{F}$ of $\mathscr{E}$ with $|\mathscr{F}|\exp(\beta d) \geq |\mathscr{E}|$ such that $\mathcal{W}_\varphi$ is the same, say $\mathcal{W}$, for every $\varphi \in \mathscr{F}$.

Let $\varphi \in \mathscr{F}$. Let us estimate how many elements in $\mathscr{F}$ are in the open ball $B_\varphi := \{\psi \in X^{[d]} : \rho_\infty(\varphi, \psi) < \varepsilon\}$. Let $\psi \in \mathscr{F} \cap B_\varphi$. For any $a \in \mathcal{W}$ and $s \in F$, we have

$$\begin{aligned} \rho(s\varphi(a), s\psi(a)) &\leq \rho(s\varphi(a), \varphi(sa)) + \rho(\varphi(sa), \psi(sa)) + \rho(\psi(sa), s\psi(a)) \\ &\leq \sqrt{\delta} + \varepsilon + \sqrt{\delta} \leq \frac{3}{2}\varepsilon, \end{aligned}$$

when δ is taken to be small enough. It follows that for any $a \in \mathcal{W}$ we have

$$\begin{aligned} \rho'(\varphi(a), \psi(a)) &\leq 2^{-k}\mathrm{diam}(X, \rho) + \sum_{n=1}^{k} 2^{-n}\rho(s_n\varphi(a), s_n\psi(a)) \\ &< \frac{1}{2}\varepsilon + \frac{3}{2}\varepsilon = 2\varepsilon. \end{aligned}$$

Then $\rho'_\infty(\varphi|_\mathcal{W}, \psi|_\mathcal{W}) < 2\varepsilon$.

Let Y be a maximal $(\rho', 2\varepsilon)$-separated subset of X. Set $\mathcal{W}^c = [d] \setminus \mathcal{W}$. For each $\psi \in \mathscr{F} \cap B_\varphi$, there exists some $f_\psi \in Y^{\mathcal{W}^c}$ with $\rho'_\infty(\psi|_{\mathcal{W}^c}, f_\psi) < 2\varepsilon$. Then we can find a subset $\mathscr{A}$ of $\mathscr{F} \cap B_\varphi$ with $|Y|^{|\mathcal{W}^c|}|\mathscr{A}| \geq |\mathscr{F} \cap B_\varphi|$ such that f_ψ is the same, say f, for every $\psi \in \mathscr{A}$. For any $\psi, \psi' \in \mathscr{A}$, we have

$$\rho'_\infty(\psi|_{\mathcal{W}^c}, \psi'|_{\mathcal{W}^c}) \leq \rho'_\infty(\psi|_{\mathcal{W}^c}, f) + \rho'_\infty(f, \psi'|_{\mathcal{W}^c}) < 4\varepsilon,$$

and

$$\rho'_\infty(\psi|_{\mathcal{W}}, \psi'|_{\mathcal{W}}) \leq \rho'_\infty(\psi|_{\mathcal{W}}, \varphi|_{\mathcal{W}}) + \rho'_\infty(\varphi|_{\mathcal{W}}, \psi'|_{\mathcal{W}}) < 4\varepsilon,$$

and hence $\rho'_\infty(\psi, \psi') < 4\varepsilon$. Since $\mathscr{A}$ is $(\rho'_\infty, 4\varepsilon)$-separated, we get $\psi = \psi'$. Therefore $|\mathscr{A}| \leq 1$, and hence

$$|\mathscr{F} \cap B_\varphi| \leq |Y|^{|\mathcal{W}^c|}|\mathscr{A}| \leq |Y|^{|F|\delta d}.$$

Let $\mathscr{B}$ be a maximal $(\rho_\infty, \varepsilon)$-separated subset of $\mathscr{F}$. Then $\mathscr{F} = \bigcup_{\varphi\in\mathscr{B}}(\mathscr{F} \cap B_\varphi)$. Thus

$$\begin{aligned} N_{4\varepsilon}(\mathrm{Map}(\rho', F, \delta', \sigma), \rho'_\infty) = |\mathscr{E}| \leq \exp(\beta d)|\mathscr{F}| &\leq \exp(\theta d/2)|\mathscr{B}||Y|^{|F|\delta d} \\ &\leq \exp(\theta d/2)|Y|^{|F|\delta d} N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty) \\ &\leq \exp(\theta d) N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty) \end{aligned}$$

when we take δ to be small enough, where in the third inequality we used $\mathscr{B} \subseteq \mathrm{Map}(\rho', F, \delta', \sigma) \subseteq \mathrm{Map}(\rho, F, \delta, \sigma)$ to conclude $|\mathscr{B}| \leq N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty)$. Therefore $h^{4\varepsilon}_{\Sigma,\infty}(\rho', F, \delta') \leq h^{\varepsilon}_{\Sigma,\infty}(\rho, F, \delta) + \theta$. It follows that $h^{4\varepsilon}_{\Sigma,\infty}(\rho') \leq h^{\varepsilon}_{\Sigma,\infty}(\rho) + \theta$ as desired. □

Note that $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X)$ was defined as the infimum over all compatible metrics ρ of $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$. However by Lemma 4.4 we get:

Proposition 4.5. *One has*

$$\mathrm{mdim}_{\Sigma,\mathrm{M}}(X) = \inf_\rho \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$$

for ρ ranging over dynamically generating continuous pseudometrics on X.

The following is the analogue of Proposition 2.11.

Proposition 4.6. *Let G act continuously on a compact metrizable space X. Let Y be a closed G-invariant subset of X. Then* $\mathrm{mdim}_{\Sigma,\mathrm{M}}(Y) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X)$.

Proof. Let ρ be a compatible metric on X. Then ρ restricts to a compatible metric ρ' on Y.

Note that $\mathrm{Map}(\rho', F, \delta, \sigma) \subseteq \mathrm{Map}(\rho, F, \delta, \sigma)$ for any nonempty finite subset F of G, any $\delta > 0$, and any map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Furthermore, the restriction of ρ_∞ on $\mathrm{Map}(\rho', F, \delta, \sigma)$ is exactly ρ'_∞. Thus $N_\varepsilon(\mathrm{Map}(\rho', F, \delta, \sigma), \rho'_\infty) \leq N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty)$ for any $\varepsilon > 0$. It follows that $\mathrm{mdim}_{\Sigma,\mathrm{M}}(Y) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(Y, \rho') \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$. Since ρ is an arbitrary compatible metric on X, we get $\mathrm{mdim}_{\Sigma,\mathrm{M}}(Y) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X)$. □

5. Sofic Metric Mean Dimension for Amenable Groups

In this section we show that the sofic metric mean dimension extends the metric mean dimension for actions of countable amenable groups:

Theorem 5.1. *Let a countable (discrete) amenable group G act continuously on a compact metrizable space X. Let Σ be a sofic approximation sequence for G. Then*

$$\mathrm{mdim}_{\Sigma,\mathrm{M}}(X,\rho) = \mathrm{mdim}_{\mathrm{M}}(X,\rho)$$

for every continuous pseudometric ρ on X. In particular,

$$\mathrm{mdim}_{\Sigma,\mathrm{M}}(X) = \mathrm{mdim}_{\mathrm{M}}(X).$$

Theorem 5.1 follows directly from Lemmas 5.2 and 5.3 below. The proof of Lemma 5.2 is similar to that of Lemma 3.5.

Lemma 5.2. *Let a countable amenable group G act continuously on a compact metrizable space X. Let ρ be a continuous pseudometric on X. Then for any $\varepsilon > 0$ we have $h^{\varepsilon}_{\Sigma,\infty}(\rho) \geq S(X, 2\varepsilon, \rho)$. In particular,* $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X,\rho) \geq \mathrm{mdim}_{\mathrm{M}}(X,\rho)$.

Proof. It suffices to show that $h^{\varepsilon}_{\Sigma,\infty}(\rho) \geq S(X, 2\varepsilon, \rho) - 2\theta$ for every $\theta > 0$.

Take a nonempty finite subset K of G and $\varepsilon' > 0$ such that for any nonempty finite subset F' of G satisfying $|KF' \setminus F'| < \varepsilon'|F'|$, one has

$$\frac{1}{|F'|}\log N_{\varepsilon}(X,\rho_{F'}) \overset{(1)}{\geq} \frac{1}{|F'|}\log \min_{\mathrm{mesh}(\mathcal{U},\rho_{F'})<2\varepsilon} |\mathcal{U}| \geq S(X,2\varepsilon,\rho) - \theta. \tag{2}$$

Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Now it suffices to show that if σ is a good enough sofic approximation then

$$\frac{1}{d}\log N_{\varepsilon}(\mathrm{Map}(\rho,F,\delta,\sigma),\rho_\infty) \geq S(X,2\varepsilon,\rho) - 2\theta.$$

Take $\delta' > 0$ such that $\sqrt{\delta'}\mathrm{diam}(X,\rho) < \delta/2$ and $(1-\delta')(S(X,2\varepsilon,\rho)-\theta) \geq S(X,2\varepsilon,\rho) - 2\theta$. By Lemma 3.2 there are an $\ell \in \mathbb{N}$ and nonempty finite subsets $F_1,\dots,F_\ell$ of G satisfying $|(K\cup F)F_k \setminus F_k| < \min(\varepsilon',\delta')|F_k|$ for all $k = 1,\dots,\ell$ such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G and every $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1-\delta'/2)d$ there exist $\mathcal{C}_1,\dots,\mathcal{C}_\ell \subseteq \mathcal{W}$ satisfying the following:

(1) for every $k = 1,\dots,\ell$, the map $(s,c) \mapsto \sigma_s(c)$ from $F_k \times \mathcal{C}_k$ to $\sigma(F_k)\mathcal{C}_k$ is bijective,
(2) the sets $\sigma(F_1)\mathcal{C}_1,\dots,\sigma(F_\ell)\mathcal{C}_\ell$ are pairwise disjoint and $|\bigcup_{k=1}^{\ell}\sigma(F_k)\mathcal{C}_k| \geq (1-\delta')d$.

Let $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ be a good enough sofic approximation for G such that $|\mathcal{W}| \geq (1-\delta'/2)d$ for

$$\mathcal{W} := \{a \in [d] : \sigma_t\sigma_s(a) = \sigma_{ts}(a) \text{ for all } t \in F, s \in \bigcup_{k=1}^{\ell} F_k\}.$$

Then we have $\mathcal{C}_1,\dots,\mathcal{C}_\ell$ as above.

For every $k \in \{1, \ldots, \ell\}$ pick an ε-separated set $E_k \subseteq X$ with respect to ρ_{F_k} of maximal cardinality. Then

$$\frac{1}{|F_k|} \log |E_k| = \frac{1}{|F_k|} \log N_\varepsilon(X, \rho_{F_k}) \overset{(2)}{\geq} S(X, 2\varepsilon, \rho) - \theta.$$

For each $h = (h_k)_{k=1}^{\ell} \in \prod_{k=1}^{\ell} (E_k)^{\mathcal{C}_k}$ take a map $\varphi_h : [d] \to X$ such that

$$\varphi_h(sc) = s(h_k(c))$$

for all $k \in \{1, \ldots, \ell\}$, $c \in \mathcal{C}_k$, and $s \in F_k$. By Lemma 3.4 one has $\varphi_h \in \mathrm{Map}(\rho, F, \delta, \sigma)$.

Now if $h = (h_k)_{k=1}^{\ell}$ and $h' = (h'_k)_{k=1}^{\ell}$ are distinct elements of $\prod_{k=1}^{\ell} (E_k)^{\mathcal{C}_k}$, then $h_k(c) \neq h'_k(c)$ for some $k \in \{1, \ldots, \ell\}$ and $c \in \mathcal{C}_k$. Since $h_k(c)$ and $h'_k(c)$ are distinct points in E_k which is ε-separated with respect to ρ_{F_k}, $h_k(c)$ and $h'_k(c)$ are ε-separated with respect to ρ_{F_k}, and thus we have $\rho_\infty(\varphi_h, \varphi_{h'}) \geq \varepsilon$. Therefore

$$\begin{aligned}
\frac{1}{d} \log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty) &\geq \frac{1}{d} \sum_{k=1}^{\ell} |\mathcal{C}_k| \log |E_k| \\
&\geq \frac{1}{d} \sum_{k=1}^{\ell} |\mathcal{C}_k||F_k|(S(X, 2\varepsilon, \rho) - \theta) \\
&\geq (1 - \delta')(S(X, 2\varepsilon, \rho) - \theta) \\
&\geq S(X, 2\varepsilon, \rho) - 2\theta,
\end{aligned}$$

as desired. □

The proof of Lemma 5.3 is similar to that of Lemma 3.7, but considerably easier.

Lemma 5.3. *Let a countable amenable group G act continuously on a compact metrizable space X. Let ρ be a continuous pseudometric on X. Then for any $\varepsilon > 0$ we have $h^\varepsilon_{\Sigma,\infty}(\rho) \leq S(X, \varepsilon/4, \rho)$. In particular,* $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) \leq \mathrm{mdim}_{\mathrm{M}}(X, \rho)$.

Proof. Let $\varepsilon > 0$. It suffices to show that $h^\varepsilon_{\Sigma,\infty}(\rho) \leq S(X, \varepsilon/4, \rho) + 3\theta$ for every $\theta > 0$.

Take a nonempty finite subset K of G and $\delta' > 0$ such that $\min_{\mathrm{mesh}(\mathcal{U}, \rho_{F'}) < \varepsilon/4} |\mathcal{U}| < \exp((S(X, \varepsilon/4, \rho) + \theta)|F'|)$ for every nonempty finite subset F' of G satisfying $|KF' \setminus F'| < \delta'|F'|$. Take an $\eta \in (0, 1)$ such that $(N_{\varepsilon/2}(X, \rho))^{2\eta} \leq \exp(\theta)$.

By Lemma 3.2 there are an $\ell \in \mathbb{N}$ and nonempty finite subsets $F_1, \ldots, F_\ell$ of G satisfying $|KF_k \setminus F_k| < \delta'|F_k|$ for all $k = 1, \ldots, \ell$ such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G and every $\mathcal{W} \subseteq [d]$ with $|\mathcal{W}| \geq (1 - \eta)d$ there exist $\mathcal{C}_1, \ldots, \mathcal{C}_\ell \subseteq \mathcal{W}$ satisfying the following:

(1) for every $k = 1, \ldots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times \mathcal{C}_k$ to $\sigma(F_k)\mathcal{C}_k$ is bijective,
(2) the sets $\sigma(F_1)\mathcal{C}_1, \ldots, \sigma(F_\ell)\mathcal{C}_\ell$ are pairwise disjoint and $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1 - 2\eta)d$.

Then

$$N_{\varepsilon/4}(X,\rho_{F_k}) \overset{(1)}{\leq} \min_{\mathrm{mesh}(\mathcal{U},\rho_{F_k})<\varepsilon/4} |\mathcal{U}| < \exp((S(X,\varepsilon/4,\rho)+\theta)|F_k|) \tag{3}$$

for every $k = 1,\dots,\ell$.

Set $F = \bigcup_{k=1}^{\ell} F_k$. Let $\delta > 0$ be a small positive number which we will determine in a moment. Let σ be a map from G to $\mathrm{Sym}(d)$ for some sufficiently large $d \in \mathbb{N}$ which is a good enough sofic approximation for G. We will show that $N_\varepsilon(\mathrm{Map}(\rho,F,\delta,\sigma),\rho_\infty) \leq \exp(S(X,\varepsilon/4,\rho)+3\theta)d$, which will complete the proof since we can then conclude that $h^{\varepsilon}_{\Sigma,\infty}(\rho,F,\delta) \leq S(X,\varepsilon/4,\rho)+3\theta$ and hence $h^{\varepsilon}_{\Sigma,\infty}(\rho) \leq S(X,\varepsilon/4,\rho)+3\theta$.

For every $\varphi \in \mathrm{Map}(\rho,F,\delta,\sigma)$, by Lemma 2.8 the set $\mathcal{W}_\varphi$ of all $a \in [d]$ satisfying

$$\rho(\varphi(sa), s\varphi(a)) \leq \sqrt{\delta}$$

for all $s \in F$ has cardinality at least $(1-|F|\delta)d$.

For each $\mathcal{W} \subseteq [d]$ we define on the set of maps from $[d]$ to X the pseudometric

$$\rho_{\mathcal{W},\infty}(\varphi,\psi) = \rho_\infty(\varphi|_{\mathcal{W}}, \psi|_{\mathcal{W}}).$$

Take a $(\rho_\infty,\varepsilon)$-separated subset $\mathscr{E}$ of $\mathrm{Map}(\rho,F,\delta,\sigma)$ of maximal cardinality.

Set $n = |F|$. Stirling's approximation says that $\frac{t!}{(2\pi t)^{1/2}(t/e)^t} \to 1$ as $t \to +\infty$. When $n\delta < 1/2$, the number of subsets of $[d]$ of cardinality no greater than $n\delta d$ is equal to $\sum_{j=0}^{\lfloor n\delta d\rfloor}\binom{d}{j}$, which is at most $n\delta d\binom{d}{n\delta d}$, which by Stirling's approximation is less than $\exp(\beta d)$ for some $\beta > 0$ depending on δ and n but not on d when d is sufficiently large with $\beta \to 0$ as $\delta \to 0$ for a fixed n. Thus when δ is small enough and d is large enough, there is a subset $\mathscr{F}$ of $\mathscr{E}$ with $\exp(\theta d)|\mathscr{F}| \geq |\mathscr{E}|$ such that the set $\mathcal{W}_\varphi$ is the same, say $\mathcal{W}$, for every $\varphi \in \mathscr{F}$, and $|\mathcal{W}|/d > 1-\eta$. Then we have $\mathcal{C}_1,\dots,\mathcal{C}_\ell \subseteq \mathcal{W}$ as above.

Let $1 \leq k \leq \ell$ and $c \in \mathcal{C}_k$. Let $\mathscr{D}_{k,c}$ be a maximal $(\varepsilon/2)$-separated subset of $\mathscr{F}$ with respect to $\rho_{\sigma(F_k)c,\infty}$. Then $\mathscr{D}_{k,c}$ is a $(\rho_{\sigma(F_k)c,\infty},\varepsilon/2)$-spanning subset of $\mathscr{F}$ in the sense that for any $\varphi \in \mathscr{F}$, there exists some $\psi \in \mathscr{D}_{k,c}$ with $\rho_{\sigma(F_k)c,\infty}(\varphi,\psi) < \varepsilon/2$. We will show that $|\mathscr{D}_{k,c}| \leq \exp((S(X,\varepsilon/4,\rho)+\theta)|F_k|)$ when δ is small enough. For any two distinct elements φ and ψ of $\mathscr{D}_{k,c}$ we have, for every $s \in F_k$, since $c \in \mathcal{W}_\varphi \cap \mathcal{W}_\psi$,

$$\begin{aligned}\rho(s\varphi(c), s\psi(c)) &\geq \rho(\varphi(sc),\psi(sc)) - \rho(s\varphi(c),\varphi(sc)) - \rho(s\psi(c),\psi(sc))\\ &\geq \rho(\varphi(sc),\psi(sc)) - 2\sqrt{\delta},\end{aligned}$$

and hence

$$\rho_{F_k}(\varphi(c),\psi(c)) = \max_{s\in F_k}\rho(s\varphi(c),s\psi(c)) \geq \max_{s\in F_k}\rho(\varphi(sc),\psi(sc)) - 2\sqrt{\delta} > \varepsilon/2 - \varepsilon/4 = \varepsilon/4,$$

granted that δ is taken small enough. Thus $\{\varphi(c) : \varphi \in \mathscr{D}_{k,c}\}$ is a $(\rho_{F_k}, \varepsilon/4)$-separated subset of X of cardinality $|\mathscr{D}_{k,c}|$, so that

$$|\mathscr{D}_{k,c}| \leq N_{\varepsilon/4}(X, \rho_{F_k}) \overset{(3)}{\leq} \exp((S(X, \varepsilon/4, \rho) + \theta)|F_k|),$$

as we wished to show.

Set

$$\mathcal{Z} = [d] \setminus \bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k,$$

and take a maximal $(\rho, \varepsilon/2)$-separated subset Y of X. Then $Y^{\mathcal{Z}}$ is a $(\rho_\infty, \varepsilon/2)$-spanning subset of $X^{\mathcal{Z}}$ in the sense that for any $\varphi \in X^{\mathcal{Z}}$ there is some $\psi \in Y^{\mathcal{Z}}$ with $\rho_\infty(\varphi, \psi) < \varepsilon/2$. We have

$$|Y^{\mathcal{Z}}| \leq (N_{\varepsilon/2}(X, \rho))^{|\mathcal{Z}|} \leq (N_{\varepsilon/2}(X, \rho))^{2\eta d}.$$

Write $\mathscr{A}$ for the set of all maps $\varphi : [d] \to X$ such that $\varphi|_{\mathcal{Z}} \in Y^{\mathcal{Z}}$ and $\varphi|_{\sigma(F_k)c} \in \mathscr{D}_{k,c}|_{\sigma(F_k)c}$ for all $1 \leq k \leq \ell$ and $c \in \mathcal{C}_k$. Then, by our choice of η,

$$\begin{aligned}
|\mathscr{A}| &= |Y^{\mathcal{Z}}| \prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} |\mathscr{D}_{k,c}| \leq (N_{\varepsilon/2}(X, \rho))^{2\eta d} \exp\left(\sum_{k=1}^{\ell} \sum_{c \in \mathcal{C}_k} (S(X, \varepsilon/4, \rho) + \theta)|F_k| \right) \\
&= (N_{\varepsilon/2}(X, \rho))^{2\eta d} \exp\left((S(X, \varepsilon/4, \rho) + \theta) \sum_{k=1}^{\ell} |F_k||C_k| \right) \\
&\leq \exp(\theta d) \exp((S(X, \varepsilon/4, \rho) + \theta)d) = \exp((S(X, \varepsilon/4, \rho) + 2\theta)d).
\end{aligned}$$

Now since every element of $\mathscr{F}$ lies within ρ_∞-distance $\varepsilon/2$ to an element of $\mathscr{A}$ and $\mathscr{F}$ is ε-separated with respect to ρ_∞, the cardinality of $\mathscr{F}$ is at most that of $\mathscr{A}$. Therefore

$$\begin{aligned}
N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty) &= |\mathscr{E}| \leq \exp(\theta d)|\mathscr{F}| \leq \exp(\theta d)|\mathscr{A}| \\
&\leq \exp(\theta d) \exp((S(X, \varepsilon/4, \rho) + 2\theta)d) \\
&= \exp((S(X, \varepsilon/4, \rho) + 3\theta)d),
\end{aligned}$$

as desired. □

6. Comparison of Sofic Mean Dimensions

In this section we prove the following relation between sofic mean dimensions:

Theorem 6.1. *Let a countable sofic group G act continuously on a compact metrizable space X. Let Σ be a sofic approximation sequence of G. Then*

$$\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X).$$

The amenable group case of Theorem 6.1 was proved by Lidenstrauss and Weiss [LW, Theorem 4.2]. We adapt their proof to our situation, by replacing the set $\{(sx)_{s\in H} : x \in X\} \subseteq X^H$ for an approximately left invariant finite subset H of G in their proof (in their case $G = \mathbb{Z}$ and $H = \{0, \dots, N-1\}$) with the set $\mathrm{Map}(\rho, F, \delta, \sigma)$.

Lemma 6.2. *Let $\mathfrak{U}$ be a finite open cover of X and ρ be a compatible metric on X. Then there exist Lipschitz functions $f_U : X \to [0,1]$ vanishing on $X \setminus U$ for each $U \in \mathfrak{U}$ such that $\max_{U\in\mathfrak{U}} f_U(x) = 1$ for every $x \in X$.*

Proof. If some elements of $\mathfrak{U}$ are equal to the whole space X, we may set f_U to be the constant function 1 for the elements U equal to X and the constant function 0 for those elements U not equal to X. Thus we mat assume that all elements of $\mathfrak{U}$ do not equal X. Note that for each $U \in \mathfrak{U}$, the function $\rho(\cdot, X \setminus U) : X \to [0, +\infty)$ is Lipschitz and vanishes on $X \setminus U$. Furthermore, $\sum_{V\in\mathfrak{U}} \rho(x, X \setminus V) > 0$ for every $x \in X$. Define $g_U, f_U : X \to [0, 1]$ by

$$g_U(x) = \frac{\rho(x, X \setminus U)}{\sum_{V\in\mathfrak{U}} \rho(x, X \setminus V)},$$

and

$$f_U(x) = |\mathfrak{U}| \min(g_U(x), 1/|\mathfrak{U}|).$$

Then g_U is Lipschitz and vanishes on $X \setminus U$, and hence so is f_U. Furthermore, for each $x \in X$, one has $\sum_{U\in\mathfrak{U}} g_U(x) = 1$ and thus $g_U(x) \geq 1/|\mathfrak{U}|$ for some $U \in \mathfrak{U}$. It follows that $f_U(x) = 1$ for some $U \in \mathfrak{U}$. $\square$

Let $\rho, \mathfrak{U}$ and f_U be as in Lemma 6.2. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We define a continuous map $\Phi_d : X^{[d]} \to [0, 1]^{[d]\times\mathfrak{U}}$ by $\Phi_d(\varphi)_{a,U} = f_U(\varphi(a))$ for $a \in [d]$ and $U \in \mathfrak{U}$.

Lemma 6.3. *Let $\theta > 0$, and set $D = \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$. Then there exist a nonempty finite subset F of G, $\delta > 0$ and $M > 0$ such that for any $i \in \mathbb{N}$ with $i \geq M$, there exists $\xi \in (0, 1)^{[d_i]\times\mathfrak{U}}$ such that for any $S \subseteq [d_i] \times \mathfrak{U}$ with $|S| \geq (D + \theta)d_i$, one has*

$$\xi|_S \notin \Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma_i))|_S.$$

Proof. Denote by C the maximum of the Lipschitz constants of f_U for all $U \in \mathfrak{U}$. Then

$$\|\Phi_d(\varphi) - \Phi_d(\psi)\|_\infty \leq C\rho_\infty(\varphi, \psi)$$

for all $d \in \mathbb{N}$ and $\varphi, \psi \in X^{[d]}$.

Take $\varepsilon > 0$ small enough, which we shall determine in a moment, satisfying

$$\frac{h^{\varepsilon}_{\Sigma,\infty}(\rho)}{|\log \varepsilon|} < D + \theta/3.$$

Then we can find a nonempty finite subset F of G, $\delta > 0$, and $M > 0$ such that for every $i \in \mathbb{N}$ with $i > M$, we have

$$\frac{1}{d_i|\log \varepsilon|} \log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_\infty) \leq D + \theta/2,$$

that is,

$$N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_\infty) \leq \varepsilon^{-(D+\theta/2)d_i}.$$

It follows that we can cover $\Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma_i))$ using $\varepsilon^{-(D+\theta/2)d_i}$ open balls in the $\|\cdot\|_\infty$ norm with radius εC.

Denote by μ the Lebesgue measure on $[0,1]^{[d_i]\times\mathcal{U}}$. For each $S \subseteq [d_i] \times \mathcal{U}$, the set $\Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma_i))|_S \subseteq [0,1]^S$ can be covered using $\varepsilon^{-(D+\theta/2)d_i}$ open balls in the $\|\cdot\|_\infty$ norm with radius εC, and hence has Lebesgue measure at most $\varepsilon^{-(D+\theta/2)d_i}(2\varepsilon C)^{|S|}$. Thus,

$$\mu(\{\xi \in [0,1]^{[d_i]\times\mathcal{U}} : \xi|_S \in \Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma_i))|_S\}) \leq \varepsilon^{-(D+\theta/2)d_i}(2\varepsilon C)^{|S|}.$$

Therefore, the set of $\xi \in [0,1]^{[d_i]\times\mathcal{U}}$ satisfying $\xi|_S \in \Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma_i))|_S$ for some $S \subseteq [d_i] \times \mathcal{U}$ with $|S| \geq (D+\theta)d_i$ has Lebesgue measure at most

$$2^{|[d_i]\times\mathcal{U}|}\varepsilon^{-(D+\theta/2)d_i}(2\varepsilon C)^{(D+\theta)d_i} = (2^{|\mathcal{U}|}\varepsilon^{\theta/2}(2C)^{D+\theta})^{d_i} < 1,$$

when we take ε to be small enough. Thus we can find ξ with desired property. $\square$

Lemma 6.4. *Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Let Ψ be a continuous map from $\Phi_d(\mathrm{Map}(\rho, F, \delta, \sigma))$ to $[0,1]^{[d]\times\mathcal{U}}$. Suppose that for any $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ and any $(a, U) \in [d] \times \mathcal{U}$, if $\Phi_d(\varphi)_{a,U} = 0$ or 1, then $\Psi(\Phi_d(\varphi))_{a,U} = 0$ or 1 accordingly. Then $\Psi \circ \Phi_d$ is $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$-compatible.*

Proof. Let $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ and $a \in [d]$. By the choice of $\{f_U\}_{U\in\mathcal{U}}$ there exists some $U \in \mathcal{U}$ such that $\Phi_d(\varphi)_{a,U} = f_U(\varphi(a)) = 1$. By our assumption on Ψ we then have $\Psi(\Phi_d(\varphi))_{a,U} = 1$.

Let $\zeta \in \Psi(\Phi_d(\mathrm{Map}(\rho, F, \delta, \sigma)))$ and $a \in [d]$. It suffices to show that there exists some $U \in \mathcal{U}$ such that one has $\varphi(a) \in U$ for every $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ satisfying $\Psi(\Phi_d(\varphi)) = \zeta$. By the above paragraph we can find some $U \in \mathcal{U}$ such that $\zeta_{a,U} = 1$. Let $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ with $\Psi(\Phi_d(\varphi)) = \zeta$. By our assumption on Ψ we have $f_U(\varphi(a)) = \Phi_d(\varphi)_{a,U} > 0$. Since f_U vanishes on $X \setminus U$, we get $\varphi(a) \in U$. $\square$

Lemma 6.5. *Let W be a finite set and Z a closed subset of $[0,1]^W$. Let $m \in \mathbb{N}$ and $\xi \in (0,1)^W$ such that for every $S \subseteq W$ with $|S| \geq m$ one has $\xi|_S \notin Z|_S$. Then there exists a continuous map Ψ from Z into $[0,1]^W$ such that $\dim(\Psi(Z)) \leq m$ and for any $z \in Z$ and any $w \in W$, if $z_w = 0$ or 1, then $\Psi(z)_w = 0$ or 1 accordingly.*

Proof. This can be proved as in the proof of [LW, Theorem 4.2]. We give a slightly different proof.

For each $S \subseteq W$, denote by Y_S the subset of $[0,1]^W$ consisting of elements whose coordinate at any $w \in W \backslash S$ is either 0 or 1. Then Y_S is a closed subset of $[0,1]^W$ with dimension $|S|$. Set $Y = \bigcup_{|S| \leq m} Y_S$. Since the union of finitely many closed subsets of dimension at most m has dimension at most m [HW, page 30 and Theorem V.8], Y has dimension at most m.

As Z is compact, by the assumption on ξ we can find $\delta > 0$ such that $0 < \xi_w - \delta < \xi_w + \delta < 1$ for every $w \in W$ and the set of $w \in W$ satisfying $\xi_w - \delta \leq z_w \leq \xi_w + \delta$ has cardinality at most m for every $z \in Z$. For each $w \in W$, take a continuous map $f_w : [0,1] \to [0,1]$ sending $[0, \xi_w - \delta]$ to 0, and $[\xi_w + \delta, 1]$ to 1.

Define a continuous map Ψ from $[0,1]^W$ into itself by setting the coordinate of $\Psi(x)$ at $w \in W$ to be $f_w(x_w)$. For any $x \in [0,1]^W$ and $w \in W$, if $x_w = 0$ or 1, then clearly $\Psi(z)_w = 0$ or 1 accordingly. From the choice of δ and f_w it is also clear that $\Psi(Z) \subseteq Y$. Thus $\dim(\Psi(Z)) \leq \dim(Y) \leq m$. $\square$

We are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $\mathcal{U}$ be a finite open cover of X, ρ be a compatible metric on X, and $\theta > 0$. Then we have f_U as in Lemma 6.2. For each $i \in \mathbb{N}$, define a continuous map $\Phi_{d_i} : X^{[d_i]} \to [0,1]^{[d_i] \times \mathcal{U}}$ by $\Phi_{d_i}(\varphi)_{a,U} = f_U(\varphi(a))$ for $a \in [d_i]$ and $U \in \mathcal{U}$. Take F, δ, M, i and ξ as in Lemma 6.3. By Lemmas 6.5 and 6.4 we can find a continuous map $\Psi : \Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma)) \to [0,1]^{[d_i] \times \mathcal{U}}$ such that $\Psi \circ \Phi_{d_i}$ is $\mathcal{U}^{d_i}|_{\mathrm{Map}(\rho,F,\delta,\sigma_i)}$-compatible and $\dim(\Psi(\Phi_{d_i}(\mathrm{Map}(\rho, F, \delta, \sigma)))) \leq (\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) + \theta)d_i$. From Lemma 3.6 we get $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \delta_i) \leq (\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) + \theta)d_i$. It follows that

$$\mathcal{D}_\Sigma(\mathcal{U}) = \mathcal{D}_\Sigma(\mathcal{U}, \rho) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho) + \theta.$$

Since $\mathcal{U}$ is an arbitrary finite open cover of X and θ is an arbitrary positive number, we get $\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$. As ρ is an arbitrary compatible metric on X, we get $\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X)$. $\square$

The Pontrjagin-Schnirelmann theorem [PS] [Nag, page 80] says that for any compact metrizable space Z, the dimension $\dim Z$ of Z is equal to the minimal value of $\underline{\dim}_B(Z, \rho)$ for ρ ranging over compatible metrics on Z. Since $\mathrm{mdim}_\Sigma(X)$ and $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)$ are dynamical analogues of $\dim(X)$ and $\underline{\dim}_B(X, \rho)$ respectively, it is natural to ask

Question 6.6. Let a countably infinite sofic group G act continuously on a compact metrizable space X. Then is there any compatible metric ρ on X satisfying

$$\mathrm{mdim}_\Sigma(X) = \mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho)?$$

Question 6.6 was answered affirmatively by Lindenstrauss in the case $G = \mathbb{Z}$ and X has a nontrivial minimal factor [Lin, Theorem 4.3].

7. Bernoulli Shifts

In this section we discuss sofic mean dimension of the Bernoulli shifts and their factors, proving Theorems 7.1 and 7.5. Throughout this section G will be a countable sofic group, and Σ will be a fixed sofic approximation sequence of G.

For any compact metrizable space Z, we have the left shift action α of G on Z^G given by $(sx)_t = x_{s^{-1}t}$ for all $x \in Z^G$ and $s, t \in G$.

Theorem 7.1. *Let Z be a compact metrizable space, and consider the left shift action of a countable sofic group G on $X = Z^G$. Then*

$$\mathrm{mdim}_\Sigma(X) \leq \mathrm{mdim}_{\Sigma,\mathrm{M}}(X) \leq \dim(Z).$$

If furthermore Z contains a copy of $[0,1]^n$ for every natural number $n \leq \dim(Z)$ (for example, Z could be any polyhedron or the Hilbert cube), then

$$\mathrm{mdim}_\Sigma(X) = \mathrm{mdim}_{\Sigma,\mathrm{M}}(X) = \dim(Z).$$

The amenable group case of Theorem 7.1 was proved by Lindenstrauss and Weiss [LW, Propositions 3.1 and 3.3], and Coornaert and Krieger [CK, Corollaries 4.2 and 5.5].

Remark 7.2. In general it is not easy to construct elements of $\mathrm{Map}(\rho, F, \delta, \sigma)$ for sofic non-amenable groups. Theorem 7.1 implies that there are plenty of such elements for Bernoulli shifts. When G is residually finite (for example free groups) and Σ comes from a sequence of finite quotient groups of G, each periodic point in X with the period being the corresponding finite-index normal subgroup of G gives rise to an element of $\mathrm{Map}(\rho, F, \delta, \sigma)$ [KL11, Theorem 7.1] [BL, Theorem 1.3].

Recall the lower box dimension recalled at the beginning of Section 4. We need the following lemma.

Lemma 7.3. *Let Z be a compact metrizable space, and consider the left shift action α of G on $X = Z^G$. Let ρ be a compatible metric on Z. Define ρ' by $\rho'(x, y) = \rho(x_{e_G}, y_{e_G})$ for $x, y \in X$. Then ρ' is a dynamically generating continuous pseudometric on X. Furthermore, for any $\varepsilon > 0$ one has*

$$\log N_\varepsilon(Z, \rho) \leq h^\varepsilon_{\Sigma,\infty}(\rho') \leq \log N_{\varepsilon/2}(Z, \rho).$$

In particular, $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X, \rho') = \underline{\dim}_B(Z, \rho)$.

Proof. Clearly ρ' is a dynamically generating continuous pseudometric on X. Let $\varepsilon > 0$.

We show first $h^\varepsilon_{\Sigma,\infty}(\rho') \geq \log N_\varepsilon(Z, \rho)$. It suffices to show $h^\varepsilon_{\Sigma,\infty}(\rho', F, \delta) \geq \log N_\varepsilon(Z, \rho)$ for every nonempty finite subset F of G and every $\delta > 0$. Take a (ρ, ε)-separated subset Y of Z with $|Y| = N_\varepsilon(Z, \rho)$.

Let σ be a map from G to $\mathrm{Sym}(d)$ from some $d \in \mathbb{N}$ which is a good enough sofic approximation for G such that $\sqrt{1 - |\mathcal{Q}|/d} \cdot \mathrm{diam}(Z, \rho) \leq \delta$, where

$$\mathcal{Q} = \{a \in [d] : \sigma_{e_G} \circ \sigma_s(a) = \sigma_s(a) \text{ for all } s \in F\}.$$

For each map $f : [d] \to Y$, we define a map $\varphi_f : [d] \to X$ by

$$(\varphi_f(a))_t = f(\sigma_{t^{-1}}(a))$$

for all $a \in [d]$ and $t \in G$. Let $s \in F$. For any $a \in \mathcal{Q}$, one has

$$(\alpha_s \circ \varphi_f(a))_{e_G} = (\varphi_f(a))_{s^{-1}} = f(\sigma_s(a)) = f(\sigma_{e_G} \circ \sigma_s(a)) = (\varphi_f \circ \sigma_s(a))_{e_G},$$

and hence $\rho'(\alpha_s \circ \varphi_f(a), \varphi_f \circ \sigma_s(a)) = 0$. Thus

$$\rho'_2(\alpha_s \circ \varphi_f, \varphi_f \circ \sigma_s) \leq \sqrt{(1 - |\mathcal{Q}|/d)(\mathrm{diam}(Z, \rho))^2} \leq \delta.$$

Therefore $\varphi_f \in \mathrm{Map}(\rho', F, \delta, \sigma)$. For any distinct maps $f, g : [d] \to Y$, say $f(a) \neq g(a)$ for some $a \in [d]$, one has

$$\begin{aligned}
\rho'_\infty(\varphi_f, \varphi_g) &\geq \rho'(\varphi_f(\sigma_{e_G}^{-1}(a)), \varphi_g(\sigma_{e_G}^{-1}(a))) \\
&= \rho((\varphi_f(\sigma_{e_G}^{-1}(a)))_{e_G}, (\varphi_g(\sigma_{e_G}^{-1}(a)))_{e_G}) \\
&= \rho(f(a), g(a)) \geq \varepsilon.
\end{aligned}$$

Thus the set $\{\varphi_f : f \in Y^{[d]}\}$ is $(\rho'_\infty, \varepsilon)$-separated. Therefore

$$N_\varepsilon(\mathrm{Map}(\rho', F, \delta, \sigma), \rho'_\infty) \geq |Y|^d = (N_\varepsilon(Z, \rho))^d.$$

It follows that $h^\varepsilon_{\Sigma,\infty}(\rho', F, \delta) \geq \log N_\varepsilon(Z, \rho)$ as desired.

Next we show $h^\varepsilon_{\Sigma,\infty}(\rho') \leq \log N_{\varepsilon/2}(Z, \rho)$. It suffices to show $h^\varepsilon_{\Sigma,\infty}(\rho', \{e_G\}, 1) \leq \log N_{\varepsilon/2}(Z, \rho)$. Take a maximal $(\rho, \varepsilon/2)$-separated subset Y of Z. Then $|Y| \leq N_{\varepsilon/2}(Z, \rho)$.

Let σ be a map from G to $\mathrm{Sym}(d)$ from some $d \in \mathbb{N}$. Let $\mathscr{E}$ be a $(\rho'_\infty, \varepsilon)$-separated subset of $\mathrm{Map}(\rho', \{e_G\}, 1, \sigma)$ with $|\mathscr{E}| = N_\varepsilon(\mathrm{Map}(\rho', \{e_G\}, 1, \sigma), \rho'_\infty)$. For each $\varphi \in \mathscr{E}$, we find some map $f_\varphi : [d] \to Y$ such that

$$\max_{a \in [d]} \rho((\varphi(a))_{e_G}, f_\varphi(a)) < \varepsilon/2.$$

If $\varphi, \psi \in \mathscr{E}$ and $f_\varphi = f_\psi$, then

$$\rho'_\infty(\varphi, \psi) = \max_{a \in [d]} \rho((\varphi(a))_{e_G}, (\psi(a))_{e_G}) < \varepsilon,$$

and hence $\varphi = \psi$, since $\mathscr{E}$ is $(\rho'_\infty, \varepsilon)$-separated. Therefore

$$N_\varepsilon(\mathrm{Map}(\rho', \{e_G\}, 1, \sigma), \rho'_\infty) = |\mathscr{E}| \leq |Y|^d \leq (N_{\varepsilon/2}(Z, \rho))^d.$$

It follows that $h^\varepsilon_{\Sigma,\infty}(\rho', \{e_G\}, 1) \leq \log N_{\varepsilon/2}(Z, \rho)$ as desired. $\square$

We also need the following version of Lebesgue's covering theorem:

Lemma 7.4. *[LW, Lemma 3.2] [HW, Theorem IV.2] Let W be a finite set. Let $\mathcal{U}$ be a finite open cover of $[0,1]^W$ such that no element of $\mathcal{U}$ intersects two opposing sides of $[0,1]^W$. Then*

$$\mathcal{D}(\mathcal{U}) \geq |W|.$$

We are ready to prove Theorem 7.1.

Proof of Theorem 7.1. By Theorem 6.1, Lemma 7.3, Proposition 4.5 and the Pontrjagin-Shnirelmann theorem as recalled at the end of Section 6, we have $\mathrm{mdim}_\Sigma(X) \le \mathrm{mdim}_{\Sigma,\mathrm{M}}(X) \le \dim(Z)$.

Now assume that Z contains a copy of $[0,1]^n$ for every natural number $n \le \dim(Z)$. It suffices to show $\mathrm{mdim}_\Sigma(X) \ge \dim(Z)$. In turn it suffices to show $\mathrm{mdim}_\Sigma(X) \ge n$ for every natural number $n \le \dim(Z)$. Since $([0,1]^n)^G$ is a closed G-invariant subset of Z^G, by Proposition 4.6 one has $\mathrm{mdim}_\Sigma(X) \ge \mathrm{mdim}_\Sigma(([0,1]^n)^G)$. Therefore it suffices to show $\mathrm{mdim}_\Sigma(([0,1]^n)^G) \ge n$.

Take a finite open cover $\mathcal{U}$ of $[0,1]^n$ such that no element of $\mathcal{U}$ intersects two opposing sides of $[0,1]^n$. For each $d \in \mathbb{N}$, note that no element of $\mathcal{U}^d$ intersects two opposing sides of $([0,1]^n)^{[d]} = [0,1]^{dn}$, and hence by Lemma 7.4 one has $\mathcal{D}(\mathcal{U}^d) \ge dn$.

Denote by π the map $([0,1]^n)^G \to [0,1]^n$ sending x to x_{e_G}. Then $\tilde{\mathcal{U}} := \pi^{-1}(\mathcal{U})$ is a finite open cover of $([0,1]^n)^G$. Let ρ be a compatible metric on $([0,1]^n)^G$. It suffices to show $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta) \ge n$ for every nonempty finite subset F of G and every $\delta > 0$.

Take a nonempty finite subset K of G such that if $x, y \in ([0,1]^n)^G$ are equal on K, then $\rho(x,y) < \delta/2$.

Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G such that $\delta^2/4 + (1 - |\mathcal{Q}|/d)(|F|+1)(\mathrm{diam}(([0,1]^n)^G, \rho))^2 \le \delta^2$, where

$$\mathcal{Q} = \{a \in [d] : \sigma_{t^{-1}} \circ \sigma_s(a) = \sigma_{t^{-1}s}(a) \text{ for all } s \in F, t \in K \text{ and } \sigma_{e_G}(a) = a\}.$$

For each map $f : [d] \to [0,1]^n$, we define a map $\varphi_f : [d] \to ([0,1]^n)^G$ by

$$(\varphi_f(a))_t = f(\sigma_{t^{-1}}(a)) \tag{4}$$

for all $a \in [d]$ and $t \in G \setminus \{e_G\}$, and

$$(\varphi_f(a))_{e_G} = f(a) \tag{5}$$

for all $a \in [d]$. Note that (4) holds for all $a \in \mathcal{Q}$ and $t \in G$. Set $\mathcal{R} = \mathcal{Q} \cap \bigcap_{s \in F} \sigma_s^{-1}(\mathcal{Q})$. For any $a \in \mathcal{R}$, $s \in F$, and $t \in K$, since $a, \sigma_s(a) \in \mathcal{Q}$, one has

$$(\alpha_s \circ \varphi_f(a))_t = (\varphi_f(a))_{s^{-1}t} = f(\sigma_{t^{-1}s}(a)) = f(\sigma_{t^{-1}} \circ \sigma_s(a)) = (\varphi_f \circ \sigma_s(a))_t,$$

and hence $\rho(\alpha_s \circ \varphi_f(a), \varphi_f \circ \sigma_s(a)) < \delta/2$ by the choice of K. Thus

$$\begin{aligned} \rho_2(\alpha_s \circ \varphi_f, \varphi_f \circ \sigma_s) &\le \sqrt{\delta^2/4 + (1 - |\mathcal{R}|/d)(\mathrm{diam}(([0,1]^n)^G, \rho))^2} \\ &\le \sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(|F|+1)(\mathrm{diam}(([0,1]^n)^G, \rho))^2} \le \delta. \end{aligned}$$

Therefore $\varphi_f \in \mathrm{Map}(\rho, F, \delta, \sigma)$. The map $\Phi : ([0,1]^n)^{[d]} \to \mathrm{Map}(\rho, F, \delta, \sigma)$ sending f to φ_f is clearly continuous. Note that $\Phi^{-1}(\tilde{\mathcal{U}}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}) = \mathcal{U}^d$ because of the equation (5). Therefore $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta, \sigma) \ge \mathcal{D}(\mathcal{U}^d) \ge dn$. It follows that $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta) \ge n$ as desired. $\square$

Theorem 7.5. *Let Z be a path-connected compact metrizable space, and consider the left shift action α of a countable sofic group G on $X = Z^G$. For any nontrivial factor Y of X, one has*

$$\mathrm{mdim}_\Sigma(Y) > 0.$$

The amenable group case of Theorem 7.5 was proved by Lindenstrauss and Weiss [LW, Theorem 3.6]. We adapt their argument to our situation. First we prove a lemma:

Lemma 7.6. *Let G act continuously on a compact metrizable space X. Let Y be a factor of X with factor map $\pi : X \to Y$. Let $\mathcal{U}$ be a finite open cover of Y. Then*

$$\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) \leq \mathcal{D}_\Sigma(\mathcal{U}).$$

Proof. Let ρ and ρ' be compatible metrics on X and Y respectively. Replacing ρ by $\rho+\pi^{-1}(\rho')$ if necessary, we may assume that $\rho(x, y) \geq \rho'(\pi(x), \pi(y))$ for all $x, y \in X$.

Let F be a nonempty finite subset of X and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. For any $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$, one has $\pi\circ\varphi \in \mathrm{Map}(\rho', F, \delta, \sigma)$. Thus we have a continuous map $\Phi : \mathrm{Map}(\rho, F, \delta, \sigma) \to \mathrm{Map}(\rho', F, \delta, \sigma)$ sending φ to $\pi\circ\varphi$. Furthermore, $\Phi^{-1}(\mathcal{U}^d|_{\mathrm{Map}(\rho',F,\delta,\sigma)}) = (\pi^{-1}(\mathcal{U}))^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$. Thus $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho, F, \delta, \sigma) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F, \delta, \sigma)$. It follows that $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho, F, \delta) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F, \delta)$. Since F is an arbitrary nonempty finite subset of G and δ is an arbitrary positive number, we get

$$\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) = \mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho') = \mathcal{D}_\Sigma(\mathcal{U}).$$

□

We are ready to prove Theorem 7.5.

Proof of Theorem 7.5. Denote by π the factor map $X \to Y$. Let $\mathcal{U} = \{U, V\}$ be an open cover of Y such that none of U and V is dense in Y. By Lemma 7.6 it suffices to show that $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) > 0$. Take compatible metrics ρ and ρ' on Z and X respectively.

Note that none of $\pi^{-1}(U)$ and $\pi^{-1}(V)$ is dense in X. Take $x_U \in X \setminus \overline{\pi^{-1}(U)}$ and $x_V \in X \setminus \overline{\pi^{-1}(V)}$. Then there exist a finite symmetric subset K of G containing e_G such that if $x \in X$ coincides with x_U (resp. x_V) on K, then $x \notin \pi^{-1}(U)$ (resp. $x \notin \pi^{-1}(V)$). Since Z is path connected, for each $s \in K$ we can take a continuous map $\gamma_s : [0, 1] \to Z$ such that $\gamma_s(0) = (x_U)_s$ and $\gamma_s(1) = (x_V)_s$.

Now it suffices to show $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho', F, \delta) \geq 1/(2|K^2|)$ for every finite subset F of G containing K and every $\delta > 0$. Take a finite symmetric subset K_1 of G such that if two points x and y of X coincide on K_1, then $\rho'(x, y) < \delta/2$.

Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G such that $\sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(\mathrm{diam}(X, \rho'))^2} \leq \delta$ and $|\mathcal{Q}|/d \geq 1/2$, where

$$\mathcal{Q} = \{a \in [d] : \sigma_s\sigma_t(a) = \sigma_{st}(a) \text{ for all } s, t \in F \cup K_1, \text{ and } \sigma_{e_G}(a) = a,$$

$$\text{and } \sigma_s(a) \neq \sigma_t(a) \text{ for all distinct } s, t \in K\}.$$

Take a maximal subset $\mathcal{W}$ of $\mathcal{Q}$ subject to the condition that the sets $\sigma(K)a$ for $a \in \mathcal{W}$ are pairwise disjoint. Then $\mathcal{Q} \subseteq \sigma(K^2)\mathcal{W}$. Thus $|\mathcal{W}|/d \geq |\mathcal{Q}|/(|K^2|d) \geq 1/(2|K^2|)$.

Now we define a map $\Phi : [0,1]^{\mathcal{W}} \to \mathrm{Map}(\rho', F, \delta, \sigma)$. Fix $z_0 \in Z$. Let $f \in [0,1]^{\mathcal{W}}$. We define $\tilde{f} \in Z^{[d]}$ by

$$\tilde{f}_{\sigma(s)a} = \gamma_{s^{-1}}(f(a))$$

for $a \in \mathcal{W}$, $s \in K$, and

$$\tilde{f}_b = z_0$$

for $b \in [d] \setminus \sigma(K)\mathcal{W}$. Then we define $\varphi_{\tilde{f}} \in X^{[d]}$ by

$$(\varphi_{\tilde{f}}(a))_s = \tilde{f}(\sigma_{s^{-1}}(a))$$

for $a \in [d]$ and $s \in G$. Let $s \in F$. For any $a \in \mathcal{Q}$ and $t \in K_1$, one has

$$(\alpha_s \circ \varphi_{\tilde{f}}(a))_t = (\varphi_{\tilde{f}}(a))_{s^{-1}t} = \tilde{f}(\sigma_{t^{-1}s}(a)) = \tilde{f}(\sigma_{t^{-1}} \circ \sigma_s(a)) = (\varphi_{\tilde{f}} \circ \sigma_s(a))_t,$$

and hence $\rho'(\alpha_s \circ \varphi_{\tilde{f}}(a), \varphi_{\tilde{f}} \circ \sigma_s(a)) < \delta/2$ by the choice of K_1. Thus

$$\rho'_2(\alpha_s \circ \varphi_{\tilde{f}}, \varphi_{\tilde{f}} \circ \sigma_s) \leq \sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(\mathrm{diam}(X, \rho'))^2} \leq \delta.$$

Therefore $\varphi_{\tilde{f}} \in \mathrm{Map}(\rho', F, \delta, \sigma)$. Set $\Phi(f) = \varphi_{\tilde{f}}$. Clearly Φ is continuous.

Set $\mathcal{V} = \Phi^{-1}((\pi^{-1}(\mathcal{U}))^d|_{\mathrm{Map}(\rho', F, \delta, \sigma)})$. Let $f \in [0,1]^{\mathcal{W}}$ and $a \in \mathcal{W}$. If $f(a) = 0$, then $(\varphi_{\tilde{f}}(a))_s = \tilde{f}_{\sigma(s^{-1})a} = \gamma_s(0) = (x_U)_s$ for all $s \in K$, and hence $\varphi_{\tilde{f}}(a) \notin \pi^{-1}(U)$ by the choice of K. Similarly, if $f(a) = 1$, then $\varphi_{\tilde{f}}(a) \notin \pi^{-1}(V)$. Thus no element of $\mathcal{V}$ intersects two opposing sides of $[0,1]^{\mathcal{W}}$. By Lemma 7.4 we conclude that $\mathcal{D}(\pi^{-1}(\mathcal{U}), \rho', F, \delta, \sigma) \geq \mathcal{D}(\mathcal{V}) \geq |\mathcal{W}| \geq d/(2|K^2|)$. It follows that $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho', F, \delta) \geq 1/(2|K^2|)$ as desired. □

8. Small-boundary Property

In this section we discuss the relation between the small-boundary property and non-positive sofic mean topological dimension.

We start with recalling the definitions of zero inductive dimensional compact metrizable spaces and actions with small-boundary property.

A compact metrizable space Y is said to have *inductive dimension* 0 if for every $y \in Y$ and every neighborhood U of Y there exists a neighborhood V of y contained in U such that the boundary ∂V of V is empty [HW, Definition II.1]. A compact metrizable space has inductive dimension 0 if and only if it has covering dimension 0 [HW, Theorem V.8].

Definition 8.1. Let a countable group Γ act continuously on a compact metrizable space X. We denote by $M(X, \Gamma)$ the set of Γ-invariant Borel probability measures on X. We say that a closed subset Z of X is *small* if $\mu(Z) = 0$ for all $\mu \in M(X, \Gamma)$. In particular, when $M(X, \Gamma)$ is empty, every closed subset of X is small. We say that the action has the *small-boundary property* (SBP) if for every point $x \in X$ and every neighborhood U of x, there is an open neighborhood $V \subseteq U$ of x with small boundary.

When Γ is amenable, for any subset Z of X it is easy to check that the function $F \mapsto \max_{x\in X} \sum_{s\in F} 1_Z(sx)$ defined on the set of nonempty finite subsets of Γ satisfies the conditions of the Ornstein-Weiss lemma [LW, Theorem 6.1]. Thus $\frac{1}{|F|} \max_{x\in X} \sum_{s\in F} 1_Z(sx)$ converges to some limit as F becomes more and more left invariant. Shub and Weiss defined Z to be *small* if this limit is 0 [SW]. It is proved in page 538 of [SW] that when Γ is amenable and Z is closed, the definition of Shub and Weiss coincides with Definition 8.1. The notion of the SBP was introduced in [Lin] and [LW].

If X has less than $2^{\aleph_0}$ ergodic G-invariant Borel probability measures, then the action has the SBP [SW] [LW, page 18].

When Γ is amenable, Lindenstrauss and Weiss showed that actions with the SBP has zero mean topological dimension [LW, Theorem 5.4]. We extend their result to sofic case:

Theorem 8.2. *Let a countable sofic group G act continuously on a compact metrizable space X. Suppose that the action has the SBP. Let Σ be a sofic approximation sequence for G. Then* $\mathrm{mdim}_\Sigma(X) \leq 0$.

Lindenstrauss showed that if a continuous action of $\mathbb{Z}$ on a compact metrizable space has a nontrivial minimal factor and has zero mean topological dimension, then it has the SBP [Lin, Theorem 6.2]. Gutman showed that if a continuous action of $\mathbb{Z}^d$ for $d \in \mathbb{N}$ on a compact metrizable space has a free zero-dimensional factor and has zero mean topological dimension, then it has the SBP [Gut, Theorem 1.11.1]. It is not clear in what generality the converse of Theorem 8.2 is true (see the discussion on page 20 of [LW]).

We shall adapt the argument of Lindenstrauss and Weiss to our case, by replacing the set $\{(sx)_{s\in H} : H \in G\} \subseteq X^H$ for an approximately left invariant finite subset H of G in their proof with the set $\mathrm{Map}(\rho, F, \delta, \sigma)$. First we need the following three lemmas.

Lemma 8.3. *Let a countable group Γ act continuously on a compact metrizable space X. Let Z be a closed subset of X. Then for any $\varepsilon > 0$ there is an open neighborhood U of Z with* $\sup_{\mu\in M(X,\Gamma)} \mu(U) < \varepsilon + \sup_{\mu\in M(X,\Gamma)} \mu(Z)$.

Proof. Suppose that this is not true. Denote by $\mathcal{S}$ the set of all open neighborhoods of Z, partially ordered by reversed inclusion. Then $M(X,\Gamma)$ is nonempty, and for each $U \in \mathcal{S}$ there is some $\mu_U \in M(X,G)$ with $\mu_U(U) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)} \mu(Z)$. Take a limit point ν of this net $\{\mu_U\}_{U\in\mathcal{S}}$ in the compact space $M(X,\Gamma)$. We claim that $\nu(Z) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)} \mu(Z)$, which is a contradiction.

To prove the claim, by the regularity of ν, it suffices to show $\nu(V) \geq \varepsilon + \sup_{\mu \in M(X,\Gamma)} \mu(Z)$ for every $V \in \mathcal{S}$. Take $U' \in \mathcal{S}$ with $\overline{U'} \subseteq V$. Take a continuous function $f : X \to [0,1]$ such that $f = 1$ on $\overline{U'}$ and $f = 0$ on $X \setminus V$. For any $U \in \mathcal{S}$ satisfying $U \subseteq U'$, one has $\int_X f \, d\mu_U \geq \mu_U(U) \geq \varepsilon + \sup_{\mu \in M(X,\Gamma)} \mu(Z)$. It follows that $\nu(V) \geq \int_X f \, d\nu \geq \varepsilon + \sup_{\mu \in M(X,\Gamma)} \mu(Z)$, as desired. □

Lemma 8.4. *Consider a continuous action of a countable group Γ on a compact metrizable space X with the SBP. Then for any finite open cover $\mathcal{U}$ of X and any $\varepsilon > 0$ there is a partition of unity $\phi_j : X \to [0,1]$ for $j = 1, \dots, |\mathcal{U}|$ subordinate to $\mathcal{U}$ such that $\sup_{\mu \in M(X,\Gamma)} \mu(\overline{\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0,1)}) < \varepsilon$.*

Proof. For each $x \in X$, take an open neighborhood V_x of x with small boundary such that $\overline{V_x}$ is contained in some element of $\mathcal{U}$. Since X is compact, we can cover X by finitely many such V_x's. Note that if two subsets Y_1 and Y_2 of X have small boundary ∂Y_1 and ∂Y_2 respectively, then $\partial(Y_1 \cup Y_2) \subseteq \partial Y_1 \cup \partial Y_2$ is also small. Thus we may take the union of those chosen V_x's whose closures are contained in one element of $\mathcal{U}$ to obtain an open cover $\mathcal{U}'$ of X such that each element of $\mathcal{U}'$ has small boundary and there is a bijection $\varphi : \mathcal{U}' \to \mathcal{U}$ with $\overline{U'} \subseteq \varphi(U')$ for every $U' \in \mathcal{U}'$.

By Lemma 8.3, for each $U' \in \mathcal{U}'$ we can find an open neighborhood U'' of the boundary $\partial U'$ of U' such that $\sup_{\mu \in M(X,\Gamma)} \mu(U'') < \varepsilon/|\mathcal{U}|$. Replacing U'' by $U'' \cap \varphi(U')$ if necessary, we may assume that $U'' \subseteq \varphi(U')$. Take an open neighborhood U''' of $\partial U'$ such that $\overline{U'''} \subseteq U''$. List the elements of $\mathcal{U}'$ as $U'_1, \dots, U'_{|\mathcal{U}|}$. For each $1 \leq j \leq |\mathcal{U}|$, take a continuous function $\psi_j : X \to [0,1]$ such that $\psi_j = 1$ on $\overline{U'_j}$ and $\psi_j = 0$ on $X \setminus (U'_j \cup U'''_j)$. Now define ϕ_j for $1 \leq j \leq |\mathcal{U}|$ inductively as $\phi_1 = \psi_1$, and $\phi_j = \min(\psi_j, 1 - \sum_{i=1}^{j-1} \phi_i)$ for $2 \leq j \leq |\mathcal{U}|$.

We claim that $\phi_1, \dots, \phi_{|\mathcal{U}|}$ is a partition of unity subordinate to $\mathcal{U}$. One has $0 \leq \phi_1 = \psi_1 \leq 1$. For each $2 \leq j \leq |\mathcal{U}|$, one also has $\phi_j \leq 1 - \sum_{i=1}^{j-1} \phi_i$. Thus $\sum_{i=1}^{j} \phi_i \leq 1$ for every $1 \leq j \leq |\mathcal{U}|$. In particular, one get $\phi_j = \min(\psi_j, 1 - \sum_{i=1}^{j-1} \phi_j) \geq 0$ for every $2 \leq j \leq |\mathcal{U}|$. For any $x \in X$, if $x \in U'_1$ then $\phi_1(x) = \psi_1(x) = 1$, otherwise $x \in U'_j$ for some $2 \leq j \leq |\mathcal{U}|$ and thus $\phi_j(x) = \min(\psi_j(x), 1 - \sum_{i=1}^{j-1} \phi_i(x)) = 1 - \sum_{i=1}^{j-1} \phi_i(x)$. It follows that $\sum_{j=1}^{|\mathcal{U}|} \phi_j = 1$. Note that for each $1 \leq j \leq |\mathcal{U}|$, the support of ϕ_j is contained in $\overline{U'_j} \cup \overline{U'''_j}$, which in turn is contained in $\varphi(U'_j)$. This proves our claim.

Now one has

$$\overline{\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0,1)} \subseteq \overline{\bigcup_{j=1}^{|\mathcal{U}|} \psi_j^{-1}(0,1)} \subseteq \overline{\bigcup_{j=1}^{|\mathcal{U}|} U'''_j} \subseteq \bigcup_{j=1}^{|\mathcal{U}|} U''_j,$$

and hence $\mu(\overline{\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0,1)}) \leq \mu(\bigcup_{j=1}^{|\mathcal{U}|} U''_j) < \varepsilon$ for every $\mu \in M(X,\Gamma)$. □

Lemma 8.5. *Let a countable group Γ act continuously on a compact metrizable space X. Let ρ be a compatible metric on X. Let Z be a closed subset of X, and*

$\varepsilon > 0$. *Then there exist a nonempty finite subset F of Γ and $\delta > 0$ such that, for any map σ from Γ to* $\mathrm{Sym}(d)$ *for some $d \in \mathbb{N}$, one has*

$$\frac{1}{d}\max_{\varphi\in\mathrm{Map}(\rho,F,\delta,\sigma)}\sum_{a\in[d]}1_Z(\varphi(a)) < \varepsilon + \sup_{\mu\in M(X,\Gamma)}\mu(Z).$$

Proof. Suppose that that for any nonempty finite subset F of Γ and any $\delta > 0$, there are a map $\sigma_{F,\delta}$ from Γ to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ and a $\varphi_{F,\delta} \in \mathrm{Map}(\rho, F, \delta, \sigma_{F,\delta})$ with

$$\frac{1}{d}\sum_{a\in[d]}1_Z(\varphi_{F,\delta}(a)) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)}\mu(Z).$$

Denote by $\mu_{F,\delta}$ the probability measure $\frac{1}{d}\sum_{a\in[d]}\delta_{\varphi_{F,\delta}(a)}$ on X. The set of all such (F, δ) is partially ordered by $(F, \delta) \geq (F', \delta)$ when $F \supseteq F', \delta \leq \delta'$. Take a limit point ν of $\{\mu_{F,\delta}\}_{F,\delta}$ in the compact set of Borel probability measures on X. Then there is a net $\{(F_j, \delta_j)\}_{j\in J}$ of pairs such that $\mu_{F_j,\delta_j} \to \nu$ as $j \to \infty$ and for any (F, δ) one has $(F_j, \delta_j) \geq (F, \delta)$ for all sufficiently large $j \in J$.

We claim that ν is G-invariant. Let g be a continuous $\mathbb{R}$-valued function on X. For each $\delta > 0$, set $C_\delta = \max_{x,y\in X,\rho(x,y)\leq\delta}|g(x) - g(y)|$. For any $j \in J$ and any $s \in F_j$, since $\varphi_{F_j,\delta_j} \in \mathrm{Map}(\rho, F_j, \delta_j, \sigma_{F_j,\delta_j})$, we have $|\mathcal{W}_s| \geq d(1 - \delta_j)$, where

$$\mathcal{W}_s = \{a \in [d] : \rho(\varphi_{F_j,\delta_j}(sa), s\varphi_{F_j,\delta_j}(a)) \leq \sqrt{\delta_j}\}.$$

Thus, for any $j \in J$ and $s \in F_j$, one has

$$\begin{aligned}
|\mu_{F_j,\delta_j}(g) - (s\mu_{F_j,\delta_j})(g)| &= \frac{1}{d}|\sum_{a\in[d]}g(\varphi_{F_j,\delta_j}(a)) - \sum_{a\in[d]}g(s\varphi_{F_j,\delta_j}(a))| \\
&= \frac{1}{d}|\sum_{a\in[d]}g(\varphi_{F_j,\delta_j}(sa)) - \sum_{a\in[d]}g(s\varphi_{F_j,\delta_j}(a))| \\
&\leq \frac{1}{d}\sum_{a\in[d]}|g(\varphi_{F_j,\delta_j}(sa)) - g(s\varphi_{F_j,\delta_j}(a))| \\
&\leq \frac{1}{d}(|\mathcal{W}_s|C_{\sqrt{\delta_j}} + (d - |\mathcal{W}_s|)2\|g\|_\infty) \\
&\leq C_{\sqrt{\delta_j}} + 2\delta_j\|g\|_\infty.
\end{aligned}$$

Letting $j \to \infty$, we have $\mu_{F_j,\delta_j}(g) \to \nu(g)$ and $(s\mu_{F_j,\delta_j})(g) \to (s\nu)(g)$ for every $s \in G$, and $\delta_j, C_{\sqrt{\delta_j}} \to 0$. It follows that $\nu(g) = (s\nu)(g)$. This proves the claim.

Next we claim that $\nu(Z) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)}\mu(Z)$, which is a contradiction.

To prove the claim, by the regularity of ν, it suffices to show $\nu(U) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)}\mu(Z)$ for every open neighborhood U of Z. Take a continuous function $f : X \to [0, 1]$ such that $f = 1$ on Z and $f = 0$ on $X \setminus U$. For any $j \in J$ one has

$$\mu_{F_j,\delta_j}(f) \geq \frac{1}{d}\sum_{a\in[d]} 1_Z(\varphi_{F_j,\delta_j}(a)) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)} \mu(Z).$$

Letting $j \to \infty$, we get $\nu(U) \geq \nu(f) \geq \varepsilon + \sup_{\mu\in M(X,\Gamma)} \mu(Z)$ as desired. □

We are ready to prove Theorem 8.2.

Proof of Theorem 8.2. Fix a compatible metric ρ on X. Let $\mathcal{U}$ be a finite open cover of X. Set $k = |\mathcal{U}|$. Let $\varepsilon > 0$. Take $\phi_1, \dots, \phi_k$ as in Lemma 8.4 for $\mathcal{U}$ and ε. Set $Z = \overline{\bigcup_{j=1}^k \phi_j^{-1}(0,1)}$. Then $\sup_{\mu\in M(X,G)} \mu(Z) \leq \varepsilon$. By Lemma 8.5 we can find a nonempty finite subset F of G and $\delta > 0$ such that, for any map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$, one has

$$\frac{1}{d}\max_{\varphi\in \mathrm{Map}(\rho,F,\delta,\sigma)} \sum_{a\in[d]} 1_Z(\varphi(a)) < \varepsilon + \sup_{\mu\in M(X,G)} \mu(Z) \leq 2\varepsilon.$$

Define $\Phi : X \to \mathbb{R}^k$ by $\Phi(x) = (\phi_1(x), \dots, \phi_k(x))$. Define $\Phi_{F,\delta,\sigma} : \mathrm{Map}(\rho, F, \delta, \sigma) \to \mathbb{R}^{kd}$ by

$$\Phi_{F,\delta,\sigma}(\varphi) = (\Phi(\varphi(1)), \dots, \Phi(\varphi(d))).$$

Let e^i_j, $i = 1, \dots, d, j = 1, \dots, k$ be the standard basis of $\mathbb{R}^{kd}$. For every $I \subseteq [d]$ with $|I| \leq 2\varepsilon d$ and every $\xi \in \{0,1\}^{kd}$, define

$$C(I,\xi) = \mathrm{span}\{e^i_j : i \in I, 1 \leq j \leq k\} + \xi.$$

Then

$$\Phi_{F,\delta,\sigma}(\mathrm{Map}(\rho, F, \delta, \sigma)) \subseteq \bigcup_{|I|\leq 2\varepsilon d,\xi\in\{0,1\}^{kd}} C(I,\xi).$$

Note that $\Phi_{F,\delta,\sigma}$ is $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)}$-compatible, and $\bigcup_{|I|\leq 2\varepsilon d,\xi\in\{0,1\}^{kd}} C(I,\xi)$ is a finite union of at most $2\varepsilon kd$ dimensional affine subspaces of $\mathbb{R}^{kd}$. Since the union of finitely many closed subsets of dimension at most $2\varepsilon kd$ has dimension at most $2\varepsilon kd$ [HW, page 30 and Theorem V.8], from Lemma 3.6 we get

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \leq 2\varepsilon kd.$$

It follows that $\mathcal{D}_\Sigma(\mathcal{U}) \leq 0$, and hence $\mathrm{mdim}_\Sigma(X) \leq 0$. □

We leave the following

Question 8.6. Could one strengthen Theorem 8.2 to get $\mathrm{mdim}_{\Sigma,\mathrm{M}}(X) = 0$?

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