SOFICITY, AMENABILITY, AND DYNAMICAL ENTROPY

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Abstract. In a previous paper the authors developed an operator-algebraic approach to Lewis Bowen's sofic measure entropy that yields invariants for actions of countable sofic groups by homeomorphisms on a compact metrizable space and by measure-preserving transformations on a standard probability space. We show here that these measure and topological entropy invariants both coincide with their classical counterparts when the acting group is amenable.

1. Introduction

In [2] Lewis Bowen introduced a notion of entropy for measure-preserving actions of a countable discrete sofic group on a standard probability space admitting a generating finite partition. By a limiting process the definition also applies more generally whenever there exists a generating partition with finite entropy. The idea is to dynamically model a given finite partition by partitions of a finite set on which the group acts in an approximate way according to the definition of soficity. Given a fixed sequence of sofic approximations, the entropy is locally defined as the exponential growth rate of the number of model partitions relative to the size of the finite sets on which the sofic approximations operate. Taking an infimum over the parameters which control the localization then defines the entropy of the original partition. This quantity is then shown to take a common value over all generating finite partitions. It may depend though on the choice of sofic approximation sequence, yielding in general a collection of entropy invariants for the system. However, in the case that the acting group is amenable and there exists a generating finite partition, Bowen showed in [3] that sofic measure entropy coincides with classical Kolmogorov-Sinai entropy for all choices of sofic approximation sequence.

Applying an operator algebra perspective, the present authors developed in [9] an alternative approach to sofic entropy that is more akin to Rufus Bowen's definition of topological entropy for $\mathbb{Z}$-actions in terms of ε-separated partial orbits. This approach furnishes both measure and topological dynamical invariants for general actions, and these entropies are related by a variational principle as in the classical case [9, Sect. 6]. For measure-preserving actions admitting a generating partition with finite entropy, our measure entropy coincides with Lewis Bowen's [9, Sect. 3].

The goal of this paper is to prove that, when the acting group is amenable, the sofic measure and topological entropies from [9] both coincide with their classical counterparts, independently of the sofic approximation sequence. In the measurable case this generalizes Bowen's result from [3] by means of a complete different type of argument. Once we have the result for measure entropy the topological version ensues by combining the variational

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principle from [9] with the classical variational principle. We will also give a direct proof in the topological case as it illustrates some of the basic ideas without the additional probabilistic complications that arise in the treatment of measure-preserving actions.

In [9] we took the operator algebra approach to defining sofic entropy because it was crucial for showing that one actually obtains a conjugacy invariant in the measurable case. However, for many purposes, including that of this paper, it is simpler to express both topological and measure entropy in terms of the dynamics on the space itself, as in Rufus Bowen's definition. In the measurable case this requires some topological structure, namely a compact metrizable space on which the group acts continuously with an invariant Borel probability measure. Since such topological models always exist, there is no loss in generality in taking this viewpoint, which we will do in this paper. We will therefore begin in Sections 2 and 3 by formulating the spatial definitions of sofic topological and measure entropy and establishing their equivalence with the original linear definitions from [9].

The basis for our analysis of the amenable case is a sofic approximation version of the Rokhlin lemma of Ornstein and Weiss, which can be extracted from Ornstein and Weiss's proof [14]. This appeared in Section 4 of [5] in a form that treats more generally the case of finite graphs. In our sofic approximation situation we will need a stronger statement that allows us to prescribe the quasitiling coverage of the finite approximation space and the set from which the tiling centres come. In Section 4 we will give a self-contained proof of this Rokhlin lemma for sofic approximations of countable discrete amenable groups following the line of argument in [14]. In Section 5 we prove that the sofic and classical topological entropies coincide for continuous actions of a countable discrete amenable group on a compact metrizable space. Finally, in Section 6 we show that the sofic measure entropy from [9] and the classical Kolmogorov-Sinai entropy coincide for measure-preserving actions of a countable discrete amenable group on a standard probability space.

We round out the introduction with some terminology concerning amenable and sofic groups and spanning and separated sets. For general information on unital commutative C^*-algebras as used in this paper and any unexplained notation and terminology see the introduction to [9]. The classical definitions of measure and topological entropy for actions of countable discrete amenable groups will be recalled in Sections 5 and 6, respectively.

For $d \in \mathbb{N}$ we write $\mathrm{Sym}(d)$ for the group of permutations of $\{1, \dots, d\}$. Let G be a countable discrete group. The identity element of such a G will always be denoted by e. The group G is said to be *amenable* if it admits a left invariant mean, i.e., a state on $\ell^\infty(G)$ which is invariant under left translation by G. This is equivalent to the existence of a Følner sequence, which is a sequence $\{F_i\}_{i=1}^\infty$ of nonempty finite subsets of G such that $|F_i|^{-1}|sF_i \Delta F_i| \to 0$ as $i \to \infty$ for all $s \in G$. We say that G is *sofic* if for $i \in \mathbb{N}$ there are a sequence $\{d_i\}_{i=1}^\infty$ of positive integers and a sequence $\{\sigma_i\}_{i=1}^\infty$ of maps $s \mapsto \sigma_{i,s}$ from G to $\mathrm{Sym}(d_i)$ which is asymptotically multiplicative and free in the sense that

$$\lim_{i\to\infty} \frac{1}{d_i} \big| \{a \in \{1, \dots, d_i\} : \sigma_{i,st}(a) = \sigma_{i,s}\sigma_{i,t}(a)\} \big| = 1$$

for all $s, t \in G$ and

$$\lim_{i\to\infty} \frac{1}{d_i} \big| \{a \in \{1, \dots, d_i\} : \sigma_{i,s}(a) \neq \sigma_{i,t}(a)\} \big| = 1$$

for all distinct $s,t \in G$. Such a sequence $\{\sigma_i\}_{i=1}^\infty$ for which $\lim_{i\to\infty} d_i = \infty$ is referred to as a *sofic approximation sequence* for G. The condition $\lim_{i\to\infty} d_i = \infty$ is assumed in order to avoid pathologies in the theory of sofic entropy (e.g., it is essential for the variational principle in [9]) and is automatic if G is infinite. Note that if G is amenable then it is sofic, as one can easily construct a sofic approximation sequence from a Følner sequence.

For a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ we will denote $\sigma_s(a)$ for $s \in G$ and $a \in \{1,\dots,d\}$ simply by sa when convenient, and also use σ to denote the induced map from G into the automorphism group of the C^*-algebra $C(\{1,\dots,d\}) \cong \mathbb{C}^d$ given by $\sigma_s(f)(a) = f(s^{-1}a)$ for all $s \in G$, $f \in \mathbb{C}^d$, and $a \in \{1,\dots,d\}$. For a $d \in \mathbb{N}$ we will invariably use ζ to denote the uniform probability measure on $\{1,\dots,d\}$, which will be regarded as a state (i.e., a unital positive linear functional) on the C^*-algebra $\mathbb{C}^d \cong C(\{1,\dots,d\})$ whenever appropriate.

Let (Y,ρ) be a pseudometric space and $\varepsilon \geq 0$. A set $A \subseteq Y$ is said to be *(ρ,ε)-separated* or *ε-separated with respect to ρ* if $\rho(x,y) \geq \varepsilon$ for all distinct $x,y \in A$, and *(ρ,ε)-spanning* or *ε-spanning with respect to ρ* if for every $y \in Y$ there is an $x \in A$ such that $\rho(x,y) < \varepsilon$. We write $N_\varepsilon(Y,\rho)$ for the maximal cardinality of a finite (ρ,ε)-separated subset of Y. If G is a group acting on Y and F is a nonempty finite subset of G then we define the pseudometric ρ_F on Y by $\rho_F(x,y) = \max_{s\in F} \rho(sx,sy)$.

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2. Topological entropy

Let G be a countable sofic group, X a compact metrizable space, and α a continuous action of G on X. The action of G on points will usually be expressed by the concatenation $(s,x) \mapsto sx$, and α will be also be used for the induced action of G on $C(X)$ by automorphisms, so that for $f \in C(X)$ and $s \in G$ the function $\alpha_s(f)$ is given by $x \mapsto f(s^{-1}x)$. A subset of $C(X)$ is said to be *dynamically generating* if it is not contained in any proper G-invariant unital C^*-subalgebra of $C(X)$.

First we recall the definition of sofic topological entropy from [9] and then show how it can be reformulated using approximately equivariant maps from the sofic approximation space into X. Throughout this section $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i=1}^\infty$ is a fixed sofic approximation sequence for G. Let $\mathcal{S} = \{p_n\}_{n\in\mathbb{N}}$ be a sequence in the unit ball of $C_{\mathbb{R}}(X)$. For a given $d \in \mathbb{N}$ we define on the set of unital homomorphisms from $C(X)$ to $\mathbb{C}^d$ the pseudometrics

$$\rho_{\mathcal{S},2}(\varphi,\psi) = \sum_{n=1}^\infty \frac{1}{2^n}\|\varphi(p_n) - \psi(p_n)\|_2,$$

$$\rho_{\mathcal{S},\infty}(\varphi,\psi) = \sum_{n=1}^\infty \frac{1}{2^n}\|\varphi(p_n) - \psi(p_n)\|_\infty,$$

where the norm $\|\cdot\|_2$ refers to the uniform probability measure ζ on $\{1,\dots,d\}$. For a nonempty finite set $F \subseteq G$, a $\delta > 0$, and a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ we define $\mathrm{Hom}(\mathcal{S}, F, \delta, \sigma)$ to be the set of all unital homomorphisms $\varphi : C(X) \to \mathbb{C}^d$ such that

$$\rho_{\mathcal{S},2}(\varphi \circ \alpha_s, \sigma_s \circ \varphi) < \delta$$

for all $s \in F$. For an $\varepsilon > 0$ we then set

$$\begin{aligned} h^{\varepsilon}_{\Sigma}(\mathcal{S}, F, \delta) &= \limsup_{i\to\infty} \frac{1}{d_i} \log N_{\varepsilon}(\mathrm{Hom}(\mathcal{S}, F, \delta, \sigma_i), \rho_{\mathcal{S},2}), \\ h^{\varepsilon}_{\Sigma}(\mathcal{S}, F) &= \inf_{\delta>0} h^{\varepsilon}_{\Sigma}(\mathcal{S}, F, \delta), \\ h^{\varepsilon}_{\Sigma}(\mathcal{S}) &= \inf_{F} h^{\varepsilon}_{\Sigma}(\mathcal{S}, F), \\ h_{\Sigma}(\mathcal{S}) &= \sup_{\varepsilon>0} h^{\varepsilon}_{\Sigma}(\mathcal{S}) \end{aligned}$$

where F in the third line ranges over all nonempty finite subsets of G. If $\mathrm{Hom}(\mathcal{S}, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h^{\varepsilon}_{\Sigma}(\mathcal{S}, F, \delta) = -\infty$. By Theorem 4.5 of [9] the quantity $h_{\Sigma}(\mathcal{S})$ is the same for all dynamically generating $\mathcal{S}$, and we define the topological entropy $h_{\Sigma}(X, G)$ of the system to be this value.

The following lemma is a version of Lemma 4.8 in [9], saying that $h_{\Sigma}(\mathcal{S})$ can also be computed by substituting $\rho_{\mathcal{S},\infty}$ for $\rho_{\mathcal{S},2}$, and can be established by a similar argument.

Lemma 2.1. *Let $\mathcal{S}$ be a sequence in the unit ball of $C_{\mathbb{R}}(X)$. Then*

$$h_{\Sigma}(\mathcal{S}) = \sup_{\varepsilon>0} \inf_{F} \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{d_i} \log N_{\varepsilon}(\mathrm{Hom}(\mathcal{S}, F, \delta, \sigma_i), \rho_{\mathcal{S},\infty})$$

where F ranges over the nonempty finite subsets of G.

Now let ρ be a continuous pseudometric on X, which will play the role of $\mathcal{S}$ in our spatial definition. For a given $d \in \mathbb{N}$, we define on the set of all maps from $\{1,\dots,d\}$ to X the pseudometrics

$$\begin{aligned} \rho_2(\varphi, \psi) &= \left(\frac{1}{d} \sum_{a=1}^{d} (\rho(\varphi(a), \psi(a)))^2 \right)^{1/2}, \\ \rho_{\infty}(\varphi, \psi) &= \max_{a=1,\dots,d} \rho(\varphi(a), \psi(a)). \end{aligned}$$

Definition 2.2. Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We define $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be the set of all maps $\varphi : \{1,\dots,d\} \to X$ such that $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) < \delta$ for all $s \in F$.

Definition 2.3. Let F be a nonempty finite subset of G and $\delta > 0$. For $\varepsilon > 0$ we define

$$\begin{aligned} h^{\varepsilon}_{\Sigma,2}(\rho, F, \delta) &= \limsup_{i\to\infty} \frac{1}{d_i} \log N_{\varepsilon}(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_2), \\ h^{\varepsilon}_{\Sigma,2}(\rho, F) &= \inf_{\delta>0} h^{\varepsilon}_{\Sigma,2}(\rho, F, \delta), \\ h^{\varepsilon}_{\Sigma,2}(\rho) &= \inf_{F} h^{\varepsilon}_{\Sigma,2}(\rho, F), \\ h_{\Sigma,2}(\rho) &= \sup_{\varepsilon>0} h^{\varepsilon}_{\Sigma,2}(\rho), \end{aligned}$$

where F in the third line ranges over the nonempty finite subsets of G. If $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h^{\varepsilon}_{\Sigma,2}(\rho, F, \delta) = -\infty$. We similarly define $h^{\varepsilon}_{\Sigma,\infty}(\rho, F, \delta)$, $h^{\varepsilon}_{\Sigma,\infty}(\rho, F)$, $h^{\varepsilon}_{\Sigma,\infty}(\rho)$ and $h_{\Sigma,\infty}(\rho)$ using $N_{\varepsilon}(\cdot, \rho_{\infty})$ in place of $N_{\varepsilon}(\cdot, \rho_2)$.

We say that ρ is *dynamically generating* [10, Sect. 4] if for any distinct points $x, y \in X$ one has $\rho(sx, sy) > 0$ for some $s \in G$.

Proposition 2.4. *Suppose that ρ is dynamically generating. Then*

$$h_{\Sigma}(X, G) = h_{\Sigma,2}(\rho) = h_{\Sigma,\infty}(\rho).$$

Proof. We will show that $h_{\Sigma}(X, G) = h_{\Sigma,2}(\rho)$. The proof for $h_{\Sigma}(X, G) = h_{\Sigma,\infty}(\rho)$ is similar, in view of Lemma 2.1.

We say that two continuous pseudometrics ρ and ρ' on X are equivalent if for any $\varepsilon > 0$ there is an $\varepsilon' > 0$ such that, for any points $x, y \in X$, if $\rho'(x, y) < \varepsilon'$ then $\rho(x, y) < \varepsilon$, and vice versa. If ρ and ρ' are equivalent, then the pseudometrics ρ_2 and ρ'_2 on the set of all maps $\{1, \dots, d\} \to X$ are uniformly equivalent in the sense that for any $\delta > 0$ there is some $\delta' > 0$ such that, for any $d \in \mathbb{N}$ and any maps Φ and Ψ from $\{1, \dots, d\}$ to X, if $\rho'_2(\Phi, \Psi) < \delta'$ then $\rho_2(\Phi, \Psi) < \delta$, and vice versa. From this one concludes easily that $h_{\Sigma,2}(\rho) = h_{\Sigma,2}(\rho')$.

Let Y be the quotient space of X modulo ρ. That is, Y is a quotient of X such that, for any points $x, y \in X$, x and y have the same image in Y if and only if $\rho(x, y) = 0$. Then ρ induces a compatible metric on Y. Let $\mathcal{S} = \{p_n\}_{n \in \mathbb{N}}$ be a sequence in the unit ball of $C_{\mathbb{R}}(Y)$ generating $C(Y)$ as a unital C^*-algebra. Then we have a compatible metric ρ' on Y defined by

$$\rho'(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |p_n(x) - p_n(y)|.$$

Via the quotient map $X \to Y$, we may think of $\mathcal{S}$ as a sequence in $C(X)$ and ρ' as a continuous pseudometric on X. Then both $\mathcal{S}$ and ρ' are dynamically generating, and, since Y is compact, ρ is equivalent to ρ'. Now it suffices to show that $h_{\Sigma}(\mathcal{S}) = h_{\Sigma,2}(\rho')$.

Note that for any $d \in \mathbb{N}$ there is a natural one-to-one correspondence between the set of unital homomorphisms $\phi : C(X) \to \mathbb{C}^d$ and the set of maps $\Phi : \{1, \dots, d\} \to X$. For each Φ, the corresponding ϕ sends $f \in C(X)$ to $f \circ \Phi$. Via this correspondence, one may think of $\rho_{\mathcal{S},2}$ as a pseudometric on the set of all maps $\{1, \dots, d\} \to X$. It is easily checked that $\rho_{\mathcal{S},2}$ and ρ'_2 are uniformly equivalent. It follows that $h_{\Sigma}(\mathcal{S}) = h_{\Sigma,2}(\rho')$. □

3. Measure entropy

Let G be a countable sofic group, (X, μ) a standard probability space, and α an action of G by measure-preserving transformations on X. As before $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ is a fixed sofic approximation sequence. The entropy $h_{\Sigma,\mu}(X, G)$ is defined as in the topological case but now using approximately multiplicative linear maps from $L^{\infty}(X, \mu)$ to $\mathbb{C}^{d_i}$ which are approximately equivariant and approximately pull back the uniform probability measure on $\{1, \dots, d_i\}$ to μ [9, Defn. 2.2]. We will not reproduce here the details of the definition, which has been formulated as such in order to show that one obtains a measure conjugacy invariant. Instead we will recall a more convenient equivalent definition that applies when μ is a G-invariant Borel probability measure for a continuous action of G on a compact metrizable space X [9, Sect. 5]. This permits us to work with homomorphisms instead of maps which are merely approximately multiplicative, which means that, as in the topological case, we can alternatively speak in terms of approximately equivariant maps at the spatial level, as we will explain.

So suppose that G acts continuously on a compact metrizable space X with a G-invariant Borel probability measure μ. Let $\mathcal{S} = \{p_n\}_{n=1}^{\infty}$ be a sequence in the unit ball of $C_{\mathbb{R}}(X)$. Recall the pseudometrics $\rho_{\mathcal{S},2}$ and $\rho_{\mathcal{S},\infty}$ defined in the second paragraph of Section 2. Let F be a nonempty finite subset of G and $m \in \mathbb{N}$. We write $\mathcal{S}_{F,m}$ for the set of all products of the form $\alpha_{s_1}(f_1)\cdots\alpha_{s_j}(f_j)$ where $1 \leq j \leq m$ and $f_1, \dots, f_j \in \{p_1, \dots, p_m\}$ and $s_1, \dots, s_j \in F$. For a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$, we write $\mathrm{Hom}_{\mu}^{X}(\mathcal{S}, F, m, \delta, \sigma)$ for the set of unital homomorphisms $\varphi : C(X) \to \mathbb{C}^d$ such that

(i) $|\zeta \circ \varphi(f) - \mu(f)| < \delta$ for all $f \in \mathcal{S}_{F,m}$, and
(ii) $\|\varphi \circ \alpha_s(f) - \sigma_s \circ \varphi(f)\|_2 < \delta$ for all $s \in F$ and $f \in \{p_1, \dots, p_m\}$.

For $\varepsilon > 0$ we set

$$\begin{aligned}
\bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F, m, \delta) &= \limsup_{i\to\infty} \frac{1}{d_i} \log N_{\varepsilon}(\mathrm{Hom}_{\mu}^{X}(\mathcal{S}, F, m, \delta, \sigma_i), \rho_{\mathcal{S},2}),\\
\bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F, m) &= \inf_{\delta>0} \bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F, m, \delta),\\
\bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F) &= \inf_{m\in\mathbb{N}} \bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F, m),\\
\bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}) &= \inf_{F} \bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F),\\
\bar{h}_{\Sigma,\mu}(\mathcal{S}) &= \sup_{\varepsilon>0} \bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}),
\end{aligned}$$

where F in the fourth line ranges over the nonempty finite subsets of G. If $\mathrm{Hom}_{\mu}^{X}(\mathcal{S}, F, m, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $\bar{h}^{\varepsilon}_{\Sigma,\mu}(\mathcal{S}, F, m, \delta) = -\infty$. In the case that $\mathcal{S}$ is dynamically generating in the sense of the first paragraph of the previous section, $\bar{h}_{\Sigma,\mu}(\mathcal{S})$ is equal to $h_{\Sigma,\mu}(X, G)$ [9, Prop. 5.4].

The following lemma is a measure-theoretic version of Proposition 4.8 in [9], saying that $\bar{h}_{\Sigma,\mu}(\mathcal{S})$ can also be computed by substituting $\rho_{\mathcal{S},\infty}$ for $\rho_{\mathcal{S},2}$, and can be proved in the same way.

Lemma 3.1. *Let $\mathcal{S}$ be a sequence in the unit ball of $C_{\mathbb{R}}(X)$. Then*

$$\bar{h}_{\Sigma,\mu}(\mathcal{S}) = \sup_{\varepsilon>0} \inf_{F} \inf_{m\in\mathbb{N}} \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{d_i} \log N_{\varepsilon}(\mathrm{Hom}_{\mu}^{X}(\mathcal{S}, F, m, \delta, \sigma_i), \rho_{\mathcal{S},\infty})$$

where F ranges over the nonempty finite subsets of G.

Now let ρ be a continuous pseudometric on X. Recall the associated pseudometrics ρ_2 and ρ_{∞} as defined before Definition 2.2.

Definition 3.2. Let F be a nonempty finite subset of G, L a finite subset of $C(X)$, and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We define $\mathrm{Map}_{\mu}(\rho, F, L, \delta, \sigma)$ to be the set of all maps $\varphi : \{1, \dots, d\} \to X$ such that

(i) $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) < \delta$ for all $s \in F$, and
(ii) $\big|(\varphi_*\zeta)(f) - \mu(f)\big| < \delta$ for all $f \in L$.

Definition 3.3. Let F be a nonempty finite subset of G, L a finite subset of $C(X)$, and $\delta > 0$. For $\varepsilon > 0$ we define

$$\begin{aligned} h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F, L, \delta) &= \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma_i), \rho_2), \\ h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F, L) &= \inf_{\delta>0} h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F, L, \delta), \\ h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F) &= \inf_{L} h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F, L), \\ h^{\varepsilon}_{\Sigma,\mu,2}(\rho) &= \inf_{F} h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F), \\ h_{\Sigma,\mu,2}(\rho) &= \sup_{\varepsilon>0} h^{\varepsilon}_{\Sigma,\mu,2}(\rho), \end{aligned}$$

where L in the third line ranges over the finite subsets of $C(X)$ and F in the fourth line ranges over the nonempty finite subsets of G. If $\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h^{\varepsilon}_{\Sigma,\mu,2}(\rho, F, L, \delta) = -\infty$. We similarly define $h^{\varepsilon}_{\Sigma,\mu,\infty}(\rho, F, L, \delta)$, $h^{\varepsilon}_{\Sigma,\mu,\infty}(\rho, F, L)$, $h^{\varepsilon}_{\Sigma,\mu,\infty}(\rho, F)$, $h^{\varepsilon}_{\Sigma,\mu,\infty}(\rho)$, and $h_{\Sigma,\mu,\infty}(\rho)$ using $N_\varepsilon(\cdot, \rho_\infty)$ in place of $N_\varepsilon(\cdot, \rho_2)$.

Recall from the previous section that ρ is said to be dynamically generating if for any distinct points $x, y \in X$ one has $\rho(sx, sy) > 0$ for some $s \in G$.

Proposition 3.4. *Suppose that ρ is dynamically generating. Then*

$$h_{\Sigma,\mu}(X, G) = h_{\Sigma,\mu,2}(\rho) = h_{\Sigma,\mu,\infty}(\rho).$$

Proof. One can argue as in the proof of Proposition 2.4, appealing to Lemma 3.1 in the case of $h_{\Sigma,\mu,\infty}(\rho)$. The only extra thing to observe is that for $\mathcal{S}$ and ρ' as in the proof of Proposition 2.4, given any finite subset L of $C(X)$ and $\delta > 0$ there exist a nonempty finite subset F of G, an $m \in \mathbb{N}$ and a $\delta' > 0$ such that, for any $d \in \mathbb{N}$ and any map $\sigma : G \to \mathrm{Sym}(d)$, if ϕ is a unital homomorphism $C(X) \to \mathbb{C}^d$ satisfying $|\zeta\circ\phi(f) - \mu(f)| < \delta'$ for all $f \in \mathcal{S}_{F,m}$, then $|(\Phi_*\zeta)(g) - \mu(g)| < \delta$ for all $g \in L$, where Φ is the corresponding map $\{1, \dots, d\} \to X$. Indeed, since $\mathcal{S}$ is dynamically generating one can find a nonempty finite subset F of G and an $m \in \mathbb{N}$ such that for each $g \in L$ there exists some $\tilde{g}$ in the linear span of $\mathcal{S}_{F,m} \cup \{1\}$ with $\|g - \tilde{g}\|_\infty < \delta/4$. Denote by M the maximum over all $g \in L$ of the sum of the absolute values of the coefficients of $\tilde{g}$ written as a linear combination of elements in $\mathcal{S}_{F,m} \cup \{1\}$. Then one may take δ' to be $\delta/(2M)$. □

4. The Rokhlin lemma for sofic approximations of countable discrete amenable groups

Here we give a proof of the Rokhlin lemma for sofic approximations of countable discrete amenable groups (Lemma 4.5), which will be used in both Sections 5 and 6. The argument is extracted from [14].

Definition 4.1. Let (X, μ) be a finite measure space and let $\delta \geq 0$. A measurable set $A \subseteq X$ is said to *δ-cover* or be a *δ-covering* of X if $\mu(A) \geq \delta\mu(X)$. A family of measurable subsets of X is said to *δ-cover* or be a *δ-covering* of X if the union of its elements δ-covers X. A collection $\{A_i\}_{i\in I}$ of positive measure subsets of X is said to be a *δ-even covering* of X if there exists a number $M > 0$ such that $\sum_{i\in I} \mathbf{1}_{A_i} \leq M$ and $\sum_{i\in I} \mu(A_i) \geq (1-\delta)M\mu(X)$. We call M a *multiplicity* of the δ-even covering.

Definition 4.2. Let (X,μ) be a finite measure space and let $\varepsilon \geq 0$. A collection $\{A_i\}_{i\in I}$ of positive measure sets is said to be *ε-disjoint* if there exist pairwise disjoint sets $\widehat{A}_i \subseteq A_i$ such that $\mu(\widehat{A}_i) \geq (1-\varepsilon)\mu(A_i)$ for all $i \in I$.

The following two lemmas are from page 23 of [14].

Lemma 4.3. *Let (X,μ) be a finite measure space. Let $\delta \in (0,1)$ and let $\{A_i\}_{i\in I}$ be a countable δ-even covering of X. Then for every positive measure $B \subseteq X$ there exists an $i \in I$ such that*

$$\frac{\mu(A_i \cap B)}{\mu(A_i)} \leq \frac{\mu(B)}{(1-\delta)\mu(X)}.$$

Proof. If for some measurable $B \subseteq X$ we had

$$\mu(A_i \cap B) > \frac{\mu(B)}{(1-\delta)\mu(X)}\mu(A_i)$$

for every $i \in I$, then taking a multiplicity M for the δ-even covering and summing over i would yield

$$\begin{aligned}
\sum_{i\in I}\mu(A_i\cap B) &> \frac{\mu(B)}{(1-\delta)\mu(X)}\sum_{i\in I}\mu(A_i) \geq \mu(B)M \\
&\geq \int_X \mathbf{1}_B(x)\Big(\sum_{i\in I}\mathbf{1}_{A_i}(x)\Big)d\mu(x) \\
&= \int_X \Big(\sum_{i\in I}\mathbf{1}_{A_i\cap B}(x)\Big)d\mu(x) = \sum_{i\in I}\Big(\int_X \mathbf{1}_{A_i\cap B}(x)\Big)d\mu(x) \\
&= \sum_{i\in I}\mu(A_i\cap B),
\end{aligned}$$

a contradiction. $\square$

Lemma 4.4. *Let (X,μ) be a finite measure space. Let $\delta,\varepsilon \in [0,1)$ and let $\{A_i\}_{i\in I}$ be a finite δ-even covering of X by positive measure sets. Then there is an ε-disjoint subcollection of $\{A_i\}_{i\in I}$ which $\varepsilon(1-\delta)$-covers X.*

Proof. Take a maximal ε-disjoint subcollection $\{A_i\}_{i\in J}$ of $\{A_i\}_{i\in I}$. If this does not $\varepsilon(1-\delta)$-cover X then by Lemma 4.3 there is an $i_0 \in I$ such that

$$\frac{\mu\big(A_{i_0}\cap\bigcup_{i\in J}A_i\big)}{\mu(A_{i_0})} \leq \frac{\mu\big(\bigcup_{i\in J}A_i\big)}{(1-\delta)\mu(X)} < \varepsilon$$

so that by adding A_{i_0} to the collection $\{A_i\}_{i\in J}$ we again have an ε-disjoint collection, contradicting maximality. $\square$

Lemma 4.5. *Let G be a countable discrete group. Let $0 \leq \tau < 1$, and $0 < \eta < 1$. Then there are an $\ell \in \mathbb{N}$ and $\eta', \eta'' > 0$ such that, whenever $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$ are finite subsets of G with $|(F_{k-1}^{-1}F_k)\setminus F_k| \leq \eta'|F_k|$ for $k = 2,\dots,\ell$, there exists a finite set $F \subseteq G$ containing e such that for every $d \in \mathbb{N}$, every map $\sigma : G \to \mathrm{Sym}(d)$ with a set $B \subseteq \{1,\dots,d\}$ satisfying $|B| \geq (1-\eta'')d$ and*

$$\sigma_{st}(a) = \sigma_s\sigma_t(a), \sigma_s(a) \neq \sigma_{s'}(a), \sigma_e(a) = a$$

for all $a \in B$ *and* $s, t, s' \in F$ *with* $s \neq s'$, *and any set* $V \subseteq \{1, \ldots, d\}$ *with* $|V| \geq (1-\tau)d$, *there exist* $C_1, \ldots, C_\ell \subseteq V$ *such that*

(i) *for every* $k = 1, \ldots, \ell$ *and* $c \in C_k$, *the map* $s \mapsto \sigma_s(c)$ *from* F_k *to* $\sigma(F_k)c$ *is bijective,*

(ii) *the sets* $\sigma(F_1)C_1, \ldots, \sigma(F_\ell)C_\ell$ *are pairwise disjoint and the family* $\bigcup_{k=1}^{\ell}\{\sigma(F_k)c : c \in C_k\}$ *is* η*-disjoint and* $(1-\tau-\eta)$*-covers* $\{1, \ldots, d\}$.

Proof. Take $\eta', \eta'' > 0$ such that $1 - \tau - 2\eta'' > 0$, $\eta(1 + \eta'/(1-\eta)) < 1$, and $(1 - \tau - 2\eta'')(1 + \eta'/(1-\eta))^{-1} > 1 - \tau - \eta$. Define an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ in $[0, +\infty)$ by setting $t_1 = \eta(1 - \tau - \eta'')$ and, for $n \in \mathbb{N}$,

$$t_{n+1} = \eta\left(1 - \tau - \eta'' - \left(1 + \frac{\eta'}{1-\eta}\right)t_n\right) + t_n$$

if $1 - \tau - \eta'' - (1 + \eta'/(1-\eta))t_n \geq 0$ and $t_{n+1} = t_n$ otherwise. It is easily checked that $1 - \tau - \eta'' - (1 + \eta'/(1-\eta))\lim_{n\to\infty} t_n \leq 0$. Thus there exists some $\ell \in \mathbb{N}$ with $1 - \tau - \eta'' - (1 + \eta'/(1-\eta))t_\ell < \eta''$ and $t_1 < t_2 < \cdots < t_\ell$. Then $t_\ell \geq (1 - \tau - 2\eta'')(1 + \eta'/(1-\eta))^{-1} > 1 - \tau - \eta$.

Suppose now that $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$ are finite subsets of G with $|(F_{k-1}^{-1}F_k) \setminus F_k| \leq \eta'|F_k|$ for $k = 2, \ldots, \ell$. Set $F = F_\ell \cup F_\ell^{-1}$. Note that for every $c \in B$ and $k = 1, \ldots, \ell$ the map $s \mapsto \sigma_s(c)$ from F_k to $\sigma(F_k)c$ is bijective. We will recursively construct the sets $C_1, \ldots, C_\ell$ in reverse order so that the sets $\sigma(F_1)C_1, \ldots, \sigma(F_\ell)C_\ell$ are pairwise disjoint and the family $\bigcup_{n=k+1}^{\ell}\{\sigma(F_n)c : c \in C_n\}$ is η-disjoint and $t_{\ell-k}$-covers $\{1, \ldots, d\}$ for each $k = 0, \ldots, \ell - 1$. Since $t_\ell \geq 1 - \tau - \eta$ we will thereby obtain condition (ii). Moreover we will choose $C_1, \ldots, C_\ell$ to be subsets of $B \cap V$ so that condition (i) holds automatically.

Note that $\sigma_{s^{-1}}\sigma_s(a) = \sigma_e(a) = a$ for all $a \in B$ and $s \in F_\ell$, and thus for all distinct $a, c \in B$ and $s \in F_\ell$ we have

$$\sigma_{s^{-1}}\sigma_s(a) \neq \sigma_{s^{-1}}\sigma_s(c),$$

and hence

$$\sigma_s(a) \neq \sigma_s(c).$$

To begin the recursive construction we observe that

$$\sum_{c \in B\cap V} |\sigma(F_\ell)c| = |F_\ell| \cdot |B \cap V| \geq |F_\ell| \cdot (1 - \tau - \eta'')d,$$

so that the family $\{\sigma(F_\ell)c\}_{c\in B\cap V}$ is a $(\tau + \eta'')$-even covering of $\{1, \ldots, d\}$ with multiplicity $|F_\ell|$. By Lemma 4.4, we can find a set $C_\ell \subseteq B \cap V$ such that the family $\{\sigma(F_\ell)c\}_{c\in C_\ell}$ is η-disjoint and $\eta(1 - \tau - \eta'')$-covers $\{1, \ldots, d\}$.

Suppose that $1 \leq k < \ell$ and we have found $C_{k+1}, \ldots, C_\ell \subseteq B \cap V$ such that the sets $\sigma(F_{k+1})C_{k+1}, \ldots, \sigma(F_\ell)C_\ell$ are pairwise disjoint and the family $\bigcup_{n=k+1}^{\ell}\{\sigma(F_n)c : c \in C_n\}$ is η-disjoint and $t_{\ell-k}$-covers $\{1, \ldots, d\}$. Set $t'_{\ell-k} = |\bigcup_{n=k+1}^{\ell}\sigma(F_n)C_n|/d$ and $E = \{c \in B\cap V : \sigma(F_k)c \cap \left(\bigcup_{n=k+1}^{\ell}\sigma(F_n)C_n\right) = \emptyset\}$. For every $c \in (B\cap V)\setminus E$ we have $\sigma_s(c) = \sigma_t(a)$ for some $n = k+1, \ldots, \ell$, $a \in C_n$, $t \in F_n$, and $s \in F_k$, and hence

$$c = \sigma_{s^{-1}}\sigma_s(c) = \sigma_{s^{-1}}\sigma_t(a) = \sigma_{s^{-1}t}(a) \in \bigcup_{n=k+1}^{\ell}\sigma(F_k^{-1}F_n)C_n.$$

Therefore

$$(B \cap V) \setminus E \subseteq \bigcup_{n=k+1}^{\ell} \sigma(F_k^{-1}F_n)C_n.$$

For every $n = k+1, \dots, \ell$, since the family $\{\sigma(F_n)c : c \in C_n\}$ is η-disjoint we have

$$(1-\eta)|F_n| \cdot |C_n| \leq |\sigma(F_n)C_n|.$$

Thus

$$\begin{aligned}
\left| \bigcup_{n=k+1}^{\ell} \sigma(F_k^{-1}F_n)C_n \right| &\leq \left| \bigcup_{n=k+1}^{\ell} \sigma((F_k^{-1}F_n) \setminus F_n)C_n \right| + \left| \bigcup_{n=k+1}^{\ell} \sigma(F_n)C_n \right| \\
&\leq \sum_{n=k+1}^{\ell} |(F_k^{-1}F_n) \setminus F_n| \cdot |C_n| + t'_{\ell-k}d \\
&\leq \sum_{n=k+1}^{\ell} |(F_{n-1}^{-1}F_n) \setminus F_n| \cdot |C_n| + t'_{\ell-k}d \\
&\leq \sum_{n=k+1}^{\ell} \eta'|F_n| \cdot |C_n| + t'_{\ell-k}d \\
&\leq \sum_{n=k+1}^{\ell} \frac{\eta'}{1-\eta}|\sigma(F_n)C_n| + t'_{\ell-k}d \\
&= \left(1 + \frac{\eta'}{1-\eta}\right)t'_{\ell-k}d,
\end{aligned}$$

where the last equality follows form the assumption that the sets $\sigma(F_{k+1})C_{k+1}, \dots, \sigma(F_\ell)C_\ell$ are pairwise disjoint. Therefore

$$\begin{aligned}
|E| = |B \cap V| - |(B \cap V) \setminus E| &\geq (1 - \tau - \eta'')d - \left| \bigcup_{n=k+1}^{\ell} \sigma(F_k^{-1}F_n)C_n \right| \\
&\geq (1 - \tau - \eta'')d - \left(1 + \frac{\eta'}{1-\eta}\right)t'_{\ell-k}d.
\end{aligned}$$

It follows that

$$\sum_{c \in E} |\sigma(F_k)c| = |F_k| \cdot |E| \geq |F_k| \cdot \left(1 - \tau - \eta'' - \left(1 + \frac{\eta'}{1-\eta}\right)t'_{\ell-k}\right)d.$$

Thus the family $\{\sigma(F_k)c\}_{c \in E}$ is a $(\tau + \eta'' + (1+\eta'(1-\eta)^{-1})t'_{\ell-k})$-even covering of $\{1, \dots, d\}$ with multiplicity $|F_k|$. By Lemma 4.4, we can find a set $C_k \subseteq E$ such that the family $\{\sigma(F_k)c\}_{c \in C_k}$ is η-disjoint and $\eta(1 - \tau - \eta'' - (1+\eta'(1-\eta)^{-1})t'_{\ell-k})$-covers $\{1, \dots, d\}$. Then the sets $\sigma(F_k)C_k, \dots, \sigma(F_\ell)C_\ell$ are pairwise disjoint, and the family $\bigcup_{n=k}^{\ell}\{\sigma(F_n)c : c \in C_n\}$ is η-disjoint. Because the family $\bigcup_{n=k+1}^{\ell}\{\sigma(F_n)c : c \in C_n\}$ $t_{\ell-k}$-covers $\{1, \dots, d\}$, we have

$t'_{\ell-k} \geq t_{\ell-k}$. Since $\eta(1+\eta'/(1-\eta)) < 1$, we get

$$\begin{aligned}\left|\bigcup_{n=k}^{\ell} \sigma(F_n)C_n\right| &= |\sigma(F_k)C_k| + \left|\bigcup_{n=k+1}^{\ell} \sigma(F_n)C_n\right| \\ &\geq \eta(1-\tau-\eta'' - (1+\eta'(1-\eta)^{-1})t'_{\ell-k})d + t'_{\ell-k}d \\ &= (\eta(1-\tau-\eta'' - (1+\eta'(1-\eta)^{-1})t'_{\ell-k}) + t'_{\ell-k})d \\ &\geq (\eta(1-\tau-\eta'' - (1+\eta'(1-\eta)^{-1})t_{\ell-k}) + t_{\ell-k})d \\ &= t_{\ell-(k-1)}d,\end{aligned}$$

completing the recursive construction. □

For an amenable countable discrete group G, by [13, Cor. 5.3], there is a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G satisfying $F_n \subseteq F_{n+1}$ and $F_n^{-1} = F_n$ for all $n \in \mathbb{N}$. In particular, this is a two-sided Følner sequence. By using η-disjointness to pass to a genuinely disjoint family, we obtain from Lemma 4.5 the following.

Lemma 4.6. *Let G be an amenable countable discrete group. Let $0 \leq \tau < 1$, $0 < \eta < 1$, K be a nonempty finite subset of G, and $\delta > 0$. Then there are an $\ell \in \mathbb{N}$, nonempty finite subsets $F_1, \dots, F_\ell$ of G with $|KF_k \setminus F_k| < \delta|F_k|$ and $|F_kK \setminus F_k| < \delta|F_k|$ for all $k = 1, \dots, \ell$, a finite set $F \subseteq G$ containing e, and an $\eta' > 0$ such that, for every $d \in \mathbb{N}$, every map $\sigma : G \to \mathrm{Sym}(d)$ for which there is a set $B \subseteq \{1, \dots, d\}$ satisfying $|B| \geq (1-\eta')d$ and*

$$\sigma_{st}(a) = \sigma_s\sigma_t(a), \sigma_s(a) \neq \sigma_{s'}(a), \sigma_e(a) = a$$

for all $a \in B$ and $s, t, s' \in F$ with $s \neq s'$, and every set $V \subseteq \{1, \dots, d\}$ with $|V| \geq (1-\tau)d$, there exist $C_1, \dots, C_\ell \subseteq V$ such that

(i) *for every $k = 1, \dots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times C_k$ to $\sigma(F_k)C_k$ is bijective,*
(ii) *the family $\{\sigma(F_1)C_1, \dots, \sigma(F_\ell)C_\ell\}$ is disjoint and $(1-\tau-\eta)$-covers $\{1, \dots, d\}$.*

5. Topological entropy in the amenable case

We begin by recalling the classical definition of topological entropy [1, 12]. Let G be an amenable countable discrete group and α a continuous action of G on a compact metrizable space X. For an open cover $\mathcal{U}$ of X we write $N(\mathcal{U})$ for the minimal cardinality of a subcover of $\mathcal{U}$. For a nonempty finite set $F \subseteq G$ we abbreviate $\bigvee_{s\in F} s^{-1}\mathcal{U}$ to $\mathcal{U}^F$. As guaranteed by the subadditivity result in Section 6 of [11], for a finite open cover $\mathcal{U}$ of X the quantities

$$\frac{1}{|F|}\log N(\mathcal{U}^F)$$

converge to a limit as the nonempty finite set $F \subseteq G$ becomes more and more left invariant in the sense that for every $\varepsilon > 0$ there are a nonempty finite set $K \subseteq G$ and a $\delta > 0$ such that the displayed quantity is within ε of the limiting value whenever $|KF\Delta F| \leq \delta|F|$. We write this limit as $h_{\mathrm{top}}(\mathcal{U})$. The classical topological entropy $h_{\mathrm{top}}(X, G)$ is defined as the supremum of the quantities $h_{\mathrm{top}}(\mathcal{U})$ over all finite open covers $\mathcal{U}$ of X.

Given a Følner sequence $\{F_k\}_{k=1}^{\infty}$ and a compatible metric ρ on X, the entropy $h_{\rm top}(X,G)$ can be alternatively expressed as

$$\sup_{\varepsilon>0}\limsup_{k\to\infty}\frac{1}{|F_k|}\log N_\varepsilon(X,\rho_{F_k})$$

using the notation established in the introduction. This approach to entropy using metrics was introduced by Rufus Bowen for $\mathbb{Z}$-actions [4], and the standard arguments showing its equivalence in that case with the open cover definition apply equally well to the general amenable setting.

Let Σ be a fixed sofic approximation sequence for G. We will prove in this section that $h_\Sigma(X,G) = h_{\rm top}(X,G)$. The basis for the argument is the fact that every good enough sofic approximation for G can be approximately decomposed into copies of Følner sets (Lemma 4.5). This decomposition implies that the maps in the definition of sofic topological entropy approximately decompose into partial orbits over Følner sets.

Lemma 5.1. *Let G be an amenable countable discrete group acting continuously on a compact metrizable space X. Then $h_\Sigma(X,G)\le h_{\rm top}(X,G)$.*

Proof. We may assume that $h_{\rm top}(X,G)<\infty$. Let ρ be a compatible metric on X. Let $\varepsilon,\kappa>0$. To establish the lemma, by Proposition 2.4 it suffices to show that $h^\varepsilon_{\Sigma,\infty}(\rho)\le h_{\rm top}(X,G)+4\kappa$.

There are a nonempty finite subset K of G and $\delta'>0$ such that $N_{\varepsilon/4}(X,\rho_{F'}) < \exp((h_{\rm top}(X,G)+\kappa)|F'|)$ for every nonempty finite subset F' of G satisfying $|KF'\setminus F'| < \delta'|F'|$.

Take an $\eta\in(0,1)$ such that $(N_{\varepsilon/4}(X,\rho))^{2\eta}\le\exp(\kappa)$ and $(1-\eta)^{-1}(h_{\rm top}(X,G)+\kappa)\le h_{\rm top}(X,G)+2\kappa$. Let $\ell\in\mathbb{N}$ and $\eta'>0$ be as given by Lemma 4.5 with respect to η and $\tau=\eta$. Take finite subsets $e\in F_1\subseteq F_2\subseteq\cdots\subseteq F_\ell$ of G such that $|(F_{k-1}^{-1}F_k)\setminus F_k|\le\eta'|F_k|$ for $k=2,\dots,\ell$ and $|KF_k\setminus F_k|<\delta'|F_k|$ for every $k=1,\dots,\ell$. Then

$$N_{\varepsilon/4}(X,\rho_{F_k})\le\exp((h_{\rm top}(X,G)+\kappa)|F_k|) \tag{1}$$

for every $k=1,\dots,\ell$.

Let $\delta>0$ be a small positive number which we will determine in a moment. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$ which is a good enough sofic approximation for G. We will show that $N_\varepsilon(\mathrm{Map}(\rho,F_\ell,\delta,\sigma),\rho_\infty)\le\exp((h_{\rm top}(X,G)+4\kappa)d$, which will complete the proof since we can then conclude that $h^\varepsilon_{\Sigma,\infty}(\rho,F_\ell,\delta)\le h_{\rm top}(X,G)+4\kappa$ and hence $h^\varepsilon_{\Sigma,\infty}(\rho)\le h_{\rm top}(X,G)+4\kappa$.

For every $\varphi\in\mathrm{Map}(\rho,F_\ell,\delta,\sigma)$, we have $\rho_2(\varphi\circ\sigma_s,\alpha_s\circ\varphi)<\delta$ for all $s\in F_\ell$. Thus the set Λ_φ of all $a\in\{1,\dots,d\}$ such that

$$\rho(\varphi(sa),s\varphi(a))<\sqrt{\delta}$$

for all $s\in F_\ell$ has cardinality at least $(1-|F_\ell|\delta)d$.

For each $J\subseteq\{1,\dots,d\}$ we define on the set of maps from $\{1,\dots,d\}$ to X the pseudometric

$$\rho_{J,\infty}(\varphi,\psi)=\rho_\infty(\varphi|_J,\psi|_J).$$

Take a $(\rho_\infty,\varepsilon)$-separated subset D of $\mathrm{Map}(\rho,F_\ell,\delta,\sigma)$ of maximal cardinality.

Set $n = |F_\ell|$. When $n\delta < 1$, the number of subsets of $\{1, \dots, d\}$ of cardinality no greater than $n\delta d$ is equal to $\sum_{j=0}^{\lfloor n\delta d\rfloor} \binom{d}{j}$, which is at most $n\delta d\binom{d}{n\delta d}$, which by Stirling's approximation is less than $\exp(\beta d)$ for some $\beta > 0$ depending on δ and n but not on d when d is sufficiently large with $\beta \to 0$ as $\delta \to 0$ for a fixed n. Thus when δ is small enough and d is large enough, there is a subset W of D with $\exp(\kappa d)|W| \geq |D|$ such that the set Λ_φ is the same, say Θ, for every $\varphi \in W$, and $|\Theta|/d > 1 - \eta$.

Since we chose ℓ and η' so that the conclusion of Lemma 4.5 holds, when σ is a good enough sofic approximation for G, there exist $C_1, \dots, C_\ell \subseteq \Theta$ such that

(i) for all $k = 1, \dots, \ell$ and $c \in C_k$ the map $s \mapsto \sigma_s(c)$ from F_k to $\sigma(F_k)c$ is bijective,

(ii) the family $\bigcup_{k=1}^{\ell} \{\sigma(F_k)c : c \in C_k\}$ is η-disjoint and $(1 - 2\eta)$-covers $\{1, \dots, d\}$.

Denote by $\mathscr{L}$ the set of all pairs (k, c) such that $k \in \{1, \dots, \ell\}$ and $c \in C_k$. By η-disjointness, for every $(k, c) \in \mathscr{L}$ we can find an $F_{k,c} \subseteq F_k$ with $|F_{k,c}| \geq (1 - \eta)|F_k|$ such that the sets $\sigma(F_{k,c})c$ for $(k, c) \in \mathscr{L}$ are pairwise disjoint.

Let $(k, c) \in \mathscr{L}$. Take an $(\varepsilon/2)$-spanning subset $V_{k,c}$ of W with respect to $\rho_{\sigma(F_{k,c})c,\infty}$ of minimal cardinality. We will show that $|V_{k,c}| \leq \exp((h_{\rm top}(X, G) + \kappa)|F_k|)$ when δ is small enough. To this end, let V be an $(\varepsilon/2)$-separated subset of W with respect to $\rho_{\sigma(F_{k,c})c,\infty}$. For any two distinct elements φ and ψ of V we have, for every $s \in F_{k,c}$, since $c \in \Lambda_\varphi \cap \Lambda_\psi$,

$$\begin{aligned}\rho(s\varphi(c), s\psi(c)) &\geq \rho(\varphi(sc), \psi(sc)) - \rho(s\varphi(c), \varphi(sc)) - \rho(s\psi(c), \psi(sc)) \\ &\geq \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta},\end{aligned}$$

and hence

$$\rho_{F_{k,c}}(\varphi(c), \psi(c)) = \max_{s\in F_{k,c}} \rho(s\varphi(c), s\psi(c)) \geq \max_{s\in F_{k,c}} \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta} > \varepsilon/2 - \varepsilon/4 = \varepsilon/4,$$

granted that δ is taken small enough. Thus $\{\varphi(c) : \varphi \in V\}$ is a $(\rho_{F_{k,c}}, \varepsilon/4)$-separated subset of X of cardinality $|V|$, so that

$$|V| \leq N_{\varepsilon/4}(X, \rho_{F_{k,c}}) \leq N_{\varepsilon/4}(X, \rho_{F_k}) \overset{(1)}{\leq} \exp((h_{\rm top}(X, G) + \kappa)|F_k|).$$

Therefore

$$|V_{k,c}| \leq N_{\varepsilon/2}(W, \rho_{\sigma(F_{k,c})c,\infty}) \leq \exp((h_{\rm top}(X, G) + \kappa)|F_k|),$$

as we wished to show.

Set

$$H = \{1, \dots, d\} \setminus \bigcup \{\sigma(F_{k,c})c : (k, c) \in \mathscr{L}\}.$$

and take an $(\varepsilon/2)$-spanning subset V_H of W with respect to $\rho_{H,\infty}$ of minimal cardinality. We have

$$|V_H| \leq (N_{\varepsilon/4}(X, \rho))^{|H|} \leq (N_{\varepsilon/4}(X, \rho))^{2\eta d}.$$

Write U for the set of all maps $\varphi : \{1, \dots, d\} \to X$ such that $\varphi|_H \in V_H|_H$ and $\varphi|_{\sigma(F_{k,c})c} \in V_{k,c}|_{\sigma(F_{k,c})c}$ for all $(k, c) \in \mathscr{L}$. Then, by our choice of η,

$$|U| = |V_H| \prod_{(k,c)\in\mathscr{L}} |V_{k,c}| \leq (N_{\varepsilon/4}(X, \rho))^{2\eta d} \exp\left(\sum_{(k,c)\in\mathscr{L}} (h_{\rm top}(X, G) + \kappa)|F_k|\right)$$

$$
\begin{aligned}
&= (N_{\varepsilon/4}(X,\rho))^{2\eta d} \exp\left((h_{\rm top}(X,G)+\kappa) \sum_{k=1}^{\ell} |F_k||C_k| \right) \\
&\leq \exp(\kappa d) \exp\left(\frac{1}{1-\eta}(h_{\rm top}(X,G)+\kappa)d \right) \\
&\leq \exp(\kappa d)\exp((h_{\rm top}(X,G)+2\kappa)d) = \exp((h_{\rm top}(X,G)+3\kappa)d).
\end{aligned}
$$

Now since every element of W lies within ρ_∞-distance $\varepsilon/2$ to an element of U and W is ε-separated with respect to ρ_∞, the cardinality of W is at most that of U. Therefore

$$
\begin{aligned}
N_\varepsilon(\mathrm{Map}(\rho,F_\ell,\delta,\sigma),\rho_\infty) &= |D| \leq \exp(\kappa d)|W| \leq \exp(\kappa d)|U| \\
&\leq \exp(\kappa d)\exp((h_{\rm top}(X,G)+3\kappa)d) \\
&= \exp((h_{\rm top}(X,G)+4\kappa)d),
\end{aligned}
$$

as desired. □

Lemma 5.2. *Let G be an amenable countable discrete group acting continuously on a compact metrizable space X. Then $h_\Sigma(X,G) \geq h_{\rm top}(X,G)$.*

Proof. Let ρ be a compatible metric on X. Let $\mathcal{U}$ be a finite open cover of X, and let $\theta > 0$. To prove the lemma it suffices to show that $h_{\Sigma,\infty}(\rho) \geq h_{\rm top}(\mathcal{U}) - 2\theta$.

Take $\varepsilon > 0$ such that every open ε-ball in X with respect to ρ is contained in some atom of $\mathcal{U}$. Then $N_\varepsilon(X,\rho_{F'}) \geq N(\mathcal{U}^{F'})$ for every nonempty finite subset F' of G. Thus, when F' is sufficiently left invariant, one has $|F'|^{-1}\log N_\varepsilon(X,\rho_{F'}) \geq h_{\rm top}(\mathcal{U}) - \theta$.

Let F be a nonempty finite subset of G and $\delta > 0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Now it suffices to show that if σ is a good enough sofic approximation then

$$
\frac{1}{d}\log N_\varepsilon(\mathrm{Map}(\rho,F,\delta,\sigma),\rho_\infty) \geq h_{\rm top}(\mathcal{U}) - 2\theta. \tag{2}
$$

Take $\delta' > 0$ such that $\sqrt{\delta'}\mathrm{diam}_\rho(X) < \delta/2$ and $(1-\delta')(h_{\rm top}(\mathcal{U})-\theta) \geq h_{\rm top}(\mathcal{U}) - 2\theta$. By Lemma 4.6 there are an $\ell \in \mathbb{N}$ and nonempty finite subsets $F_1,\dots,F_\ell$ of G which are sufficiently left invariant so that

$$
\inf_{k=1,\dots,\ell} \frac{1}{|F_k|}\log N_\varepsilon(X,\rho_{F_k}) \geq h_{\rm top}(\mathcal{U}) - \theta
$$

such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G there exist $C_1,\dots,C_\ell \subseteq \{1,\dots,d\}$ satisfying the following:

(i) for every $k = 1,\dots,\ell$, the map $(s,c) \mapsto \sigma_s(c)$ from $F_k \times C_k$ to $\sigma(F_k)C_k$ is bijective,
(ii) the family $\{\sigma(F_1)C_1,\dots,\sigma(F_\ell)C_\ell\}$ is disjoint and $(1-\delta')$-covers $\{1,\dots,d\}$.

For every $k \in \{1,\dots,\ell\}$ pick an ε-separated set $E_k \subseteq X$ with respect to ρ_{F_k} of maximal cardinality. For each $h = (h_k)_{k=1}^{\ell} \in \prod_{k=1}^{\ell}(E_k)^{C_k}$ take a map $\varphi_h : \{1,\dots,d\} \to X$ such that

$$
\varphi_h(sc) = s(h_k(c))
$$

for all $k \in \{1,\dots,\ell\}$, $c \in C_k$, and $s \in F_k$. Observe that if $\max_{k=1,\dots,\ell}|FF_k\Delta F_k|/|F_k|$ is small enough, as will be the case if we take $F_1,\dots,F_\ell$ to be sufficiently left invariant, and

σ is a good enough sofic approximation for G, then we will have $\rho_2(\alpha_s \circ \varphi_h, \varphi_h \circ \sigma_s) < \delta$ for all $s \in F$, so that $\varphi_h \in \mathrm{Map}(\rho, F, \delta, \sigma)$.

Now if $h = (h_k)_{k=1}^{\ell}$ and $h' = (h'_k)_{k=1}^{\ell}$ are distinct elements of $\prod_{k=1}^{\ell}(E_k)^{C_k}$, then $h_k(c) \neq h'_k(c)$ for some $k \in \{1, \dots, \ell\}$ and $c \in C_k$. Since $h_k(c)$ and $h'_k(c)$ are distinct points in E_k which is ε-separated with respect to ρ_{F_k}, $h_k(c)$ and $h'_k(c)$ are ε-separated with respect to ρ_{F_k}, and thus we have $\rho_\infty(\varphi_h, \varphi_{h'}) \geq \varepsilon$. Therefore

$$\begin{aligned}\frac{1}{d}\log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty) &\geq \frac{1}{d}\sum_{k=1}^{\ell}|C_k|\log|E_k| \\ &\geq \frac{1}{d}\sum_{k=1}^{\ell}|C_k||F_k|(h_{\mathrm{top}}(\mathcal{U}) - \theta) \\ &\geq (1-\delta')(h_{\mathrm{top}}(\mathcal{U}) - \theta) \\ &\geq h_{\mathrm{top}}(\mathcal{U}) - 2\theta,\end{aligned}$$

as desired. □

Combining Lemmas 5.1 and 5.2 we obtain the desired equality of entropies:

Theorem 5.3. *Let G be an amenable countable discrete group acting continuously on a compact metrizable space X. Let Σ be a sofic approximation sequence for G. Then*

$$h_\Sigma(X, G) = h_{\mathrm{top}}(X, G).$$

6. Measure entropy in the amenable case

Let G be an amenable countable discrete group acting on a standard probability space (X, μ) by measure-preserving transformations. The entropy of a measurable partition $\mathcal{Q}$ of X is defined by

$$H_\mu(\mathcal{Q}) = -\sum_{Q \in \mathcal{Q}} \mu(Q)\log\mu(Q).$$

For a nonempty finite set $F \subseteq G$ we abbreviate $\bigvee_{s \in F} s^{-1}\mathcal{Q}$ to $\mathcal{Q}^F$. By the subadditivity result in Section 6 of [11], for a finite measurable partition $\mathcal{Q}$ of X the quantities

$$\frac{1}{|F|}\log H_\mu(\mathcal{Q}^F)$$

converge to a limit as the nonempty finite set $F \subseteq G$ becomes more and more left invariant in the sense that for every $\varepsilon > 0$ there are a nonempty finite set $K \subseteq G$ and a $\delta > 0$ such that the displayed quantity is within ε of the limiting value whenever $|KF\Delta F| \leq \delta|F|$. We write this limit as $h_\mu(\mathcal{Q})$. The classical Kolmogorov-Sinai measure entropy $h_\mu(X, G)$ is defined as the supremum of the quantities $h_\mu(\mathcal{Q})$ over all finite measurable partitions $\mathcal{Q}$ of X.

Throughout this section $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i=1}^{\infty}$ is a fixed but arbitrary sofic approximation sequence for G. Our objective in this section is to show that the sofic entropy $h_{\Sigma,\mu}(X, G)$ agrees with the classical measure entropy $h_\mu(X, G)$. The proof of the topological analogue of this equality in Section 5 provides a basis for the argument, but the measure-preserving condition requires us in addition to keep track of statistical distributions along orbits. For this we will need a particular form of the Shannon-McMillan theorem which asserts, for infinite G, the L^1-convergence of the mean information functions to the entropy function (Lemma 6.1). In the proof of Lemma 6.1 and elsewhere we also require the ergodic decomposition of entropy, which relies on the affineness of the entropy function [12] (see [17, Thm. 8.4] for the $\mathbb{Z}$-action case) and hence requires G to be infinite. The proof of Lemma 6.4 also requires G to be infinite for different reasons. We will therefore need to handle the case of finite G separately, which we do in Lemmas 6.5 and 6.6.

Given that the sofic measure entropy $h_{\Sigma,\mu}(X, G)$ essentially amounts to counting unions of partial orbits over Følner sets in the case that G is amenable, our arguments will pass through some of the ideas in the proof of Theorem 1.1 of [6], which gives a formula for the entropy of an ergodic measure-preserving transformation in terms of orbit growth in the spirit of Rufus Bowen's definition of topological entropy. Note however that we do not assume our actions to be ergodic.

Consider a Borel action of a countable group G on a standard Borel space $(X, \mathcal{B}_X)$. We consider the σ-algebra

$$\mathcal{B}_{X,G} = \{A \in \mathcal{B}_X : sA = A \text{ for all } s \in G\}.$$

Denote by $\mathcal{M}(X, G)$ the set of G-invariant probability measures on $(X, \mathcal{B}_X)$ and by $\mathcal{M}^{\mathrm{e}}(X, G)$ the set of G-invariant ergodic probability measures on $(X, \mathcal{B}_X)$. Assume that $\mathcal{M}(X, G)$ is nonempty. Endow $\mathcal{M}^{\mathrm{e}}(X, G)$ with the smallest σ-algebra making the functions $\mu \mapsto \mu(A)$ on $\mathcal{M}^{\mathrm{e}}(X, G)$ measurable for all $A \in \mathcal{B}_X$. Then $\mathcal{M}^{\mathrm{e}}(X, G)$ is a standard Borel space (in particular, $\mathcal{M}^{\mathrm{e}}(X, G)$ is nonempty) and there is a surjective Borel map $X \to \mathcal{M}^{\mathrm{e}}(X, G)$ sending x to μ_x [16, Thm. 4.2 and p. 204] satisfying the following conditions:

(i) $\mu_{sx} = \mu_x$ for all $x \in X$ and $s \in G$,
(ii) for each $\nu \in \mathcal{M}^{\mathrm{e}}(X, G)$ if we set $X_\nu = \{x \in X : \mu_x = \nu\}$ then ν is the unique μ in $\mathcal{M}(X, G)$ satisfying $\mu(X_\nu) = 1$,
(iii) for every $\mu \in \mathcal{M}(X, G)$ and $A \in \mathcal{B}_X$ we have $\mu(A) = \int_X \mu_x(A)\, d\mu(x)$.

Furthermore, this map is essentially unique in the sense that if $x \mapsto \mu'_x$ is another map satisfying the above conditions then there exists an $A \in \mathcal{B}_{X,G}$ such that $\mu(A) = 0$ for every $\mu \in \mathcal{M}(X, G)$ and $\mu_x = \mu'_x$ for all $x \in X \setminus A$. It follows that $\mu = \int_X \mu_x\, d\mu(x)$ is the ergodic decomposition of μ for every $\mu \in \mathcal{M}(X, G)$, and that for each $\mu \in \mathcal{M}(X, G)$ and each $\mathbb{C}$-valued bounded Borel function f on X one has $\mathbb{E}_\mu(f|\mathcal{B}_{X,G})(x) = \int_X f\, d\mu_x$ for μ-a.e. x, where $\mathbb{E}_\mu(f|\mathcal{B}_{X,G})$ denotes the conditional expectation of f in $L^\infty(f, \mathcal{B}_{X,G}, \mu)$.

When G is an amenable countably infinite discrete group, for any finite measurable partition $\mathcal{Q}$ of X and any $\mu \in \mathcal{M}(X, G)$, one has $h_\mu(\mathcal{Q}) = \int_X h_{\mu_x}(\mathcal{Q})\, d\mu(x)$, as one can deduce from [12, Propositions 5.3.2 and 5.3.5] and the proof in the case $G = \mathbb{Z}$ in [17, Theorem 8.4.(i)].

For a finite measurable partition $\mathcal{Q}$ of X and a $\mu \in \mathcal{M}(X, G)$, the information function $I_\mu(\mathcal{Q})$ is defined by

$$I_\mu(\mathcal{Q})(x) = -\sum_{Q \in \mathcal{Q}} 1_Q(x) \log \mu(Q)$$

for all $x \in X$.

Lemma 6.1. *Consider a Borel action of an amenable countably infinite discrete group G on a standard Borel space $(X, \mathcal{B}_X)$. Let $\mathcal{Q}$ be a finite measurable partition of X and $\mu \in \mathcal{M}(X, \mathcal{B}_X)$. Then the functions $\frac{1}{|F|} I_\mu(\mathcal{Q}^F)$ converge to the function $x \mapsto h_{\mu_x}(\mathcal{Q})$ in $L^1(X, \mathcal{B}_X, \mu)$ as the nonempty finite set $F \subseteq G$ becomes more and more left invariant in the sense that for every $\varepsilon > 0$ there are a nonempty finite set $K \subseteq G$ and a $\delta > 0$ such that $\frac{1}{|F|} I_\mu(\mathcal{Q}^F)$ is within ε of the function $x \mapsto h_{\mu_x}(\mathcal{Q})$ in the L^1-norm whenever $|KF \Delta F| \leq \delta |F|$.*

Proof. By the Shannon-McMillian theorem [12, Thm. 4.4.2], there exists an $f \in L^1(X, \mathcal{B}_{X,G}, \mu)$ such that the function $\frac{1}{|F|} I_\mu(\mathcal{Q}^F)$ converges to f in $L^1(X, \mathcal{B}_X, \mu)$ as the nonempty finite set $F \subseteq G$ becomes more and more left invariant. Set $g(x) = h_{\mu_x}(\mathcal{Q})$ for all $x \in X$. Then g is a bounded $\mathcal{B}_{X,G}$-measurable function on X. We just need to show that $f(x) = g(x)$ for μ-a.e. x.

We claim that $\int_A f \, d\mu \geq \int_A g \, d\mu$ for all $A \in \mathcal{B}_{X,G}$. Let $A \in \mathcal{B}_{X,G}$. We may assume that $\mu(A) > 0$. Define $\nu \in \mathcal{M}(X, G)$ by $\nu(B) = \frac{1}{\mu(A)} \mu(B \cap A)$ for all $B \in \mathcal{B}_X$. Let F be a nonempty finite subset of G. Set $\xi(t) = -t \log t$ for $t \geq 0$. Then ξ is concave on $[0, +\infty)$. Thus

$$\begin{aligned}
H_\nu(\mathcal{Q}^F) - \frac{1}{\mu(A)} \int_A I_\mu(\mathcal{Q}^F) \, d\mu &= \sum_{B \in \mathcal{Q}^F} -\nu(B) \log \nu(B) - \sum_{B \in \mathcal{Q}^F} -\nu(B) \log \mu(B) \\
&= \sum_{B \in \mathcal{Q}^F} -\nu(B) \log \frac{\nu(B)}{\mu(B)} \\
&= \sum_{B \in \mathcal{Q}^F} \mu(B) \xi\Big(\frac{\nu(B)}{\mu(B)}\Big) \\
&\leq \xi\Big(\sum_{B \in \mathcal{Q}^F} \mu(B) \frac{\nu(B)}{\mu(B)}\Big) \\
&= \xi(1) = 0,
\end{aligned}$$

where the inequality comes from the concavity of ξ. Dividing the above inequality by $|F|$ and taking limits with F becoming more and more left invariant, we get $h_\nu(\mathcal{Q}) - \frac{1}{\mu(A)} \int_A f \, d\mu \leq 0$. Thus

$$\int_A g \, d\mu = \mu(A) \int_X h_{\mu_x}(\mathcal{Q}) \, d\nu(x) = \mu(A) h_\nu(\mathcal{Q}) \leq \int_A f \, d\mu.$$

This proves our claim. It follows that $f(x) - g(x) \geq 0$ for μ-a.e. x.

For $A = X$ the argument in the above paragraph shows that $\int_X f \, d\mu = \int_X g \, d\mu$. Thus $f(x) - g(x) = 0$ for μ-a.e. x. $\square$

Lemma 6.2. *Let $\kappa > 0$. Then there are $\delta_0 > 0$, $M \in \mathbb{N}$, and $\omega : \mathbb{N} \to (0, 1)$ such that if F is a finite subset of a group G with $|F| \geq M$, $\delta \in (0, \delta_0)$, $d \in \mathbb{N}$, and $\sigma : G \to \mathrm{Sym}(d)$ is a map with $\big|\bigcup_{s,t \in F, s \neq t} \{k \in \{1, \ldots, d\} : \sigma_s(k) = \sigma_t(k)\}\big| \leq \delta d$, then the number of subsets $A \subseteq \{1, \ldots, d\}$ such that $\max_{s \in F} |A \Delta \sigma_s(A)| \leq \omega(|F|) d$ is at most $\exp(\kappa d)$.*

Proof. Partition $\{1,\dots,d\}$ into sets $Q_1,\dots,Q_n$ each of which is invariant under the subgroup $\langle\sigma(F)\rangle$ of $\mathrm{Sym}(d)$ generated by $\sigma(F)$ and has no nonempty proper subset with this property. Write I for the set of all $i\in\{1,\dots,n\}$ such that $|Q_i|\geq|F|$ and set $I'=\{1,\dots,n\}\setminus I$. Then $|I|\leq d/|F|$. For each $i\in I$ fix an element a_i of Q_i.

Set $R=\bigcup_{i\in I}Q_i$ and $R'=\bigcup_{i\in I'}Q_i$. For every $a\in R'$ we can find distinct $s,t\in F$ such that $\sigma_s(a)=\sigma_t(a)$. Since $\big|\bigcup_{s,t\in F,s\neq t}\{k\in\{1,\dots,d\}:\sigma_s(k)=\sigma_t(k)\}\big|\leq\delta d$, it follows that $|R'|\leq\delta d$.

Now let us estimate the number of sets $A\subseteq\{1,\dots,d\}$ such that $\max_{s\in F}|A\Delta\sigma_s(A)|\leq\eta d$. Let A be an arbitrary such set. For each $s\in F$ define the function $\gamma_s:R\to\{0,1\}$ by $\gamma_s(a)=1$ if either (i) $a\in A$ and $\sigma_s(a)\notin A$ or (ii) $a\notin A$ and $\sigma_s(a)\in A$, and $\gamma_s(a)=0$ otherwise. For each $s\in F$ define the function $\tilde{\gamma}_s:R\to\{0,1\}$ by $\tilde{\gamma}_s(a)=1$ if either (i) $a\in A$ and $\sigma_s^{-1}(a)\notin A$ or (ii) $a\notin A$ and $\sigma_s^{-1}(a)\in A$, and $\tilde{\gamma}_s(a)=0$ otherwise. Also define a function $\beta:I\to\{0,1\}$ by $\beta(i)=1$ if $a_i\in A$, and $\beta(i)=0$ otherwise.

Let $s\in F$. Since $|A\Delta\sigma_s(A)|\leq\eta d$, the number of $a\in R$ such that $\gamma_s(a)=1$ can be at most ηd. Similarly, the number of $a\in R$ such that $\tilde{\gamma}_s(a)=1$ can be at most ηd.

Note that the collection of functions $\{\gamma_s:s\in F\}\cup\{\tilde{\gamma}_s:s\in F\}\cup\{\beta\}$ uniquely specifies $A\cap R$, for if $i\in I$ and $a\in Q_i$ then for some $t_1,\dots,t_k\in F$ and $e_1,\dots,e_k\in\{0,-1\}$ the permutation $\omega=\sigma_{t_1}^{e_1}\cdots\sigma_{t_k}^{e_k}$ will send a_i to a, which enables us to determine whether or not a belongs to A by using the functions from $\{\gamma_s:s\in F\}\cup\{\tilde{\gamma}_s:s\in F\}\cup\{\beta\}$. Thus the number of possibilities for $A\cap R$ is at most the number of possible collections $\{\gamma_s:s\in F\}\cup\{\tilde{\gamma}_s:s\in F\}\cup\{\beta\}$ and hence is bounded above by $\big(\sum_{k=0}^{\lfloor\eta d\rfloor}\binom{d}{k}\big)^{2|F|}2^{d/|F|}$. By Stirling's approximation this is bounded above by $\exp(\beta|F|d)2^{d/|F|}$ for some $\beta>0$ not depending on d or $|F|$ with $\beta\to 0$ as $\eta\to 0$.

For the number of possibilities for the intersection $A\cap R'$ we have the crude upper bound of $2^{|R'|}$, which by the second paragraph is at most $2^{\delta d}$. We deduce that the number of possibilities for A is at most $\exp(\beta|F|d)2^{d/|F|+\delta d}$, yielding the lemma. $\square$

Lemma 6.3. *Let G be an amenable countably infinite discrete group acting continuously on a compact metrizable space X and μ a G-invariant Borel probability measure on X. Let ρ be a compatible metric on X. Let $\varepsilon>0$. Let $\mathcal{Q}$ be a finite Borel partition of X with $\max_{Q\in\mathcal{Q}}\mathrm{diam}_\rho(Q)<\varepsilon/16$. Then $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\leq h_\mu(\mathcal{Q})$.*

Proof. Let $\kappa>0$. Take a finite G-invariant Borel partition $\mathcal{R}'$ of X such that $\sup_{x\in R}h_{\mu_x}(\mathcal{Q})-\inf_{x\in R}h_{\mu_x}(\mathcal{Q})<\kappa$ for every $R\in\mathcal{R}'$, where $x\mapsto\mu_x$ is the Borel map from X to $\mathcal{M}^{\mathrm{e}}(X,G)$ described at the beginning of the section. Denote by $\mathcal{R}$ the set of atoms in $\mathcal{R}'$ with positive μ-measure. For each $R\in\mathcal{R}$, set $\xi_R=\sup_{x\in R}h_{\mu_x}(\mathcal{Q})$. We will show that $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\leq\sum_{R\in\mathcal{R}}\xi_R\mu(R)+5\kappa$. Since

$$\begin{aligned}h_\mu(\mathcal{Q})&=\int_X h_{\mu_x}(\mathcal{Q})\,d\mu(x)\geq\sum_{R\in\mathcal{R}}\mu(R)\inf_{x\in R}h_{\mu_x}(\mathcal{Q})\\&\geq\sum_{R\in\mathcal{R}}\mu(R)(\xi_R-\kappa)=\sum_{R\in\mathcal{R}}\xi_R\mu(R)-\kappa,\end{aligned}$$

this will imply $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\leq h_\mu(\mathcal{Q})+6\kappa$. As κ is an arbitrary positive number, the latter will imply that $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\leq h_\mu(\mathcal{Q})$.

By Lemma 6.2, there exist an $M' \in \mathbb{N}$ and a function $\omega : \mathbb{N} \to (0,1)$ such that, for any finite subset F' of G with $|F'| \geq M'$, whenever a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ is a good enough sofic approximation for G the number of sets $A \subseteq \{1, \dots, d\}$ satisfying $\max_{s \in F'} |A \Delta \sigma_s(A)|/d \leq \omega(|F'|)$ is at most $\exp(|\mathcal{R}|^{-1}\kappa d)$.

Take an $\eta > 0$ such that $(N_{\varepsilon/4}(X, \rho))^{3\eta|\mathcal{R}|} < \exp(\kappa)$, $\eta < 2^{-1} \min_{R \in \mathcal{R}} \mu(R)$,

$$\frac{1}{1-\eta}\bigg(\sum_{R \in \mathcal{R}} \xi_R \mu(R) + \kappa + 2\kappa|\mathcal{R}|^2\eta + 2|\mathcal{R}|\eta \sum_{R \in \mathcal{R}} \xi_R \bigg) < \sum_{R \in \mathcal{R}} \xi_R \mu(R) + 2\kappa,$$

and for every $R \in \mathcal{R}$ and finite set Υ the number of sets $\Upsilon' \subseteq \Upsilon$ satisfying $|\Upsilon'| \geq |\Upsilon|(\mu(R) - \eta)/(\mu(R) + \eta)$ is at most $\exp(\kappa|\Upsilon|)$, as is possible by Stirling's approximation.

By Lemma 4.5, there are an $\ell \in \mathbb{N}$ and an $\eta' > 0$ such that, whenever $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$ are finite subsets of G with $|(F_{k-1}^{-1}F_k)/F_k| \leq \eta'|F_k|$ for $k = 2, \dots, \ell$, for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G and every $Y_R \subseteq \{1, \dots, d\}$ with $|Y_R|/d \geq \mu(R) - \eta$ for all $R \in \mathcal{R}$, there exist, for every $R \in \mathcal{R}$, sets $C_{R,1}, \dots, C_{R,\ell} \subseteq Y_R$ such that

(i) for all $k = 1, \dots, \ell$ and $c \in C_{R,k}$ the map $s \mapsto \sigma_s(c)$ from F_k to $\sigma(F_k)c$ is bijective,
(ii) the family $\bigcup_{k=1}^{\ell} \{\sigma(F_k)c : c \in C_{R,k}\}$ is η-disjoint and $(\mu(R) - 2\eta)$-covers $\{1, \dots, d\}$.

Take $0 < \tau < \eta/4$. Let $R \in \mathcal{R}$. Note that $\mu(\cdot \cap R)/\mu(R)$ is a Borel probability measure on X, which we denote by μ_R. One has $h_{\mu_x}(\mathcal{Q}) \leq \xi_R$ for μ_R-almost every x. By Lemma 6.1, there exist a nonempty finite subset K_R of G and a $\delta_R > 0$ such that for every nonempty finite subset F' of G satisfying $|K_R F' \setminus F'| < \delta_R |F'|$ there exists an $\mathcal{A}_{R,F'} \subseteq \mathcal{Q}^{F'}$ such that $\mu_R(\bigcup \mathcal{A}_{R,F'}) > 1 - \tau/\ell$, and for every $A \in \mathcal{A}_{R,F'}$ we have $\mu_R(A) > 0$ and $-|F'|^{-1} \log \mu_R(A) \leq \xi_R + \kappa$, that is

$$\mu_R(A) \geq \exp(-(\xi_R + \kappa)|F'|). \tag{3}$$

For each $A \in \mathcal{A}_{R,F'}$ pick a point $x_A \in A \cap R$ and set $E_{R,F'} = \{x_A : A \in \mathcal{A}_{R,F'}\}$. Since $\max_{Q \in \mathcal{Q}} \mathrm{diam}_\rho(Q) < \varepsilon/16$, the set $E_{R,F'}$ is an $(\varepsilon/16)$-spanning subset of $\bigcup \mathcal{A}_{R,F'}$ with respect to $\rho_{F'}$.

Now we fix finite subsets $F_1, \dots, F_\ell$ of G such that $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$, $|F_\ell| \geq M'$, $|(F_{k-1}^{-1}F_k)/F_k| \leq \eta'|F_k|$ for $k = 2, \dots, \ell$, and $|K_R F_k \setminus F_k| < \delta_R |F_k|$ for every $R \in \mathcal{R}$ and $k = 1, \dots, \ell$. Then we have $\mathcal{A}_{R,F_k}$ and E_{R,F_k} for every $R \in \mathcal{R}$ and $k = 1, \dots, \ell$.

Let $\lambda > 0$ be a small number to be determined in a moment. Let $R \in \mathcal{R}$. Then $\mu_R\big(\bigcap_{k=1}^{\ell} \bigcup \mathcal{A}_{R,F_k}\big) > 1 - \tau$. By the regularity of μ_R [17, Thm. 6.1], we can find a closed subset Z_R of $R \cap \bigcap_{k=1}^{\ell} \bigcup \mathcal{A}_{R,F_k}$ such that $\mu_R(Z_R) > 1 - \tau - \lambda$ and a closed subset Z'_R of R such that $Z_R \subseteq Z'_R$ and $\mu_R(Z'_R) > 1 - \lambda$. Then $F_\ell Z'_R$ is a closed subset of the G-invariant set R.

Since the closed sets $F_\ell Z'_R$ for $R \in \mathcal{R}$ are pairwise disjoint, we can find an open neighborhood U_R of $F_\ell Z'_R$ for every $R \in \mathcal{R}$ such that the sets U_R for $R \in \mathcal{R}$ are pairwise disjoint.

Let $R \in \mathcal{R}$. By the continuity of the action of G on X, we can find open neighborhoods B_R and B'_R of Z_R and Z'_R respectively, such that $B_R \subseteq B'_R$, $F_\ell B'_R \subseteq U_R$ and E_{R,F_k} is a $(\rho_{F_k}, \varepsilon/8)$-spanning subset of $B_R \cup \bigcup \mathcal{A}_{R,F_k}$ for every $k = 1, \dots, \ell$. For each $k = 1, \dots, \ell$

we have

$$N_{\varepsilon/4}(B_R,\rho_{F_k}) \leq N_{\varepsilon/4}\Big(B_R\cup\bigcup\mathcal{A}_{R,F_k},\rho_{F_k}\Big) \leq |E_{R,F_k}| = |\mathcal{A}_{R,F_k}| \\ \leq \mu_R\Big(\bigcup\mathcal{A}_{R,F_k}\Big)/\exp(-(\xi_R+\kappa)|F_k|) \\ \leq \exp((\xi_R+\kappa)|F_k|). \tag{4}$$

Take an $h_R\in C(X)$ such that $0\leq h_R\leq 1$, $h_R=1$ on Z'_R, and $h_R=0$ outside of B'_R. Also, take a $g_R\in C(X)$ such that $0\leq g_R\leq 1$, $g_R=1$ on Z_R, and $g_R=0$ outside of B_R. Replacing g_R by $\min(g_R,h_R)$ if necessary, we may assume that $g_R\leq h_R$.

Set $L=\bigcup_{R\in\mathcal{R}}\{h_R,g_R\}$. Let $\delta>0$ be a small number which we will determine in a moment. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$ which is a good enough sofic approximation for G. We will show that $N_\varepsilon(\mathrm{Map}_\mu(\rho,F_\ell,L,\delta,\sigma),\rho_\infty)\leq\exp((\sum_{R\in\mathcal{R}}\xi_R\mu(R)+5\kappa)d)$, which will complete the proof since we can then conclude that $h^\varepsilon_{\Sigma,\mu,\infty}(\rho,F_\ell,L,\delta)\leq\sum_{R\in\mathcal{R}}\xi_R\mu(R)+5\kappa$ and hence $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\leq\sum_{R\in\mathcal{R}}\xi_R\mu(R)+5\kappa$.

Denote by Λ the set of all $a\in\{1,\dots,d\}$ satisfying $\sigma_e(a)=a$. Let $\varphi\in\mathrm{Map}_\mu(\rho,F_\ell,L,\delta,\sigma)$. Denote by Λ_φ the set of all $a\in\{1,\dots,d\}$ such that

(i) $|h_R(\varphi(a))-h_R(s^{-1}\varphi(sa))|<1/2$ for all $R\in\mathcal{R}$ and $s\in F_\ell$, and
(ii) $\rho(\varphi(sa),s\varphi(a))<\sqrt{\delta}$ for all $s\in F_\ell$.

Set

$$\begin{aligned}\Omega'_{R,\varphi}&=\{a\in\{1,\dots,d\}:h_R(\varphi(a))>0\},\\ \Omega''_{R,\varphi}&=\{a\in\{1,\dots,d\}:h_R(\varphi(a))>1/2\},\\ \Omega_{R,\varphi}&=\Omega''_{R,\varphi}\cap\Lambda_\varphi\cap\Lambda,\end{aligned}$$

and

$$\begin{aligned}\Theta'_{R,\varphi}&=\{a\in\{1,\dots,d\}:g_R(\varphi(a))>0\},\\ \Theta''_{R,\varphi}&=\{a\in\{1,\dots,d\}:g_R(\varphi(a))>1/2\},\\ \Theta_{R,\varphi}&=\Theta''_{R,\varphi}\cap\Lambda_\varphi\cap\Lambda.\end{aligned}$$

Claim I: Assuming λ,δ are small enough and σ is a good enough sofic approximation for G, for every $\varphi\in\mathrm{Map}_\mu(\rho,F_\ell,L,\delta,\sigma)$ we have that $|\Omega_{R,\varphi}|/d\leq\mu(R)+\eta$ for every $R\in\mathcal{R}$, the sets $\sigma(F_\ell)\Omega_{R,\varphi}$ for $R\in\mathcal{R}$ are pairwise disjoint, and

$$\frac{1}{d}\max_{s\in F_\ell}\Big|\Omega_{R,\varphi}\Delta\sigma_s(\Omega_{R,\varphi})\Big|\leq\omega(|F_\ell|).$$

To verify Claim I, note first that if σ is a good enough sofic approximation for G then $|\Lambda|/d\geq 1-\lambda$. Consider the continuous pseudometric ρ' on X defined by

$$\rho'(x,y)=\max_{s\in F_\ell}\max_{R\in\mathcal{R}}|h_R(s^{-1}x)-h_R(s^{-1}y)|.$$

When δ is small enough, for any $x,y\in X$ with $\rho(x,y)<\sqrt{\delta}$, one has $\rho'(x,y)<1/2$. It follows that for any $a\in\{1,\dots,d\}$ and $s\in F_\ell$ with $\rho(\varphi(sa),s\varphi(a))<\sqrt{\delta}$, one has

$|h_R(\varphi(a)) - h_R(s^{-1}\varphi(sa))| < 1/2$ for all $R \in \mathcal{R}$. Since $\varphi \in \mathrm{Map}_\mu(\rho, F_\ell, L, \delta, \sigma)$, for each $s \in F_\ell$ one has $\rho_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s) < \delta$ and hence

$$\left|\left\{a \in \{1, \dots, d\} : \rho(\varphi(sa), s\varphi(a)) < \sqrt{\delta}\right\}\right| \geq (1-\delta)d.$$

Therefore $|\Lambda_\varphi|/d \geq 1 - |F_\ell|\delta$.

Now let $R \in \mathcal{R}$, $a \in \Omega_{R,\varphi}$ and $s \in F_\ell$. One has

$$h_R(s^{-1}\varphi(sa)) \geq h_R(\varphi(a)) - |h_R(\varphi(a)) - h_R(s^{-1}\varphi(sa))| > 1/2 - 1/2 = 0.$$

Therefore $s^{-1}\varphi(sa) \in B'_R$, and hence $\varphi(sa) \in F_\ell B'_R \subseteq U_R$. Since the sets U_R for $R \in \mathcal{R}$ are pairwise disjoint, the sets $\sigma(F_\ell)\Omega_{R,\varphi}$ for $R \in \mathcal{R}$ are pairwise disjoint.

Let $R \in \mathcal{R}$. We have

$$(\varphi_*\zeta)(h_R) \geq \mu(h_R) - \delta \geq \mu(Z'_R) - \delta \geq \mu(R)(1-\lambda) - \delta. \tag{5}$$

Since $h_R \leq 1$, we have

$$\frac{|\Omega'_{R,\varphi}|}{d} \geq (\varphi_*\zeta)(h_R) \geq \mu(R)(1-\lambda) - \delta.$$

Since $h_R h_{R'} = 0$ for all distinct $R, R' \in \mathcal{R}$, the sets $\{\Omega'_{R,\varphi}\}_{R\in\mathcal{R}}$ are pairwise disjoint. Therefore

$$\begin{aligned} \frac{|\Omega'_{R,\varphi}|}{d} &\leq 1 - \sum_{R' \in \mathcal{R}\setminus\{R\}} \frac{|\Omega'_{R',\varphi}|}{d} \\ &\leq 1 - \sum_{R' \in \mathcal{R}\setminus\{R\}} (\mu(R')(1-\lambda) - \delta) \\ &\leq \mu(R)(1-\lambda) + \lambda + |\mathcal{R}|\delta, \end{aligned} \tag{6}$$

and hence

$$\begin{aligned} \frac{|\Omega_{R,\varphi}|}{d} \leq \frac{|\Omega'_{R,\varphi}|}{d} &\leq \mu(R)(1-\lambda) + \lambda + |\mathcal{R}|\delta \\ &\leq \mu(R) + \eta \end{aligned}$$

when λ, δ are small enough. We have

$$\begin{aligned} (\varphi_*\zeta)(h_R) &\leq \frac{|\Omega''_{R,\varphi}|}{d} + \frac{|\Omega'_{R,\varphi} \setminus \Omega''_{R,\varphi}|}{2d} = \frac{|\Omega'_{R,\varphi}|}{2d} + \frac{|\Omega''_{R,\varphi}|}{2d} \\ &\leq \frac{\mu(R)(1-\lambda) + \lambda + |\mathcal{R}|\delta}{2} + \frac{|\Omega''_{R,\varphi}|}{2d}. \end{aligned}$$

Thus using (5) we get

$$\frac{|\Omega''_{R,\varphi}|}{d} \geq \mu(R)(1-\lambda) - \lambda - (2 + |\mathcal{R}|)\delta, \tag{7}$$

and hence

$$\begin{aligned} \frac{|\Omega_{R,\varphi}|}{d} &\geq \frac{|\Omega''_{R,\varphi}|}{d} - \left(1 - \frac{|\Lambda_\varphi|}{d}\right) - \left(1 - \frac{|\Lambda|}{d}\right) \\ &\geq \mu(R)(1-\lambda) - 2\lambda - (2 + |\mathcal{R}| + |F_\ell|)\delta. \end{aligned}$$

Since the sets $\sigma(F_\ell)\Omega_{R,\varphi}$ for $R \in \mathcal{R}$ are pairwise disjoint, for every $R \in \mathcal{R}$ we have

$$\begin{aligned}\frac{|\sigma(F_\ell)\Omega_{R,\varphi}|}{d} - \frac{|\Omega_{R,\varphi}|}{d} &\leq 1 - \sum_{R' \in \mathcal{R}} \frac{|\Omega_{R',\varphi}|}{d} \\ &\leq (1 + 2|\mathcal{R}|)\lambda + |\mathcal{R}|(2 + |\mathcal{R}| + |F_\ell|)\delta,\end{aligned}$$

and hence, using the fact that $\sigma(F_\ell)\Omega_{R,\varphi} \supseteq \sigma_e\Omega_{R,\varphi} = \Omega_{R,\varphi}$,

$$\begin{aligned}\max_{s \in F_\ell} \frac{|\Omega_{R,\varphi}\Delta\sigma_s\Omega_{R,\varphi}|}{d} &\leq 2\bigg(\frac{|\sigma(F_\ell)\Omega_{R,\varphi}|}{d} - \frac{|\Omega_{R,\varphi}|}{d}\bigg) \\ &\leq 2(1 + 2|\mathcal{R}|)\lambda + 2|\mathcal{R}|(2 + |\mathcal{R}| + |F_\ell|)\delta \\ &\leq \omega(|F_\ell|),\end{aligned}$$

when λ, δ are small enough. This proves Claim I.

Claim II: Assuming λ, δ are small enough and σ is a good enough sofic approximation for G, for every $\varphi \in \mathrm{Map}_\mu(\rho, F_\ell, L, \delta, \sigma)$ and $R \in \mathcal{R}$ we have $\Theta_{R,\varphi} \subseteq \Omega_{R,\varphi}$, $|\Theta_{R,\varphi}|/d \geq \mu(R) - \eta$, and for every $a \in \Theta_{R,\varphi}$ one has $\varphi(a) \in B_R$.

To verify Claim II, first observe that, for every $R \in \mathcal{R}$, since $h_R \geq g_R$ we have $\Theta'_{R,\varphi} \subseteq \Omega'_{R,\varphi}$ and $\Theta_{R,\varphi} \subseteq \Omega_{R,\varphi}$. Also, for $R \in \mathcal{R}$ and $a \in \Theta_{R,\varphi}$, since $g_R(\varphi(a)) > 0$ we see that $\varphi(a)$ lies in B_R.

Now let $R \in \mathcal{R}$. One has

$$(\varphi_*\zeta)(g_R) \geq \mu(g_R) - \delta \geq \mu(Z_R) - \delta \geq \mu(R)(1 - \tau - \lambda) - \delta.$$

We also have

$$\frac{|\Theta'_{R,\varphi}|}{d} \leq \frac{|\Omega'_{R,\varphi}|}{d} \overset{(6)}{\leq} \mu(R)(1 - \lambda) + \lambda + |\mathcal{R}|\delta.$$

Similarly to (7), we get

$$\frac{|\Theta''_{R,\varphi}|}{d} \geq \mu(R)(1 - 2\tau - \lambda) - \lambda - (2 + |\mathcal{R}|)\delta,$$

and hence, using $\tau < \eta/4$,

$$\begin{aligned}\frac{|\Theta_{R,\varphi}|}{d} &\geq \frac{|\Theta''_{R,\varphi}|}{d} - \bigg(1 - \frac{|\Lambda_\varphi|}{d}\bigg) - \bigg(1 - \frac{|\Lambda|}{d}\bigg) \\ &\geq \mu(R)(1 - 2\tau - \lambda) - 2\lambda - (2 + |\mathcal{R}| + |F_\ell|)\delta \\ &\geq \mu(R) - \eta,\end{aligned}$$

when λ, δ are small enough and σ is a good enough sofic approximation for G. This proves Claim II.

For each $J \subseteq \{1, \dots, d\}$ we define on the set of maps from $\{1, \dots, d\}$ to X the pseudometric

$$\rho_{J,\infty}(\varphi, \psi) = \rho_\infty(\varphi|_J, \psi|_J).$$

Take a $(\rho_\infty, \varepsilon)$-separated subset D of $\mathrm{Map}_\mu(\rho, F_\ell, L, \delta, \sigma)$ of maximal cardinality.

By Claim I and the second paragraph of the proof of this lemma, when σ is a good enough sofic approximation for G there is a subset D' of D with $\exp(\kappa d)|D'| \geq |D|$ such that, for each $R \in \mathcal{R}$, the set $\Omega_{R,\varphi}$ is the same, say Ω_R, for every $\varphi \in D'$. By Claims I and II and the third paragraph of the proof, when σ is a good enough sofic approximation for

G, there is a subset W of D' with $|W|\exp(\kappa d) \geq |W|\prod_{R\in\mathcal{R}}\exp(\kappa|\Omega_R|) \geq |D'|$ such that, for each $R \in \mathcal{R}$, the set $\Theta_{R,\varphi}$ is the same, say Θ_R, for every $\varphi \in W$.

Now take Y_R to be Θ_R in our application of Lemma 4.5 in the fourth paragraph of the proof in order to obtain sets $C_{R,1}, \dots, C_{R,\ell} \subseteq \Theta_R$ satisfying the two conditions listed there. Denote by $\mathscr{L}_R$ the set of all pairs (k, c) such that $k \in \{1, \dots, \ell\}$ and $c \in C_{R,k}$. By η-disjointness, for every $(k, c) \in \mathscr{L}_R$ we can find an $F_{k,c} \subseteq F_k$ with $|F_{k,c}| \geq (1-\eta)|F_k|$ such that the sets $\sigma(F_{k,c})c$ for $(k, c) \in \mathscr{L}_R$ are pairwise disjoint.

Let $(k, c) \in \mathscr{L}_R$. Take an $(\varepsilon/2)$-spanning subset $V_{k,c}$ of W with respect to $\rho_{\sigma(F_{k,c})c,\infty}$ of minimal cardinality. We record here the relation between these sets:

$$V_{k,c} \subseteq W \subseteq D' \subseteq D \subseteq \mathrm{Map}_\mu(\rho, F_\ell, L, \delta, \sigma).$$

Claim III: Assuming δ is small enough, one has

$$|V_{k,c}| \leq \exp((\xi_R + \kappa)|F_k|).$$

To verify Claim III, let V be an $(\varepsilon/2)$-separated subset of W with respect to $\rho_{\sigma(F_{k,c})c,\infty}$. For each $\varphi \in V$, since $c \in C_{R,k} \subseteq \Theta_R$ the point $\varphi(c)$ lies in B_R. Let φ and ψ be distinct elements of V. Then for every $s \in F_{k,c}$, since $c \in \Lambda_\varphi \cap \Lambda_\psi$ we have

$$\begin{aligned}\rho(s\varphi(c), s\psi(c)) &\geq \rho(\varphi(sc), \psi(sc)) - \rho(s\varphi(c), \varphi(sc)) - \rho(s\psi(c), \psi(sc)) \\ &\geq \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta},\end{aligned}$$

and hence

$$\rho_{F_{k,c}}(\varphi(c), \psi(c)) = \max_{s\in F_{k,c}} \rho(s\varphi(c), s\psi(c)) \geq \max_{s\in F_{k,c}} \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta} > \varepsilon/2 - \varepsilon/4 = \varepsilon/4,$$

granted that δ is taken small enough. Thus $\{\varphi(c) : \varphi \in V\}$ is a $(\rho_{F_{k,c}}, \varepsilon/4)$-separated subset of B_R of cardinality $|V|$, so that

$$|V| \leq N_{\varepsilon/4}(B_R, \rho_{F_{k,c}}) \leq N_{\varepsilon/4}(B_R, \rho_{F_k}) \overset{(4)}{\leq} \exp((\xi_R + \kappa)|F_k|).$$

Therefore

$$|V_{k,c}| \leq N_{\varepsilon/2}(W, \rho_{\sigma(F_{k,c})c,\infty}) \leq \exp((\xi_R + \kappa)|F_k|).$$

This proves Claim III.

Set

$$H = \{1, \dots, d\} \setminus \bigcup_{R\in\mathcal{R}} \bigcup \{\sigma(F_{k,c})c : (k, c) \in \mathscr{L}_R\},$$

and take an $(\varepsilon/2)$-spanning subset V_H of W with respect to $\rho_{H,\infty}$ of minimal cardinality.

Claim IV:

$$|V_H| \leq (N_{\varepsilon/4}(X, \rho))^{3|\mathcal{R}|\eta d}.$$

To verify Claim IV, first note that for each $R \in \mathcal{R}$ we have

$$\left|\bigcup\{\sigma(F_{k,c})c : (k, c) \in \mathscr{L}_R\}\right| \geq (1-\eta)\left|\bigcup_{k=1}^{\ell} \sigma(F_k)C_{R,k}\right| \geq (1-\eta)(\mu(R) - 2\eta)d.$$

Since the sets $\bigcup_{k=1}^{\ell} \sigma(F_k)C_{R,k}$ for $R \in \mathcal{R}$ are pairwise disjoint, we get

$$\left|\bigcup_{R\in\mathcal{R}} \bigcup\{\sigma(F_{k,c})c : (k, c) \in \mathscr{L}_R\}\right| \geq \sum_{R\in\mathcal{R}} (1-\eta)(\mu(R) - 2\eta)d = (1-\eta)(1 - 2|\mathcal{R}|\eta)d.$$

Therefore

$$|H| \le (\eta + 2(1-\eta)|\mathcal{R}|\eta)d \le (1+2|\mathcal{R}|)\eta d \le 3|\mathcal{R}|\eta d,$$

and thus

$$|V_H| \le (N_{\varepsilon/4}(X,\rho))^{|H|} \le (N_{\varepsilon/4}(X,\rho))^{3|\mathcal{R}|\eta d}.$$

This proves Claim IV.

Write U for the set of all maps $\varphi : \{1,\dots,d\} \to X$ such that $\varphi|_H \in V_H|_H$ and $\varphi|_{\sigma(F_{k,c})c} \in V_{k,c}|_{\sigma(F_{k,c})c}$ for all $R \in \mathcal{R}$ and $(k,c) \in \mathscr{L}_R$.

Claim V:

$$|U| \le \exp\left(\left(\sum_{R\in\mathcal{R}} \xi_R \mu(R) + 3\kappa\right)d\right).$$

To verify Claim V, observe that, since the sets $\bigcup_{k=1}^{\ell} \sigma(F_k)C_{R,k}$ for $R \in \mathcal{R}$ are pairwise disjoint, for each $R \in \mathcal{R}$ we have

$$\begin{aligned}\left|\bigcup_{k=1}^{\ell} \sigma(F_k)C_{R,k}\right| &\le d - \sum_{R'\in\mathcal{R}\setminus\{R\}} \left|\bigcup_{k=1}^{\ell} \sigma(F_k)C_{R',k}\right| \\ &\le d - \sum_{R'\in\mathcal{R}\setminus\{R\}} (\mu(R') - 2\eta)d \\ &\le \mu(R)d + 2|\mathcal{R}|\eta d.\end{aligned}$$

Therefore, by our choice of η in the third paragraph of the proof,

$$\begin{aligned}|U| = |V_H| \prod_{R\in\mathcal{R}} \prod_{(k,c)\in\mathscr{L}_R} |V_{k,c}| &\le (N_{\varepsilon/4}(X,\rho))^{3|\mathcal{R}|\eta d} \exp\left(\sum_{R\in\mathcal{R}} \sum_{(k,c)\in\mathscr{L}_R} (\xi_R+\kappa)|F_k|\right) \\ &= (N_{\varepsilon/4}(X,\rho))^{3|\mathcal{R}|\eta d} \exp\left(\sum_{R\in\mathcal{R}} (\xi_R+\kappa) \sum_{k=1}^{\ell} |F_k|\cdot|C_{R,k}|\right) \\ &\le \exp(\kappa d)\exp\left(\frac{1}{1-\eta}\sum_{R\in\mathcal{R}}(\xi_R+\kappa)\left|\bigcup_{k=1}^{\ell}\sigma(F_k)C_{R,k}\right|\right) \\ &\le \exp(\kappa d)\exp\left(\frac{1}{1-\eta}\sum_{R\in\mathcal{R}}(\xi_R+\kappa)(\mu(R)d + 2|\mathcal{R}|\eta d)\right) \\ &\le \exp(\kappa d)\exp\left(\frac{1}{1-\eta}\left(\sum_{R\in\mathcal{R}}\xi_R\mu(R) + \kappa + 2\kappa|\mathcal{R}|^2\eta + 2|\mathcal{R}|\eta\sum_{R\in\mathcal{R}}\xi_R\right)d\right) \\ &\le \exp(\kappa d)\exp\left(\left(\sum_{R\in\mathcal{R}}\xi_R\mu(R) + 2\kappa\right)d\right) = \exp\left(\left(\sum_{R\in\mathcal{R}}\xi_R\mu(R) + 3\kappa\right)d\right).\end{aligned}$$

This proves Claim V.

Note that every element of W lies within ρ_∞-distance $\varepsilon/2$ to an element of U, and since W is ε-separated with respect to ρ_∞ this means that the cardinality of W is at most that of U. Therefore

$$N_\varepsilon(\mathrm{Map}_\mu(\rho, F_\ell, L, \delta, \sigma), \rho_\infty) = |D| \le \exp(\kappa d)|D'| \le \exp(2\kappa d)|W| \le \exp(2\kappa d)|U|$$

$$\leq \exp(2\kappa d)\exp\bigg(\bigg(\sum_{R\in\mathcal{R}}\xi_R\mu(R)+3\kappa\bigg)d\bigg)$$
$$= \exp\bigg(\bigg(\sum_{R\in\mathcal{R}}\xi_R\mu(R)+5\kappa\bigg)d\bigg),$$

as we aimed to show. □

Lemma 6.4. *Let G be an amenable countably infinite discrete group acting continuously on a compact metrizable space X and μ a G-invariant Borel probability measure on X. Let ρ be a compatible metric on X. Then*

$$h_{\Sigma,\mu,\infty}(\rho) \geq h_\mu(X,G).$$

Proof. By [12, Lemmas 5.3.6 and 5.3.4], the entropy $h_\mu(X,G)$ is equal to the supremum of $h_\mu(\mathcal{Q})$ for $\mathcal{Q}$ ranging over finite Borel partitions of X with $\max_{Q\in\mathcal{Q}}\mu(\partial Q)=0$, where ∂Q denotes the boundary of Q. Thus it suffices to show that $h_{\Sigma,\mu,\infty}(\rho)\geq h_\mu(\mathcal{Q})$ for every such $\mathcal{Q}$. Let $\mathcal{Q}$ be such a partition.

Since $h_\mu(\mathcal{Q})=\int_X h_{\mu_x}(\mathcal{Q})\,d\mu(x)$ and the function $x\mapsto h_{\mu_x}(\mathcal{Q})$ is $X_{\mathcal{B},G}$-measurable, where $\mathcal{B}_{X,G}$ denotes the σ-algebra of G-invariant Borel subsets of X and $x\mapsto\mu_x$ is the Borel map from X to $\mathcal{M}^{\mathrm{e}}(X,G)$ described at the beginning of the section, it suffices to show that, for every nonnegative simple $\mathcal{B}_{X,G}$-measurable function g on X with $g(x)\leq h_{\mu_x}(\mathcal{Q})$ for every $x\in X$, one has $h_{\Sigma,\mu,\infty}(\rho)\geq\int_X g\,d\mu$. Let g be such a function.

It is enough to show that for every $\theta>0$ there is an $\varepsilon>0$ such that $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\geq\int_X g\,d\mu-2\theta$. So let $\theta>0$.

Let $0<\eta<\theta/8$. Also let $\kappa>0$, which we will determine in a moment. For every $\varepsilon>0$ and $Q\in\mathcal{Q}$ we write Q^ε for the open ε-neighbourhood of Q with respect to ρ. Thus $\bigcap_{\varepsilon>0}Q^{2\varepsilon}=\overline{Q}$ for each $Q\in\mathcal{Q}$ where $\overline{Q}$ denotes the closure of Q. As $\max_{Q\in\mathcal{Q}}\mu(\partial Q)=0$, we can find a particular ε, which we now fix, such that $\sum_{Q\in\mathcal{Q}}\mu(Q^{2\varepsilon}\setminus Q)<\kappa^2$. Set $D=X\setminus\bigcup_{Q\in\mathcal{Q}}(Q^{2\varepsilon}\setminus Q)$. Now it suffices to show $h^\varepsilon_{\Sigma,\mu,\infty}(\rho)\geq\int_X g\,d\mu-2\theta$.

Let F be a nonempty finite subset of G, L a finite subset of $C(X)$, and $\delta>0$. Let σ be a map from G to $\mathrm{Sym}(d)$ for some $d\in\mathbb{N}$. It suffices to show that if σ is a good enough sofic approximation then

$$\frac{1}{d}\log N_\varepsilon(\mathrm{Map}_\mu(\rho,F,L,\delta,\sigma),\rho_\infty)\geq\int_X g\,d\mu-2\theta. \tag{8}$$

Given a nonempty finite subset F' of G, set

$$D_{F'}=\bigg\{x\in X:\sum_{s\in F'}1_D(sx)\geq(1-\kappa)|F'|\bigg\}.$$

Then

$$\kappa|F'|\mu(X\setminus D_{F'})\leq\int_{X\setminus D_{F'}}\sum_{s\in F'}1_{X\setminus D}(sx)\,d\mu$$
$$\leq\int_X\sum_{s\in F'}1_{X\setminus D}(sx)\,d\mu=\mu(X\setminus D)|F'|<\kappa^2|F'|,$$

so that $\mu(D_{F'}) > 1 - \kappa$. By Lemma 6.1, when F' is sufficiently left invariant there is a Borel subset $V_{F'}$ of X with $\mu(V_{F'}) \geq 1 - \kappa$ such that if $x \in V_{F'}$ and the atom of $\mathcal{Q}^{F'}$ containing x is A then

$$\mu(A) \leq \exp(-(h_{\mu_x}(\mathcal{Q}) - \eta)|F'|) \leq \exp(-(g(x) - \eta)|F'|).$$

Since g and the functions $x \mapsto \mu_x(f)$ on X for $f \in L$ are all $\mathcal{B}_{X,G}$-measurable, we can take a finite $\mathcal{B}_{X,G}$-measurable partition $\tilde{\mathcal{B}}$ of X such that g is a constant function on each atom of $\tilde{\mathcal{B}}$ and for every $B \in \tilde{\mathcal{B}}$ one has

$$\max_{f \in L} \Big(\sup_{x \in B} \mu_x(f) - \inf_{y \in B} \mu_y(f) \Big) < \frac{\delta}{8}.$$

Denote by $\mathcal{B}$ the set of atoms of $\tilde{\mathcal{B}}$ with positive μ-measure. Set $\tau = \min_{B \in \mathcal{B}} \mu(B)$. The mean ergodic theorem [12, page 44] states that, as the nonempty finite set $F' \subseteq G$ becomes more and more left invariant, $|F'|^{-1} \sum_{s \in F'} \alpha_s(f)$ converges to $\mathbb{E}_\mu(f|\mathcal{B}_{X,G})$ in $L^2(X, \mathcal{B}_X, \mu)$ for every $f \in L^2(X, \mathcal{B}_X, \mu)$, where $\mathcal{B}_X$ denotes the σ-algebra of Borel subsets of X. Thus, when F' is sufficiently right invariant, in other words when $(F')^{-1}$ is sufficiently left invariant, we can find a Borel subset $W_{F'}$ of X such that $\mu(W_{F'}) > 1 - \min(\kappa, \tau/2)$ and $\big||F'|^{-1} \sum_{s \in F'} f(sx) - \mu_x(f)\big| < \delta/8$ for all $f \in L$ and $x \in W_{F'}$. In particular, when F' is sufficiently right invariant we can find, for each $B \in \mathcal{B}$, a point $x_{B,F'} \in B$ such that $\big||F'|^{-1} \sum_{s \in F'} f(sx_{B,F'}) - \mu_{x_{B,F'}}(f)\big| < \delta/8$ for all $f \in L$.

Now consider a nonempty finite subset F' of G which is sufficiently two-sided invariant so that both $V_{F'}$ and $W_{F'}$ exist. For each $B \in \mathcal{B}$, write $\mathcal{A}_{B,F'}$ for the collection of all $A \in \mathcal{Q}^{F'}$ such that $\mu(A \cap B \cap D_{F'} \cap V_{F'} \cap W_{F'}) > 0$. For each $A \in \mathcal{A}_{B,F'}$ pick a point $x_A \in A \cap B \cap D_{F'} \cap V_{F'} \cap W_{F'}$ and set $E'_{B,F'} = \{x_A : A \in \mathcal{A}_{B,F'}\}$. Denote by g_B the constant value of g on B.

Claim I: Assuming κ is small enough there is, for each $B \in \mathcal{B}$, a $(\rho_{F'}, \varepsilon)$-separated subset $E_{B,F'}$ of $E'_{B,F'}$ such that

$$|E_{B,F'}| \geq \mu(B \cap D_{F'} \cap V_{F'} \cap W_{F'}) \exp(\max(g_B - 2\eta, 0)|F'|). \tag{9}$$

To verify Claim I, note first that

$$\mu\Big(\bigcup \mathcal{A}_{B,F'}\Big) \geq \mu(B \cap D_{F'} \cap V_{F'} \cap W_{F'}),$$

and since $\mu(A) \leq \exp(-(g(x_A) - \eta)|F'|) = \exp(-(g_B - \eta)|F'|)$ for every $A \in \mathcal{A}_{B,F'}$, one has

$$\begin{aligned}|\mathcal{A}_{B,F'}| &\geq \mu\Big(\bigcup \mathcal{A}_{B,F'}\Big) / \exp(-(g_B - \eta)|F'|) \\ &\geq \mu(B \cap D_{F'} \cap V_{F'} \cap W_{F'}) \exp((g_B - \eta)|F'|).\end{aligned}$$

For each $x \in E'_{B,F'}$, since $x \in D_{F'}$ there exists a $J_x \subseteq F'$ with $|J_x| = |F'| - \lfloor \kappa |F'| \rfloor$ such that $sx \in D$ for every $s \in J_x$, where $\lfloor t \rfloor$ denotes the largest integer no bigger than t. Then there exists an $E''_{B,F'} \subseteq E'_{B,F'}$ with $\binom{|F'|}{\kappa|F'|} |E''_{B,F'}| \geq |E'_{B,F'}|$ such that J_x is the same, say $J_{B,F'}$, for all $x \in E''_{B,F'}$.

Let $x \in E''_{B,F'}$. Let $y \in E''_{B,F'}$ be such that $\rho_{F'}(x, y) \leq \varepsilon$. Then sx and sy lie in the same atom of $\mathcal{Q}$ for each $s \in J_{B,F'}$, for if sx and sy were contained in different atoms of $\mathcal{Q}$

for some $s \in J_{B,F'}$ then since $sx, sy \in D$ we would have $\rho(sx, sy) \geq 2\varepsilon$, which is impossible since $s \in F'$. It follows that there are at most $|\mathcal{Q}|^{|F'|-|J_{B,F'}|}$ many $y \in E''_{B,F'}$ satisfying $\rho_{F'}(x, y) \leq \varepsilon$. Hence there must exist a $(\rho_{F'}, \varepsilon)$-separated subset $E_{B,F'}$ of $E''_{B,F'}$ such that $|\mathcal{Q}|^{|F'|-|J_{B,F'}|}|E_{B,F'}| \geq |E''_{B,F'}|$. We then have

$$\begin{aligned}
|E_{B,F'}| &\geq |\mathcal{Q}|^{|J_{B,F'}|-|F'|}|E''_{B,F'}| \geq |\mathcal{Q}|^{-\kappa|F'|}|E'_{B,F'}|\binom{|F'|}{\kappa|F'|}^{-1} \\
&= |\mathcal{Q}|^{-\kappa|F'|}|\mathcal{A}_{B,F'}|\binom{|F'|}{\kappa|F'|}^{-1} \\
&\geq \mu(B \cap D_{F'} \cap V_{F'} \cap W_{F'}) \exp((g_B - \eta)|F'|)|\mathcal{Q}|^{-\kappa|F'|}\binom{|F'|}{\kappa|F'|}^{-1}.
\end{aligned}$$

Stirling's approximation then implies that when κ is small enough we have the inequality (9) for all sufficiently right invariant F'. This proves Claim I.

Let $\delta' > 0$ be such that $\delta' < \tau$, $4\delta'|\mathcal{B}| \max_{f \in L} \|f\|_\infty < \delta$ and $\delta' \sum_{B \in \mathcal{B}} g_B < \theta/4$. Let M be a large positive integer to be specified below.

Let $\delta'' > 0$, which we will determine in a moment. It follows from Lemma 4.6 that there are an $\ell \in \mathbb{N}$ and sufficiently two-sided invariant nonempty finite subsets $F_1, \dots, F_\ell$ of G such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ which is a good enough sofic approximation for G there exist $C_1, \dots, C_\ell \subseteq \{1, \dots, d\}$ such that

(i) for every $k = 1, \dots, \ell$, the map $(s, c) \mapsto \sigma_s(c)$ from $F_k \times C_k$ to $\sigma(F_k)C_k$ is bijective,
(ii) the family $\{\sigma(F_1)C_1, \dots, \sigma(F_\ell)C_\ell\}$ is disjoint and $(1 - \delta'')$-covers $\{1, \dots, d\}$.

Write Λ for the set of all $k \in \{1, \dots, \ell\}$ such that $|C_k| \geq M$. Taking M to be large enough, for every $k \in \Lambda$ we can find a partition $\{C_{k,B}\}_{B \in \mathcal{B}}$ of C_k such that $||C_{k,B}|/|C_k| - \mu(B)| < \delta'$ for every $B \in \mathcal{B}$.

For each $k \in \Lambda$, set $\mathcal{B}'_k = \{B \in \mathcal{B} : \mu(B \cap D_{F_k} \cap V_{F_k} \cap W_{F_k}) \geq \tau/2\}$. If $B \in \mathcal{B} \setminus \mathcal{B}'_k$, then $\mu(B \setminus (D_{F_k} \cap V_{F_k} \cap W_{F_k})) > \tau/2 > \mu(B \cap D_{F_k} \cap V_{F_k} \cap W_{F_k})$, and hence $\mu(B) < 2\mu(B \setminus (D_{F_k} \cap V_{F_k} \cap W_{F_k}))$. Since $\mu(X \setminus (D_{F_k} \cap V_{F_k} \cap W_{F_k})) < 3\kappa$, we have

$$\mu\Big(\bigcup(\mathcal{B} \setminus \mathcal{B}'_k)\Big) \leq 2\mu(X \setminus (D_{F_k} \cap V_{F_k} \cap W_{F_k})) < 6\kappa.$$

Taking κ to be small enough, we may require that $\int_Y g\, d\mu < \theta/4$ for every Borel set $Y \subseteq X$ with $\mu(Y) < 6\kappa$. Then

$$\int_{\bigcup(\mathcal{B} \setminus \mathcal{B}'_k)} g\, d\mu < \theta/4. \tag{10}$$

For each $h = (h_{k,B})_{k,B} \in \prod_{k \in \Lambda} \prod_{B \in \mathcal{B}'_k} (E_{B,F_k})^{C_{k,B}}$, take a map φ_h from $\{1, \dots, d\}$ to X such that for every $k \in \Lambda$, $B \in \mathcal{B}$, $c \in C_{k,B}$, and $s \in F_k$ the point $\varphi_h(sc)$ is equal to $s(h_{k,B}(c))$ or $s(x_{B,F_k})$ depending on whether $B \in \mathcal{B}'_k$ or $B \in \mathcal{B} \setminus \mathcal{B}'_k$.

Claim II: Assuming $F_1, \dots, F_\ell$ are sufficiently left invariant, δ'' is small enough, and σ is a good enough sofic approximation for G, one has $|\bigcup_{k \in \Lambda} \sigma(F_k)C_k| > (1 - 2\delta'')d$ and the map φ_h lies in $\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma)$ for every $h \in \prod_{k \in \Lambda} \prod_{B \in \mathcal{B}'_k} (E_{B,F_k})^{C_{k,B}}$.

To verify Claim II, suppose we are given $k \in \Lambda$, $B \in \mathcal{B}'_k$ and $f \in L$. Since $B \in \mathcal{B}_{X,G}$, we have

$$\int_B f \, d\mu = \int_B \mathbb{E}_\mu(f|\mathcal{B}_{X,G}) \, d\mu = \int_B \mu_x(f) \, d\mu(x).$$

For each $c \in C_{k,B}$, since $h_{k,B}(c) \in E_{B,F_k} \subseteq W_{F_k} \cap B$ one has

$$\begin{aligned} &\bigg| \frac{1}{|F_k|} \sum_{s \in F_k} f(\varphi_h(sc)) - \frac{1}{\mu(B)} \int_B f \, d\mu \bigg| \\ &\qquad \leq \bigg| \frac{1}{|F_k|} \sum_{s \in F_k} f(s(h_{k,B}(c))) - \mu_{h_{k,B}(c)}(f) \bigg| + \bigg| \mu_{h_{k,B}(c)}(f) - \frac{1}{\mu(B)} \int_B \mu_x(f) \, d\mu(x) \bigg| \\ &\qquad < \frac{\delta}{8} + \frac{\delta}{8} = \frac{\delta}{4}. \end{aligned}$$

As $|\sigma(F_k)C_{k,B}|^{-1} \sum_{a \in \sigma(F_k)C_{k,B}} f(\varphi_h(a))$ is a convex combination of the quantities $|F_k|^{-1} \sum_{s \in F_k} f(\varphi_h(sc))$ for $c \in C_{k,B}$, we get

$$\bigg| \frac{1}{|\sigma(F_k)C_{k,B}|} \sum_{a \in \sigma(F_k)C_{k,B}} f(\varphi_h(a)) - \frac{1}{\mu(B)} \int_B f \, d\mu \bigg| < \delta/4. \tag{11}$$

Inequality (11) also holds similarly for all $k \in \Lambda$, $B \in \mathcal{B} \setminus \mathcal{B}'_k$, and $f \in L$. Thus, for all $k \in \Lambda$ and $f \in L$ we have

$$\begin{aligned} &\bigg| \frac{1}{|\sigma(F_k)C_k|} \sum_{a \in \sigma(F_k)C_k} f(\varphi_h(a)) - \int_X f \, d\mu \bigg| \\ &\qquad \leq \bigg| \sum_{B \in \mathcal{B}} \frac{|\sigma(F_k)C_{k,B}|}{|\sigma(F_k)C_k|} \cdot \frac{1}{|\sigma(F_k)C_{k,B}|} \sum_{a \in \sigma(F_k)C_{k,B}} f(\varphi_h(a)) \\ &\qquad\qquad - \sum_{B \in \mathcal{B}} \frac{|\sigma(F_k)C_{k,B}|}{|\sigma(F_k)C_k|} \cdot \frac{1}{\mu(B)} \int_B f \, d\mu \bigg| \\ &\qquad\qquad + \bigg| \sum_{B \in \mathcal{B}} \frac{|\sigma(F_k)C_{k,B}|}{|\sigma(F_k)C_k|} \cdot \frac{1}{\mu(B)} \int_B f \, d\mu - \sum_{B \in \mathcal{B}} \mu(B) \cdot \frac{1}{\mu(B)} \int_B f \, d\mu \bigg| \\ &\qquad < \frac{\delta}{4} + \delta' |\mathcal{B}| \max_{f \in L} \|f\|_\infty \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}. \end{aligned}$$

For all $f \in L$, as $|\bigcup_{k \in \Lambda} \sigma(F_k)C_k|^{-1} \sum_{a \in \bigcup_{k \in \Lambda} \sigma(F_k)C_k} f(\varphi_h(a))$ is a convex combination of the quantities $|\sigma(F_k)C_k|^{-1} \sum_{a \in \sigma(F_k)C_k} f(\varphi_h(a))$ for $k \in \Lambda$, we get

$$\bigg| \frac{1}{|\bigcup_{k \in \Lambda} \sigma(F_k)C_k|} \sum_{a \in \bigcup_{k \in \Lambda} \sigma(F_k)C_k} f(\varphi_h(a)) - \int_X f \, d\mu \bigg| < \frac{\delta}{2}.$$

Note that if σ is a good enough sofic approximation for G then d will be large enough so that the family $\{\sigma(F_k)C_k : k \in \Lambda\}$ is a $(1 - 2\delta'')$-covering of $\{1, \dots, d\}$. It follows that, when $F_1, \dots, F_\ell$ are sufficiently left invariant, δ'' is small enough, and σ is a good enough sofic approximation for G, the map φ_h lies in $\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma)$. This proves Claim II.

Claim III:

$$\frac{1}{d}\log N_\varepsilon(\mathrm{Map}_\mu(\rho,F,L,\delta,\sigma),\rho_\infty) \geq \frac{1}{\min_{1\leq k\leq \ell}|F_k|}\log\frac{\tau}{2}+(1-2\delta'')\Big(\int_X g\,d\mu-\theta\Big). \tag{12}$$

To verify Claim III, note first that if $h=(h_{k,B})_{k,B}$ and $h'=(h'_{k,B})_{k,B}$ are distinct elements of $\prod_{k\in\Lambda}\prod_{B\in\mathcal{B}'_k}(E_{B,F_k})^{C_{k,B}}$, then $h_{k,B}(c)\neq h'_{k,B}(c)$ for some $k\in\Lambda$, $B\in\mathcal{B}'_k$, and $c\in C_{k,B}$. Since $h_{k,B}(c)$ and $h'_{k,B}(c)$ are (ρ_{F_k},ε)-separated, we see that $\rho_\infty(\varphi_h,\varphi_{h'})\geq\varepsilon$. Therefore

$$\begin{aligned}
&N_\varepsilon(\mathrm{Map}_\mu(\rho,F,L,\delta,\sigma),\rho_\infty)\\
&\quad\geq\Big|\prod_{k\in\Lambda}\prod_{B\in\mathcal{B}'_k}(E_{B,F_k})^{C_{k,B}}\Big|\\
&\quad\overset{(9)}{\geq}\prod_{k\in\Lambda}\prod_{B\in\mathcal{B}'_k}(\mu(B\cap D_{F_k}\cap V_{F_k}\cap W_{F_k}))^{|C_{k,B}|}\exp(\max(g_B-2\eta,0)|F_k|\cdot|C_{k,B}|)\\
&\quad\geq\prod_{k\in\Lambda}\prod_{B\in\mathcal{B}'_k}\Big(\frac{\tau}{2}\Big)^{|C_{k,B}|}\exp(\max(g_B-2\eta,0)|F_k|\cdot|C_{k,B}|)\\
&\quad\geq\prod_{k\in\Lambda}\Big(\frac{\tau}{2}\Big)^{\sum_{B\in\mathcal{B}'_k}|C_{k,B}|}\exp\Big(|F_k|\cdot|C_k|\sum_{B\in\mathcal{B}'_k}\max(g_B-2\eta,0)(\mu(B)-\delta')\Big)\\
&\quad\geq\prod_{k\in\Lambda}\Big(\frac{\tau}{2}\Big)^{|C_k|}\exp\Big(|F_k|\cdot|C_k|\sum_{B\in\mathcal{B}'_k}(g_B-2\eta)(\mu(B)-\delta')\Big)\\
&\quad\geq\Big(\Big(\frac{\tau}{2}\Big)^{1/\min_{1\leq k\leq\ell}|F_k|}\Big)^d\prod_{k\in\Lambda}\exp\Big(|F_k|\cdot|C_k|\sum_{B\in\mathcal{B}'_k}(g_B-2\eta)(\mu(B)-\delta')\Big).
\end{aligned}$$

For $k\in\Lambda$, one has

$$\begin{aligned}
\sum_{B\in\mathcal{B}'_k}(g_B-2\eta)(\mu(B)-\delta')&\geq\sum_{B\in\mathcal{B}'_k}g_B\mu(B)-\delta'\sum_{B\in\mathcal{B}'_k}g_B-2\eta\sum_{B\in\mathcal{B}'_k}(\mu(B)-\delta')\\
&\geq\int_{\bigcup\mathcal{B}'_k}g\,d\mu-\delta'\sum_{B\in\mathcal{B}}g_B-2\eta\\
&>\Big(\int_X g\,d\mu-\theta/4\Big)-\theta/4-\theta/4>\int_X g\,d\mu-\theta,
\end{aligned}$$

where (10) has been used to obtain the second last inequality. Thus

$$\begin{aligned}
&N_\varepsilon(\mathrm{Map}_\mu(\rho,F,L,\delta,\sigma),\rho_\infty)\\
&\qquad\geq\Big(\Big(\frac{\tau}{2}\Big)^{1/\min_{1\leq k\leq\ell}|F_k|}\Big)^d\prod_{k\in\Lambda}\exp\Big(|F_k|\cdot|C_k|\Big(\int_X g\,d\mu-\theta\Big)\Big)
\end{aligned}$$

$$\geq \Big(\Big(\frac{\tau}{2}\Big)^{1/\min_{1\leq k\leq \ell}|F_k|}\Big)^d \exp\Big(d(1-2\delta'')\Big(\int_X g\,d\mu-\theta\Big)\Big),$$

and hence the inequality (12) holds. This proves Claim III.

Now by taking δ'' to be small enough and $F_1, \dots, F_\ell$ to be sufficiently left invariant, we obtain

$$\frac{1}{\min_{1\leq k\leq \ell}|F_k|}\log\frac{\tau}{2}+(1-2\delta'')\Big(\int_X g\,d\mu-\theta\Big)\geq \int_X g\,d\mu-2\theta,$$

yielding (8). □

When a finite group G acts on a standard probability space (X, μ) by measure-preserving transformations, from the definition of $h_\mu(X, G)$ given at the beginning of this section we have $h_\mu(X, G) = +\infty$ when $\sum_{x\in X}\mu(\{x\}) < 1$, and $h_\mu(X, G) = |G|^{-1}\sum_{x\in X}\xi(\mu(\{x\}))$ when $\sum_{x\in X}\mu(\{x\}) = 1$, where $\xi(t) = -t\log t$ for $t \in [0, 1]$.

Lemma 6.5. *Let G be a finite group acting on a standard probability space (X, μ) by measure-preserving transformations. Then*

$$h_{\Sigma,\mu}(X, G) \leq h_\mu(X, G).$$

Proof. Set $\xi(t) = -t\log t$ for $t \in [0, 1]$. We may assume that $h_\mu(X, G) < +\infty$. Then there is a G-invariant countable subset Z of X with $\sum_{z\in Z}\mu(\{z\}) = 1$, $\mu(\{z\}) > 0$ for every $z \in Z$, and $h_\mu(X, G) = |G|^{-1}\sum_{z\in Z}\xi(\mu(\{z\}))$. Let $\kappa > 0$. It suffices to show that $h_{\Sigma,\mu}(X, G) \leq h_\mu(X, G) + 3\kappa$.

Up to measure conjugacy, we may assume that X is a compact metrizable space such that each point of Z is isolated in X, G acts on X continuously, and μ is a G-invariant Borel probability measure on X. Let ρ be a compatible metric on X with $\mathrm{diam}_\rho(X) \leq 1$. By Proposition 3.4 it suffices to show that $h^\varepsilon_{\Sigma,\mu,2}(\rho) \leq h_\mu(X, G) + 3\kappa$ for every $\varepsilon > 0$.

Let $\varepsilon > 0$. Take a finite subset Z' of Z such that the orbits Gz for $z \in Z'$ are pairwise disjoint, $1 - \mu(GZ') < \varepsilon^2/2$, and $\xi(1 - \mu(GZ')) < \kappa$. Say $Z' = \{z_1, \dots, z_n\}$. For each $k = 1, \dots, n$ write p_k for the characteristic function of $\{z_k\}$ in $C(X)$, and write c_k and G_k for $\mu(Gz_k)$ and $\{s \in G : sz_k = z_k\}$, respectively. Let τ be a strictly positive number to be specified in a moment. Let $\delta > 0$ be such that the ρ-distance of each point in GZ' from any other point in X is bigger than $\sqrt{\delta}$, $(c_k + \delta|G/G_k|)/(1-\delta) < c_k + \tau$ for every $k = 1, \dots, n$, $(2 + |G|)\delta|G| < \tau$, and $n|G|(2 + |G|)\delta < \varepsilon^2/2$. Set $L = \{p_1, \dots, p_n\}$. Now it suffices to show that $h^\varepsilon_{\Sigma,\mu,2}(\rho, G, L, \delta) \leq h_\mu(X, G) + 3\kappa$.

By Lemma 4.6, when a map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ is a good enough sofic approximation for G, there exists a subset C of $\{1, \dots, d\}$ such that the map $(s, c) \mapsto \sigma_s(c)$ from $G \times C$ to $\sigma(G)C$ is bijective, $|\sigma(G)C|/d \geq 1 - \delta$, and $\sigma_e(c) = c$ and $\sigma_s\sigma_t(c) = \sigma_{st}(c)$ for all $c \in C$ and $s, t \in G$.

Let $\varphi \in \mathrm{Map}_\mu(\rho, G, L, \delta, \sigma)$. Let $1 \leq k \leq n$ and $s \in G$. Set $Y_{k,s,\varphi} = \varphi^{-1}(sz_k)$. When $s = e$, we write $Y_{k,\varphi}$ for $Y_{k,e,\varphi}$. If $a \in Y_{k,\varphi}$ and $sa \notin Y_{k,s,\varphi}$, then $\rho(\varphi(sa), s\varphi(a)) > \sqrt{\delta}$. It follows that

$$\frac{1}{d}|(sY_{k,\varphi}) \setminus Y_{k,s,\varphi}|\delta \leq (\rho_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s))^2 < \delta^2,$$

and hence $|(sY_{k,\varphi}) \setminus Y_{k,s,\varphi}|/d < \delta$. Set $Y'_{k,\varphi} = Y_{k,\varphi} \cap \sigma(G)C \cap \bigcap_{s\in G}\sigma_s^{-1}(Y_{k,s,\varphi})$. Then $|Y_{k,\varphi} \setminus Y'_{k,\varphi}|/d \leq (1 + |G|)\delta \leq \tau$.

For each $k = 1, \dots, n$ set $C_{k,\varphi} = \{c \in C : \sigma(G)c \subseteq \sigma(G)Y'_{k,\varphi}\}$. Then $\sigma(G)C_{k,\varphi} = \sigma(G)Y_{k',\varphi}$. Note that if $s \in G$ and $t \in sG_k$, then $tz_k = sz_k$ and hence $Y_{k,t,\varphi} = Y_{k,s,\varphi}$. If $t \in G \setminus sG_k$, then $tz_k \neq sz_k$ and hence $Y_{k,t,\varphi} \cap Y_{k,s,\varphi} = \emptyset$. Let $a \in Y'_{k,\varphi}$. Then $\sigma(sG_k)a \subseteq \bigcup_{t \in sG_k} Y_{k,t,\varphi} = Y_{k,s,\varphi}$ for every $s \in G$. It follows that $Y_{k,s,\varphi} \cap \sigma(G)a = \sigma(sG_k)a$ for every $s \in G$. In particular, for each $c \in C_{k,\varphi}$, the set $Y'_{k,\varphi} \cap \sigma(G)c$ is of the form $\sigma(G_k s)c$ for some $s \in G$. Thus, given $(C_{1,\varphi}, \dots, C_{n,\varphi})$, the number of possibilities for $(Y'_{1,\varphi}, \dots, Y'_{n,\varphi})$ is at most

$$M_1 := \prod_{k=1}^{n} |G/G_k|^{|C_{k,\varphi}|}.$$

Let $k = 1, \dots, n$. We have

$$\big||Y_{k,\varphi}|/d - c_k|G/G_k|^{-1}\big| = |(\varphi_*\zeta)(p_k) - \mu(p_k)| < \delta,$$

and hence

$$|Y'_{k,\varphi}|/d \leq |Y_{k,\varphi}|/d < c_k|G/G_k|^{-1} + \delta$$

and

$$|Y'_{k,\varphi}|/d \geq |Y_{k,\varphi}|/d - (1 + |G|)\delta \geq c_k|G/G_k|^{-1} - (2 + |G|)\delta. \tag{13}$$

Since $|Y'_{k,\varphi}| = |C_{k,\varphi}| \cdot |G_k|$, we get

$$|C_{k,\varphi}|/|C| = |G/G_k| \cdot |Y'_{k,\varphi}|/|\sigma(G)C| < (c_k + \delta|G/G_k|)/(1 - \delta) < c_k + \tau,$$

and

$$|C_{k,\varphi}|/|C| \geq |G/G_k| \cdot |Y'_{k,\varphi}|/d > c_k - (2 + |G|)\delta|G| > c_k - \tau.$$

Thus

$$M_1 \leq \prod_{k=1}^{n} |G/G_k|^{(c_k+\tau)|C|} \leq \exp\left(\left(\kappa + \sum_{k=1}^{n} c_k \log|G/G_k|\right)\frac{d}{|G|}\right)$$

granted that τ is small enough.

For each $k = 1, \dots, n$, one has $\varphi(C_{k,\varphi}) \subseteq \varphi(\sigma(G)Y'_{k,\varphi}) \subseteq Gz_k$. Thus the sets $C_{1,\varphi}, \dots, C_{n,\varphi}$ are pairwise disjoint. Therefore the number of possibilities for the collection $(C_{1,\varphi}, \dots, C_{n,\varphi})$ is at most

$$M_2 := \sum_{j_1,\dots,j_n} \binom{|C|}{j_1}\binom{|C| - j_1}{j_2} \cdots \binom{|C| - \sum_{k=1}^{n-1} j_k}{j_n},$$

where the sum ranges over all nonnegative integers $j_1, \dots, j_n$ such that $|j_k/|C| - c_k| < \tau$ for all $1 \leq k \leq n$ and $\sum_{k=1}^{n} j_k \leq |C|$. By Stirling's approximation, for such $j_1, \dots, j_n$ one has

$$\begin{aligned} &\binom{|C|}{j_1}\binom{|C| - j_1}{j_2} \cdots \binom{|C| - \sum_{k=1}^{n-1} j_k}{j_n} \\ &\qquad \leq b\exp\left(\left(\sum_{k=1}^{n} \xi(j_k/|C|) + \xi\left(1 - \sum_{k=1}^{n} j_k/|C|\right)\right)|C|\right) \end{aligned}$$

for some $b > 0$ independent of $|C|$ and $j_1, \dots, j_n$. Since the function ξ is continuous, when τ is small enough one has

$$\begin{aligned}\sum_{k=1}^n \xi(t_k) + \xi\Big(1 - \sum_{k=1}^n t_k\Big) &< \sum_{k=1}^n \xi(c_k) + \xi\Big(1 - \sum_{k=1}^n c_k\Big) + \kappa \\ &= \sum_{k=1}^n \xi(c_k) + \xi\Big(1 - \mu(GZ')\Big) + \kappa \\ &< \sum_{k=1}^n \xi(c_k) + 2\kappa\end{aligned}$$

whenever $t_k \geq 0$, $|t_k - c_k| < \tau$ for all $k = 1, \dots, n$ and $\sum_{k=1}^n t_k \leq 1$. Therefore

$$M_2 \leq b(2\tau d)^n \exp\bigg(\Big(\sum_{k=1}^n \xi(c_k) + 2\kappa\Big)\frac{d}{|G|}\bigg).$$

Let D be a (ρ_2, ε)-separated subset of $\mathrm{Map}_\mu(\rho, G, L, \delta, \sigma)$ of maximal cardinality. Then there is a (ρ_2, ε)-separated subset W of D with $M_1 M_2 |W| \geq |D|$ such that the collection $(Y'_{1,\varphi}, \dots, Y'_{n,\varphi})$ is the same, say $(Y'_1, \dots, Y'_n)$, for every $\varphi \in W$. Note that the elements of W are all equal on $\bigcup_{1 \leq k \leq n} \sigma(G) Y'_k$. By our choice of δ,

$$\begin{aligned}\frac{1}{d}\Big|\bigcup_{1\leq k\leq n}\sigma(G)Y'_k\Big| &= \frac{1}{d}\sum_{1\leq k\leq n}|G/G_k|\cdot|Y'_k| \overset{(13)}{\geq} \sum_{1\leq k\leq n}(c_k - (2+|G|)|G/G_k|\delta) \\ &\geq \sum_{1\leq k\leq n}(c_k - |G|(2+|G|)\delta) = \mu(GZ') - n|G|(2+|G|)\delta > 1 - \varepsilon^2.\end{aligned}$$

It follows that any two elements of W have ρ_2-distance less than ε. Thus $|W| \leq 1$. Therefore

$$\begin{aligned}N_\varepsilon(\mathrm{Map}_\mu(\rho, G, L, \delta, \sigma), \rho_2) = |D| &\leq M_1 M_2 \\ &\leq b\exp\bigg(\Big(\kappa + \sum_{k=1}^n c_k \log|G/G_k|\Big)\frac{d}{|G|}\bigg) \\ &\qquad\qquad \times (2\tau d)^n \exp\bigg(\Big(\sum_{k=1}^n \xi(c_k) + 2\kappa\Big)\frac{d}{|G|}\bigg) \\ &\leq b(2\tau d)^n \exp\bigg(\Big(3\kappa + |G|^{-1}\sum_{z\in GZ'}\xi(\mu(\{z\}))\Big)d\bigg) \\ &\leq b(2\tau d)^n \exp((3\kappa + h_\mu(X,G))d).\end{aligned}$$

It follows that

$$h^\varepsilon_{\Sigma,\mu,2}(\rho, G, L, \delta) \leq 3\kappa + h_\mu(X, G).$$

□

Lemma 6.6. *Let G be a finite group acting on a standard probability space (X,μ) by measure-preserving transformations. Then*

$$h_{\Sigma,\mu}(X,G) \geq h_\mu(X,G).$$

Proof. List the subgroups of G as $H_1,\dots,H_\ell$. For $x \in X$ we write G_x for $\{g \in G : gx = x\}$.

Since G is finite, we can find a measurable subset Y of X such that $|Y \cap Gx| = 1$ for every $x \in X$ [8, Ex. 6.1 and Prop. 6.4]. For every $k = 1,\dots,\ell$ set $Y_k = \{x \in Y : G_x = H_k\}$. We may add or remove a measure zero subset of Y_k without changing either $h_{\Sigma,\mu}(X,G)$ or $h_\mu(X,G)$. Then we may identify Y_k with a closed subset of the interval $[2k, 2k+1]$ in such a way that there exist $2k \leq t_k \leq 2k+1$ and $2k+1 \geq a_{k,1} > a_{k,2} > \dots \geq t_k$ with $Y_k = [2k, t_k] \cup \{a_{k,1}, a_{k,2}, \dots\}$, $\mu(E)$ is the Lebesgue measure of E for every Borel $E \subseteq [2k, t_k]$, and $\mu(a_{k,n}) > 0$ for every n [7, Thm. 17.41]. Here we allow the set $\{a_{k,1}, a_{k,2}, \dots\}$ to be finite or even empty. Reordering $H_1,\dots,H_\ell$ if necessary, we may assume that there is some $0 \leq \ell' \leq \ell$ such that $t_k > 2k$ for all $1 \leq k \leq \ell'$ and $t_k = 2k$ for all $\ell' < k \leq \ell$.

Now we may identify X with the disjoint union $\bigsqcup_{k=1}^{\ell} Y_k \times (G/H_k)$ in a natural way. Equip $\bigsqcup_{k=1}^{\ell} Y_k \times (G/H_k)$ with its natural topology coming from the product topology of $Y_k \times (G/H_k)$. Then G acts continuously on the compact metrizable space X. To simply the notation, we shall also identify Y_k with $Y_k \times \{eH_k\} \subseteq X$, and hence think of Y as a subset of X. Let ρ be a compatible metric on X such that $\rho(x,y) = |x-y|$ for all $x, y \in Y$. By Proposition 3.4 one has $h_{\Sigma,\mu}(X,G) = h_{\Sigma,\mu,\infty}(\rho)$.

We consider first the case $t_k > 2k$ for some $1 \leq k \leq \ell$, i.e. $\ell' \neq 0$. In this case we will show that $h_{\Sigma,\mu,\infty}(\rho) = +\infty$. Let $N \in \mathbb{N}$. Take an $\varepsilon > 0$ such that $\varepsilon < (t_1 - 2)/N$. Let L be a finite subset of $C(X)$ and let $\delta > 0$.

Let η be a strictly positive number satisfying $\eta < \min_{1 \leq k \leq \ell'} \mu(G[2k, t_k])/2$ to be further specified in a moment. By Lemma 4.6, when a map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ is a good enough sofic approximation for G, there exists a subset C of $\{1,\dots,d\}$ such that the map $(s,c) \mapsto \sigma_s(c)$ from $G \times C$ to $\sigma(G)C$ is bijective, $|\sigma(G)C|/d \geq 1-\eta$, and $\sigma_e(c) = c$ and $\sigma_s\sigma_t(c) = \sigma_{st}(c)$ for all $c \in C$ and $s, t \in G$.

For each $k = 1,\dots,\ell$ take an $n_k \in \mathbb{N} \cup \{0\}$ such that the points $a_{k,1},\dots,a_{k,n_k}$ are defined and $\sum_{k=1}^{\ell}\sum_{j>n_k} \mu(Ga_{k,j}) < \eta$. Denote by Λ the set of (k,j) such that either $1 \leq k \leq \ell'$ and $j = 0$ or $1 \leq k \leq \ell$ and $1 \leq j \leq n_k$. Note that $|C| \to +\infty$ as $d \to +\infty$. Thus when d is large enough we can find a partition $\{C_{k,j}\}_{(k,j)\in\Lambda}$ of C with $\sum_{k=1}^{\ell'} ||C_{k,0}|/|C| - \mu(G[2k,t_k])| < \eta$ and $\sum_{k=1}^{\ell}\sum_{j=1}^{n_k} ||C_{k,j}|/|C| - \mu(Ga_{k,j})| < \eta$. Set $x_{k,j} = 2k + j(t_k - 2k)/|C_{k,0}|$ for $1 \leq k \leq \ell'$ and $1 \leq j \leq |C_{k,0}|$. For each $h = (h_k)_{k=1}^{\ell'}$ consisting of a bijection from $C_{k,0}$ to $\{x_{k,j} : 1 \leq j \leq |C_{k,0}|\}$ for each $1 \leq k \leq \ell'$, we take a map $\varphi_h : \{1,\dots,d\} \to X$ sending sc to $s(h_k(c))$ for $1 \leq k \leq \ell'$, $c \in C_{k,0}$, and $s \in G$, and sending sc to $sa_{k,j}$ for $1 \leq k \leq \ell$, $c \in C_{k,j}$, and $s \in G$. It is readily checked that, when η is small enough and d is large enough, every such φ_h belongs to $\mathrm{Map}_\mu(\rho, G, L, \delta, \sigma)$.

Note that $\rho(x_{1,j}, x_{1,j'}) > \varepsilon$ for any $1 \leq j, j' \leq |C_{1,0}|$ with $|j - j'| \geq |C_{1,0}|/N$. When d is large enough, we may require that N divides $|C_{1,0}|$. Denote by Γ the set of permutations of $\{x_{1,j} : 1 \leq j \leq |C_{1,0}|\}$ preserving the subset $\{x_{1,j+k|C_{1,0}|/N} : 0 \leq k < N\}$ for each $1 \leq j \leq |C_{1,0}|/N$. Fix one h as above. For each $\gamma \in \Gamma$, set $h_\gamma = (\gamma \circ h_1, h_2, \dots, h_{\ell'})$. Then the set $\{\varphi_{h_\gamma} : \gamma \in \Gamma\}$ is $(\rho_\infty, \varepsilon)$-separated. Therefore

$$N_\varepsilon(\mathrm{Map}_\mu(\rho, G, L, \delta, \sigma), \rho_\infty) \geq |\Gamma| = (N!)^{|C_{1,0}|/N}.$$

It follows that

$$\begin{aligned}\liminf_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}_\mu(\rho, G, L, \delta, \sigma_i), \rho_\infty) &\geq \frac{(\mu(G[2,t_1]) - \eta)(1-\eta)}{|G|N} \log(N!) \\ &\geq \frac{\mu(G[2,t_1])(1 - \mu(G[2,t_1])/2)}{2|G|N} \log(N!).\end{aligned}$$

Since N can be taken to be arbitrarily large, we conclude that $h_{\Sigma,\mu,\infty}(\rho) = \sup_{\varepsilon>0} h^\varepsilon_{\Sigma,\mu,\infty}(\rho) = +\infty$.

In the case that X is atomic, i.e. $\ell' = 0$, one can see how to proceed by rewinding through the proof of Lemma 6.5. We will simply outline the argument and leave the details to the reader. The goal is to construct sufficiently many approximately equivariant maps from a given sofic approximation space into X in order to get the desired lower bound for the sofic measure entropy. Such a map is constructed as follows. Fixing a partition of X into orbits, if the action is free then we can pair off each base point from a finite collection of orbits with sets of base points of the decomposition of a fixed sofic approximation as given by Lemma 4.5, subject to the requirement that the measures approximately match up. If the action is not free then the components of the sofic approximation decomposition must be further partitioned as necessary by means of cosets in order to enable the pairing off with base points in X having nontrivial isotropy subgroup. The choices involved in this pairing procedure are controlled, up to some error, by the product of the quantities M_1 and M_2 as in the proof of Lemma 6.5. From this we obtain the desired lower bound. $\square$

Theorem 6.7. *Let G be an amenable countable discrete group acting on a standard probability space (X, μ) by measure-preserving transformations. Let Σ be a sofic approximation sequence for G. Then*

$$h_{\Sigma,\mu}(X, G) = h_\mu(X, G).$$

Proof. By Lemmas 6.5 and 6.6, we may assume that G is infinite. Since (X, μ) is standard, up to measure conjugacy we may assume that X is a compact metrizable space on which G acts continuously and μ is a G-invariant Borel probability measure on X. Then $h_{\Sigma,\mu}(X, G) \geq h_\mu(X, G)$ by Lemma 6.4 and Proposition 3.4, while the reverse inequality follows from Lemma 6.3 and Proposition 3.4. $\square$

We remark that, when the action is ergodic, Theorem 6.7 follows from Theorem 5.3, the Jewett-Krieger theorem for actions of amenable groups [15], and the variational principle [12, Thm. 5.2.7][9, Thm. 6.1]. Note however that much of the complication of the proof of Theorem 6.7 is due to the fact that we are not assuming the action to be ergodic.

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