STRONG MORITA EQUIVALENCE OF HIGHER-DIMENSIONAL NONCOMMUTATIVE TORI

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Abstract. We show that matrices in the same orbit of the $SO(n, n|\mathbb{Z})$ action on the space of $n \times n$ skew-symmetric matrices give strongly Morita equivalent noncommutative tori, both at the C^*-algebra level and at the smooth algebra level. This proves a conjecture of Rieffel and Schwarz.

1. Introduction

Let $n \geq 2$ and $\mathcal{T}_n$ be the space of $n \times n$ real skew-symmetric matrices. For each $\theta \in \mathcal{T}_n$ the corresponding n-dimensional noncommutative torus A_θ is defined as the universal C^*-algebra generated by unitaries $U_1, \cdots, U_n$ satisfying the relation

$$U_k U_j = e(\theta_{kj}) U_j U_k,$$

where $e(t) = e^{2\pi i t}$. Noncommutative tori are one of the canonical examples in noncommutative differential geometry [12, 2].

One may also consider the smooth version A_θ^∞ of a noncommutative torus, which is the algebra of formal series

$$\sum c_{j_1, \cdots, j_n} U_1^{j_1} \cdots U_n^{j_n},$$

where the coefficient function $\mathbb{Z}^n \ni (j_1, \cdots, j_n) \mapsto c_{j_1, \cdots, j_n}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ i.e. the space of $\mathbb{C}$-valued functions on $\mathbb{Z}^n$ which vanish at infinity more rapidly than any polynomial grows. This is the space of smooth elements of A_θ for the canonical action of $\mathbb{T}^n$ on A_θ.

The notion of (strong) Morita equivalence of C^*-algebras was introduced by Rieffel [8, 10]. Strongly Morita equivalent C^*-algebras share a lot of important properties such as equivalent categories of modules, isomorphic K-groups, etc., and hence are usually thought to have the same geometry. In [14] Schwarz also introduced the notion of complete Morita equivalence of smooth noncommutative tori (see Section 2 below), which is stronger than strong Morita equivalence and has important application in M(atrix) theory [14, 4].

A natural question is to classify noncommutative tori up to strong Morita equivalence. Such results have important application to physics [3, 14]. For $n = 2$ this was done by Rieffel [9]. In this case there is a (densely defined) action of the group $GL(2, \mathbb{Z})$ on $\mathcal{T}_2$, and two matrices in $\mathcal{T}_2$ give strongly Morita equivalent noncommutative tori if and only if they are in the same orbit of this action. The higher dimensional case is much more complicated. In [13] Rieffel and Schwarz found a (densely defined) action of $SO(n, n|\mathbb{Z})$ on $\mathcal{T}_n$ generalizing the above $GL(2, \mathbb{Z})$-action. Here $O(n, n|\mathbb{R})$ is the group of linear transformations of the space $\mathbb{R}^{2n}$ preserving

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the quadratic form $x_1x_{n+1}+x_2x_{n+2}+\cdots+x_nx_{2n}$, and $SO(n,n|\mathbb{Z})$ is the subgroup of $O(n,n|\mathbb{R})$ consisting of matrices with integer entries and determinant 1.

Following [13] we write the elements of $O(n,n|\mathbb{R})$ in 2×2 block form:

$$g=\begin{pmatrix} A & B \\ C & D\end{pmatrix}.$$

Here A,B,C,D are $n\times n$ matrices satisfying

$$A^tC+C^tA=0=B^tD+D^tB,\quad A^tD+C^tB=I. \tag{1}$$

The action of $SO(n,n|\mathbb{Z})$ is then defined as

$$g\theta=(A\theta+B)(C\theta+D)^{-1}, \tag{2}$$

whenever $C\theta+D$ is invertible. For each $g\in SO(n,n|\mathbb{Z})$ this action is defined on a dense open subset of $\mathcal{T}_n$.

Rieffel and Schwarz conjectured that if two matrices in $\mathcal{T}_n$ are in the same orbit of this action then they give strongly Morita equivalent noncommutative tori, both at the C^*-algebra level and at the smooth algebra level. They proved it for matrices restricted to a certain subset of $\mathcal{T}_n$ of second category. They also showed that the converse of their conjecture at the C^*-algebra level fails for $n=3$ [13, page 297], in contrast to the case $n=2$, using the classification results of G. A. Elliott and Q. Lin [6].

The main goal of this paper is to prove their conjecture:

Theorem 1.1. *For any $\theta\in\mathcal{T}_n$ and $g\in SO(n,n|\mathbb{Z})$, if $g\theta$ is defined then A_θ and $A_{g\theta}$ are strongly Morita equivalent. Also A^∞_θ and $A^\infty_{g\theta}$ are completely Morita equivalent.*

Schwarz has proved that if two matrices in $\mathcal{T}_n$ give completely Morita equivalent smooth noncommutative tori then they are in the same orbit of the $SO(n,n|\mathbb{Z})$-action [14, Section 5]. Thus we get

Theorem 1.2. *Two matrices in $\mathcal{T}_n$ give completely Morita equivalent smooth noncommutative tori if and only if they are in the same orbit of the $SO(n,n|\mathbb{Z})$-action.*

We have learned recently that using classification theory N. C. Phillips has been able to show that two simple noncommutative tori A_θ and $A_{\theta'}$ are strongly Morita equivalent if and only if their ordered K_0-groups are isomorphic [7, Remark 7.9]. It would be interesting to see directly from the matrices why the ordered K_0-groups of A_θ and $A_{g\theta}$ are isomorphic.

This paper is organized as follows. Our proof of Theorem 1.1 is constructive, and we shall use the Heisenberg equivalence modules constructed by Rieffel in [11]. So we recall briefly Rieffel's construction first in Section 2. In order to apply Rieffel's construction we need to reduce an arbitrary matrix in $\mathcal{T}_n$ to one satisfying certain nice properties. This is done in Section 3. We prove Theorem 1.1 in Section 4.

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2. Heisenberg Equivalence Modules

In this section we recall Schwarz's definition of complete Morita equivalence and Rieffel's construction of Heisenberg equivalence modules for noncommutative tori.

Let $L = \mathbb{R}^n$. We shall think of $\mathbb{Z}^n$ as the standard lattice in L^*, and θ as in $\wedge^2 L$. One may also describe A_θ as the universal C^*-algebra generated by unitaries $\{U_x\}_{x\in\mathbb{Z}^n}$ satisfying the relation

$$U_x U_y = \sigma_\theta(x, y) U_{x+y}, \tag{3}$$

where we write x, y as column vectors, and $\sigma_\theta(x, y) = e((x \cdot \theta y)/2)$. Under this description the smooth algebra A_θ^∞ becomes $\mathcal{S}(\mathbb{Z}^n, \sigma_\theta)$, the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ equipped with the convolution induced by (3). There is a canonical action of the Lie algebra L as derivations on A_θ^∞, which is induced by the canonical action of $\mathbb{T}^n$ on A_θ and is given explicitly by

$$\delta_X(U_x) = 2\pi i \langle X, x\rangle U_x$$

for all $X \in L$ and $x \in \mathbb{Z}^n$, where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between L and L^*.

Given a right A_θ^∞-module E, a *connection* on E is a linear map $\nabla : L \to Hom_{\mathbb{C}}(E)$ satisfying the Leibniz rule:

$$\nabla_X(fU_x) = (\nabla_X f)U_x + f \cdot \delta_X(U_x)$$

for all $X \in L$, $f \in E$ and $x \in \mathbb{Z}^n$. For each $X \in L$ the connection ∇ induces a derivation $\hat{\delta}_X$ on $End_{A_\theta^\infty}(E)$ by

$$(\hat{\delta}_X a)(f) = \nabla_X(af) - a \cdot \nabla_X f$$

for all $a \in End_{A_\theta^\infty}(E)$ and $f \in E$. If ∇ has *constant curvature*, i.e. there is skew-symmetric bilinear map $\Omega : L \times L \to \mathbb{C}$ such that $[\nabla_X, \nabla_Y] = \Omega(X, Y) \cdot 1$ for all $X, Y \in L$, then $X \mapsto \hat{\delta}_X$ is a Lie algebra homomorphism from L to the derivation space $Der(End_{A_\theta^\infty}(E))$ of $End_{A_\theta^\infty}(E)$. When E is equipped with an A_θ^∞-valued inner product, we shall consider only *Hermitian* connections, i.e. $\delta_X\langle f, g\rangle = \langle\nabla_X f, g\rangle + \langle f, \nabla_X g\rangle$ for $X \in L$ and $f, g \in A_\theta^\infty$.

We refer to [10] for the definition and standard facts about strong Morita equivalence of C^*-algebras. Let E be a strong Morita equivalence $A_{\theta'}^\infty$-A_θ^∞-bimodule. For clarity we let L_θ and $L_{\theta'}$ denote the space L for θ and θ' respectively. We say that E is a *complete Morita equivalence* $A_{\theta'}^\infty$-A_θ^∞*-bimodule* [14, page 729] if there is a constant-curvature connection ∇ on $E_{A_\theta^\infty}$ and a linear isomorphism $\phi : L_\theta \to L_{\theta'}$ such that the induced Lie algebra homomorphism $L_\theta \to Der(End_{A_\theta^\infty}(E)) = Der(A_{\theta'}^\infty)$ coincides with the composition homomorphism $L_\theta \xrightarrow{\phi} L_{\theta'} \to Der(A_{\theta'}^\infty)$. Intuitively, this means that the equivalence bimodule E is "smooth", i.e. it transfers the tangent spaces (L_θ and $L_{\theta'}$) of the noncommutative differentiable manifolds A_θ^∞ and $A_{\theta'}^\infty$ back and forth.

Next we recall Rieffel's construction of Heisenberg equivalence modules in [11, Sections 2-4]. Let M be a locally compact abelian group, let $\hat{M}$ be its dual group, and let $G = M \times \hat{M}$. There is a canonical Heisenberg cocycle on G defined by

$$\beta((m, s), (l, t)) = \langle m, t\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between M and $\hat{M}$. There is also a skew bicharacter, ρ, on G defined by

$$\rho(x,y) = \beta(x,y)\bar{\beta}(y,x). \tag{4}$$

We'll concentrate on the case $M = \mathbb{R}^p \times \mathbb{Z}^q \times W$, where $p, q \in \mathbb{Z}_{\geq 0}$ with $2p+q = n$ and W is a finite abelian group. Say $W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ for some $n_1, \cdots, n_k \in \mathbb{N}$. We shall write G as $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. Let

$$P_1 = diag(\frac{1}{n_1}, \cdots, \frac{1}{n_k}), \qquad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}, \tag{5}$$

$$J_2 = \begin{pmatrix} 0 & P_1 \\ -P_1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

Then J is a square matrix of size $n+q+2k$, and we shall think of it as a 2-form on $H^* := \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q \times \mathbb{R}^{*q} \times \mathbb{R}^k \times \mathbb{R}^{*k}$. Let J' be the matrix obtained by replacing negative entries of J by 0. Then $J = J' - (J')^t$. For any $x, y \in G$ we have

$$\beta(x,y) = e(x \cdot J'y) \quad \text{and} \quad \rho(x,y) = e(x \cdot Jy),$$

where we use the natural covering map $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$ to write x and y as column vectors in $\mathbb{R}^{n+q+2k}$ (notice that though $J'y$ depends on the choice of the representative of y in $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$, the values $e(x \cdot J'y)$ and $e(x \cdot Jy)$ do not depend on such choice).

Definition 2.1. [11, Definition 4.1] By an *embedding map* for $\theta \in \mathcal{T}_n$ we mean a linear map T from L^* to H^* such that:

(1) $T(\mathbb{Z}^n) \subseteq \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$. Then we can think of $T(\mathbb{Z}^n)$ as in G via composing $T|_{\mathbb{Z}^n}$ with the natural covering map $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$.

(2) $T(\mathbb{Z}^n)$ is a lattice in G.

(3) The form J on H^* is pulled back by T to the form θ on L^*, i.e. $T^t J T = \theta$.

The condition (2) above is equivalent to

(2') The map $\tilde{T} := \gamma \circ T : L^* \to \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q$ is invertible, where γ is the projection of H^* onto $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q$.

The bimodule Rieffel constructed is the Schwartz space $\mathcal{S}(M)$, i.e. the space of smooth functions on M which, together with all their derivatives, vanish at infinity more rapidly than any polynomial grows.

Proposition 2.2. [11, Theorem 2.15, Corollary 3.8] *Let $\theta, \theta' \in \mathcal{T}_n$, and let T, S be embedding maps of L^* into H^* for θ and $-\theta'$ respectively such that $S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp$, where $(T(\mathbb{Z}^n))^\perp = \{z \in G : \rho(z,y) = 1 \text{ for all } y \in T(\mathbb{Z}^n)\}$. Let T' and T'' be the composition maps $\mathbb{Z}^n \xrightarrow{T} G \to M$ and $\mathbb{Z}^n \xrightarrow{T} G \to \hat{M}$ respectively. Define S' and S'' similarly. Fix a Haar measure on M. Then $\mathcal{S}(M)$ is a strong Morita equivalence $A^\infty_{\theta'}$-A^∞_θ-bimodule with the module structure and inner products defined by:*

$$\begin{aligned}
(fU_x)(m) &= e(-T(x) \cdot J'T(x)/2) \langle m, T''(x) \rangle f(m - T'(x)), \\
\langle f, g \rangle_{\mathcal{S}(\mathbb{Z}^n, \sigma_\theta)}(x) &= e(-T(x) \cdot J'T(x)/2) \int_G \langle m, -T''(x) \rangle g(m + T'(x)) \bar{f}(m)\, dm, \\
(V_x f)(m) &= e(-S(x) \cdot J'S(x)/2) \langle m, -S''(x) \rangle f(m + S'(x)), \\
{}_{\mathcal{S}(\mathbb{Z}^n, \sigma_{\theta'})}\langle f, g \rangle(x) &= K \cdot e(S(x) \cdot J'S(x)/2) \int_G \langle m, S''(x) \rangle f(m) \bar{g}(m + S'(x))\, dm,
\end{aligned}$$

where K is a positive constant and for clarity V_x denotes the unitary in $\mathcal{S}(\mathbb{Z}^n, \sigma_{\theta'})$. Moreover, there is a linear map $Q : \mathbb{R}^{*p} \times \mathbb{R}^p \times \mathbb{R}^{*q} \to Hom_{\mathbb{C}}(\mathcal{S}(M))$ such that $\nabla_X = Q_{(\tilde{T}^{-1})^*(X)}$ and $\nabla'_X = Q_{(\tilde{S}^{-1})^*(-X)}$ are connections with respect to $\mathcal{S}(M)_{A^\infty_\theta}$ and ${}_{A^\infty_{\theta'}}\mathcal{S}(M)$ respectively. The connection ∇ has constant curvature

$$\Omega = 2\pi i \tilde{T}^{-1}(\sum_{j=1}^{p} \bar{e}_j \wedge e_j),$$

where $e_1, \cdots, e_p$ are the standard basis of $\mathbb{R}^p$ and $\bar{e}_1, \cdots, \bar{e}_p$ are the dual basis of $\mathbb{R}^{*p}$. Thus $\mathcal{S}(M)$ is a complete Morita equivalence $A^\infty_{\theta'}$-A^∞_θ-bimodule. When completed with the norm $\| f \| := \| \langle f, f \rangle_{A^\infty_\theta} \|^{\frac{1}{2}} = \| {}_{A^\infty_{\theta'}}\langle f, f \rangle \|^{\frac{1}{2}}$, $\mathcal{S}(M)$ becomes a strong Morita equivalence $A_{\theta'}$-A_θ-bimodule.

Remark 2.3. (1) The definition of embedding maps in Definition 2.1 differs from that in [11, Definition 4.1] by a sign of θ. This is because Rieffel's A_θ is our $A_{-\theta}$ (see the discussion at the end of page 285 of [11]).

(2) In [11, Section 4] the definition of embedding maps and the part of Proposition 2.2 above concerning connections and curvatures are only given for the case $W = 0$ [11, Definition 4.1] [11, pages 290-291]. The general case was discussed there in terms of tensor products with finite dimensional representations [11, Section 5]. For our purpose it's better to deal with $\mathbb{R}^p \times \mathbb{Z}^q \times W$ directly. The proofs in [11, pages 290-291] for the case $W = 0$ are easily checked to hold for the general case.

3. Decomposition of Matrices

In Proposition 4.1 we shall use the construction in [1] to find the appropriate finite abelian group W. To this goal we need the matrix $g \in SO(n, n|\mathbb{Z})$ to be of the special form in Lemma 3.3 below. We shall prove in Lemma 3.5 that every g can be reduced to such a special one.

Lemma 3.1. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Then DC^t is skew-symmetric.*

Proof. Since $g \in O(n, n|\mathbb{R})$ we have that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Hence

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}.$$

Since $O(n, n|\mathbb{R})$ is a group we have that $g^{-1} \in O(n, n|\mathbb{R})$. By (1) the matrix $(D^t)^t C^t = DC^t$ is skew-symmetric. $\square$

Using Lemma 3.1 simple calculations yield:

Lemma 3.2. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Let $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible. Then $(C\theta + D)^{-1}C$ is skew-symmetric.*

Lemma 3.3. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n|\mathbb{Z})$, and let $p \in \mathbb{Z}_{\geq 0}$. Then the following are equivalent:*

(i) there is some $\theta \in \mathcal{T}_n$ *such that* $(C\theta + D)^{-1}C$ *is of the form* $\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$ *for some* $F_{11} \in GL(2p|\mathbb{R})$*;*

(ii) there exists a $Z \in \mathcal{T}_{2p}$ *such that*

$$C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} -C_{11}Z & D_{12} \\ -C_{21}Z & D_{22} \end{pmatrix},$$

where $C_{11} \in M_{2p}(\mathbb{Z})$*.*

In this event, the matrix $\begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}$ *is invertible. The matrix* Z *is unique, and its entries are all rational numbers. Also for any* $\theta' \in \mathcal{T}_n$ *in the block form* $\begin{pmatrix} \theta'_{11} & \theta'_{12} \\ \theta'_{21} & \theta'_{22} \end{pmatrix}$*, where* θ'_{11} *has size* $2p \times 2p$*, the matrix* $C\theta' + D$ *is invertible if and only if* $\theta'_{11} - Z$ *is invertible. In this case*

$$(C\theta' + D)^{-1}C = \begin{pmatrix} (\theta'_{11} - Z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Proof. (i)⇒(ii). From the assumption we have $C\begin{pmatrix} I_{2p} & 0 \\ 0 & 0 \end{pmatrix} = C$. Thus C has the desired form in (ii). Notice that

$$\begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} = C = (C\theta + D)\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (C_{11}\theta_{11} + D_{11})F_{11} & 0 \\ (C_{21}\theta_{11} + D_{21})F_{11} & 0 \end{pmatrix},$$

where we are writing both θ and D in block forms. Thus $C_{j1} = (C_{j1}\theta_{11} + D_{j1})F_{11}$ for $j = 1, 2$. Let $Z = \theta_{11} - (F_{11})^{-1}$. Then $D_{j1} = -C_{j1}Z$. By Lemma 3.2 the matrix F_{11} is skew-symmetric. Then so is Z.

(ii)⇒(i). For any $\theta' \in \mathcal{T}_n$ we have

$$C\theta' + D = \begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}\begin{pmatrix} \theta'_{11} - Z & \theta'_{12} \\ 0 & I \end{pmatrix}.$$

Take $\theta \in \mathcal{T}_n$ such that $C\theta + D$ is invertible. Then $\begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}$ is invertible. Therefore $C\theta' + D$ is invertible if and only if $\theta'_{11} - Z$ is invertible. In this case simple computations yield (6). In particular $(C\theta + D)^{-1}C$ has the form described in (i). By varying θ slightly we may assume that θ is rational, *i.e.* the entries of θ are all rational numbers. Then so are F_{11} and $Z = \theta_{11} - (F_{11})^{-1}$. □

Notation 3.4. For any $R \in GL(n|\mathbb{Z})$ we denote by $\rho(R)$ the matrix $\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$. For any $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$ we denote by $\mu(N)$ the matrix $\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$.

Notice that the noncommutative tori corresponding to the matrices $\rho(R)\theta = R\theta R^t$ and $\mu(N)\theta = \theta + N$ are both isomorphic to A_θ.

Lemma 3.5. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *in* $SO(n, n|\mathbb{Z})$*. Then there exists an* $R \in GL(n|\mathbb{Z})$ *such that* $g \cdot \rho(R)$ *satisfies the condition (1) in Lemma 3.3 for some* $p \in \mathbb{Z}_{\geq 0}$*.*

Proof. Let $V = \{X \in \mathbb{R}^n | CX = 0\}$, and let $K = V \cap \mathbb{Z}^n$. Since the entries of C are all integers, K spans V. By the elementary divisors theorem [5, page 153, Theorem III.7.8] we can find a basis $\beta_1, \cdots, \beta_n$ of $\mathbb{Z}^n$, some integer $1 \leq k \leq n$ and positive integers $c_k, \cdots, c_n$ such that K is generated by $c_k\beta_k, \cdots, c_n\beta_n$. Then $V = \mathrm{span}(\beta_k, \cdots, \beta_n)$. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{Z}^n$. Then $(\beta_1, \cdots, \beta_n) = (e_1, \cdots, e_n)R$ for some $R \in GL(n|\mathbb{Z})$. Let

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rho(R) \in SO(n, n|\mathbb{Z}).$$

Choose $\theta \in \mathcal{T}_n$ such that $C\theta + D$ is invertible. Let $\theta' = R^{-1}\theta(R^{-1})^t \in \mathcal{T}_n$. Now we need

Lemma 3.6. *$(C'\theta' + D')^{-1}C'$ is of the form $\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$ for some $F_{11} \in GL(k-1|\mathbb{R})$.*

Proof. In view of Lemma 3.2 this is clearly equivalent to saying that the vectors $X = (x_1, \cdots, x_n)^t$ in $\mathbb{R}^n$ satisfying $(C'\theta' + D')^{-1}C'X = 0$ are exactly those with $x_1 = \cdots = x_{k-1} = 0$. Notice that $(C'\theta' + D')^{-1}C' = R^t(C\theta + D)^{-1}CR$. Hence $(C'\theta' + D')^{-1}C'X = 0$ if and only if $CRX = 0$, if and only if $RX \in V$, if and only if $(\beta_1, \cdots, \beta_n)X \in V$, if and only if $x_1 = \cdots = x_{k-1} = 0$. □

Back to the proof of Lemma 3.5. By Lemma 3.2 the matrix F_{11} is skew-symmetric. Since $F_{11} \in GL(k-1|\mathbb{Z})$ we see that $k-1$ is even. This completes the proof of Lemma 3.5. □

4. Strong Morita Equivalence

In this section we prove Theorem 1.1. We shall employ the notation in Section 2 and Lemma 3.3. In view of Proposition 2.2 the key is to find embedding maps. This is established in the following

Proposition 4.1. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$ satisfying the conditions (1) and (2) in Lemma 3.3 for some $p \in \mathbb{Z}_{\geq 0}$. Then there exist an $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$, an $R \in GL(n|\mathbb{Z})$, a $g' \in SO(n, n|\mathbb{Z})$ and a finite abelian group W such that $g = \nu(N)\rho(R)g'$ and for any $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible there are embedding maps $T, S : L^* \to H^*$ for θ and $-g'\theta$ respectively satisfying $S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp$ (see Definition 2.1(1) and Proposition 2.2 for the meaning of $(T(\mathbb{Z}^n))^\perp$).*

Proof. Let Z be as in Lemma 3.3 for g. Then Z is rational, and hence there is some $m \in \mathbb{N}$ such that mZ is integral. Thinking of mZ as a bilinear alternating form on $\mathbb{Z}^n$, by [5, page 598, Exercise XV.17] we can find an $R \in GL(2p|\mathbb{Z})$, some integer $1 \leq k \leq p$ and integers $h_1, \cdots, h_k$ such that

$$mZ = R^t \begin{pmatrix} 0 & P & 0 \\ -P & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R,$$

where $P = diag(h_1, \cdots, h_k)$. Let $m_j/n_j = h_j/m$ with $(m_j, n_j) = 1$ and $n_j > 0$ for each $1 \leq j \leq k$. Set $W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

Let $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible. We are ready to construct an embedding map for θ now. Our method is similar to that in the proof of the proposition in [13]. But our situation is more complicated since we have to deal with the torsion part W. Write θ in block form as in Lemma 3.3. By Lemmas 3.3 and 3.2 the matrix $\theta_{11} - Z$ is invertible and skew-symmetric. So we can find a $T_{11} \in GL(2p|\mathbb{R})$ such that $T_{11}^t J_0 T_{11} = \theta_{11} - Z$, where J_0 is defined in (5). Let $T_{31} = \theta_{21}$, and let T_{32} be any $q \times q$ matrix such that $T_{32} - T_{32}^t = \theta_{22}$, where $q = n - 2p$. Let $P_2 = diag(m_1, \cdots, m_k)$, and let

$$T_1 = \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix}, T_2 = \begin{pmatrix} P_2 & 0 & 0 \\ 0 & I_k & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & I_q \end{pmatrix}, T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}.$$

Then T_1, T_2 and T have sizes $(n+q) \times n$, $2k \times n$ and $(n+q+2k) \times n$ respectively. Simple calculations yield $T^t J T = \theta$, where J is defined in (5). Notice that as a linear map from $L^* = \mathbb{R}^{*n}$ to $H^* = \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q \times \mathbb{R}^{*q} \times \mathbb{R}^k \times \mathbb{R}^{*k}$, T carries the lattice $\mathbb{Z}^n = \mathbb{Z}^{2p} \times \mathbb{Z}^q$ into $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$. Also observe that $\tilde{T}$ (see Definition 2.1(2')) is given by the invertible matrix $\begin{pmatrix} T_{11} & 0 \\ 0 & I_q \end{pmatrix}$. Thus the conditions in Definition 2.1 are satisfied and hence T is an embedding map for θ.

Let $\mathcal{D} = T(\mathbb{Z}^n)$. By Definition 2.1(1) we may think of $\mathcal{D}$ as in $G = \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. We need to find some embedding map of $\mathbb{Z}^n$ into G with image being exactly $\mathcal{D}^\perp = \{z \in G : \rho(z, y) = 1 \text{ for all } y \in \mathcal{D}\}$, where ρ is defined in (4).

For any $x \in G$, it is in $\mathcal{D}^\perp$ exactly if $x \cdot JTz \in \mathbb{Z}$ for all $z \in \mathbb{Z}^n$, exactly if $T^t Jx \in \mathbb{Z}^n$. Let $T_3 = \begin{pmatrix} 0 \\ -I_q \end{pmatrix}$ be an $(n+q) \times q$ matrix. Let $T_4 = diag(n_1, \cdots, n_k, n_1, \ldots, n_k)$. Set

$$\bar{T} = \begin{pmatrix} T_1 & T_3 & 0 \\ T_2 & 0 & T_4 \end{pmatrix},$$

a square matrix of size $n + q + 2k$. It is easy to check that $T^t Jx \in \mathbb{Z}^n$ exactly if $\bar{T}^t Jx \in \mathbb{Z}^{n+q+2k}$. Also it is easy to see that $\bar{T}$ is invertible. Thus

$$\mathcal{D}^\perp = (\bar{T}^t J)^{-1}(\mathbb{Z}^{n+q+2k}).$$

Recall the matrices J_0 and J_1 defined in (5). Straight-forward calculations show that

$$(\bar{T}^t J)^{-1} = \begin{pmatrix} -J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} & J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} \begin{pmatrix} T_2^t \\ 0 \end{pmatrix} T_4^{-1} \\ 0 & \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \end{pmatrix},$$

and

$$J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} = \begin{pmatrix} J_0 (T_{11}^t)^{-1} & 0 & -J_0 (T_{11}^t)^{-1} T_{31}^t \\ 0 & 0 & I_q \\ 0 & -I_q & T_{32}^t \end{pmatrix}.$$

Thus

$$(\bar{T}^t J)^{-1}(0^{2p} \times \mathbb{Z}^q \times 0^q \times 0^{2k}) = 0^{2p} \times 0^q \times \mathbb{Z}^q \times 0^{2k},$$

which is 0 in G. So $(\bar{T}^t J)^{-1}(\mathbb{Z}^{n+q+2k}) = (\bar{T}^t J)^{-1}(\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k})$. Let $\triangle$ be the set of all vectors $y = \begin{pmatrix} y_1 \\ 0 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^q \times \mathbb{Z}^{2k}$ satisfying $y_3 \in n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \times$

$n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z}$ and $\begin{pmatrix} y_1 \\ 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} T_2^t \\ 0 \end{pmatrix} T_4^{-1} y_3$. Then $(\bar{T}^t J)^{-1}(\triangle) = 0$ in G. For each $1 \leq j \leq k$ since $(m_j, n_j) = 1$ we can find $c_j, d_j \in \mathbb{Z}$ such that $c_j m_j + d_j n_j = 1$. Now we need

Lemma 4.2. *Let φ_1 be the embedding $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k}$ sending $(x_1, \cdots, x_n)^t$ to*

$$(-d_1 x_1, \cdots, -d_k x_k, 0, \cdots, 0, x_{2k+1}, \cdots, x_{2p})^t \times 0^q \times \\ (x_{2p+1}, \cdots, x_n, c_1 x_1, \cdots, c_k x_k, x_{k+1}, \ldots, x_{2k})^t.$$

Let φ be the composition of φ_1 and $\begin{pmatrix} R^t & 0 \\ 0 & I_{2q+2k} \end{pmatrix} : \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} \to \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k}$. Then $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n)$.

Proof. Let $y = \begin{pmatrix} y_1 \\ 0 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^q \times \mathbb{Z}^{2k}$ satisfying $y_3 \in n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \times n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z}$. Say $y_3 = (n_1 z_1 \cdots, n_k z_k, n_1 z_{k+1}, \cdots, n_k z_{2k})^t$. Then it is easy to see that $y \in \triangle$ exactly if

$$(R^t)^{-1} y_1 \quad = \quad (m_1 z_1, \cdots, m_k z_k, z_{k+1}, \cdots, z_{2k}, 0, \cdots, 0)^t \quad \text{and} \quad y_2 = 0.$$

It is clear from this that $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n)$. □

Back to the proof of Proposition 4.1. Putting $\varphi : \mathbb{Z}^n \to (\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k})$ and $(\bar{T}^t J)^{-1} : \mathbb{Z}^{n+q+2k} \to H^*$ together, we get a map $S := (\bar{T}^t J)^{-1} \circ \varphi : \mathbb{Z}^n \to H^*$ with $S(\mathbb{Z}^n) = \mathcal{D}^{\perp}$. Let

$$Q_1 = diag(d_1, \cdots, d_k), \quad Q_2 = diag(c_1, \cdots, c_k).$$

A routine calculation shows that

$$S = \begin{pmatrix} W_1 & W_2 \\ \begin{pmatrix} 0 & -I_k & 0 \\ Q_2 & 0 & 0 \end{pmatrix} & 0 \end{pmatrix},$$

where W_1 and W_2 have sizes $(n+q) \times 2p$ and $(n+q) \times q$ respectively:

$$W_1 = \begin{pmatrix} J_0 (T_{11}^t)^{-1} R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -J_0 (T_{11}^t)^{-1} T_{31}^t \\ I_q \\ T_{32}^t \end{pmatrix}.$$

Clearly S satisfies Definition 2.1(1)(2'). Then S is an embedding map for

$$-\theta' = S^t J S = -\begin{pmatrix} \theta'_{11} & \theta'_{12} \\ \theta'_{21} & \theta'_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\theta'_{11} &= \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} RF_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -Q_2P_1 & 0 \\ Q_2P_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta'_{12} &= \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} RF_{11}\theta_{12}, \\
\theta'_{21} &= -\theta_{21}F_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix}, \\
\theta'_{22} &= -\theta_{21}F_{11}\theta_{12} + \theta_{22}.
\end{aligned}$$

Proposition 2.2 tells us that $\mathcal{S}(M)$ is a complete Morita equivalence $A^\infty_{\theta'}$-A^∞_{θ}-bimodule. Clearly the dual $\phi^* : L^*_{\theta'} \to L^*_{\theta}$ of $\phi : L_\theta \to L_{\theta'}$ is just $-\tilde{T}^{-1} \circ \tilde{S}$. A routine calculation shows that ϕ^* is given by the matrix

$$\mathscr{A} = -\begin{pmatrix} -F_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} & -F_{11}\theta_{12} \\ 0 & I_q \end{pmatrix}.$$

It is also easy to see that the matrix form of the normalized curvature $\frac{1}{2\pi i}\Omega$ is

$$\Phi = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Now that we have the matrices $\theta, \theta', \mathscr{A}$, and Ω, Schwarz [14, page 733] has shown how to find $g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in SO(n, n|\mathbb{Z})$ such that $\theta' = g'\theta$. Actually we have the formulas:

$$\begin{aligned}
C' = \mathscr{A}^{-1}\Phi, \quad D' = \mathscr{A}^{-1} - C'\theta, \\
A' = \mathscr{A}^t + \theta'C', \quad B' = \theta'\mathscr{A}^{-1} - A'\theta.
\end{aligned} \tag{7}$$

Our formulas (7) are exactly the equation (53) of [14], in slightly different form. Straight-forward calculations yield

$$C' = \begin{pmatrix} \begin{pmatrix} T_4 & 0 \\ 0 & -I_{2p-2k} \end{pmatrix} (R^t)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D' = \begin{pmatrix} \begin{pmatrix} 0 & -P_2 & 0 \\ P_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R & 0 \\ 0 & I_q \end{pmatrix},$$

$$A' = \begin{pmatrix} \begin{pmatrix} 0 & -Q_2 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (R^t)^{-1} & 0 \\ 0 & I_q \end{pmatrix}, \quad B' = \begin{pmatrix} \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} := g(g')^{-1}.$$

Then $\tilde{g} \in SO(n,n|\mathbb{Z})$. A routine calculation shows that $\tilde{C} = 0$. By (1) we have $I = \tilde{A}^t\tilde{D} + \tilde{C}^t\tilde{B} = \tilde{A}^t\tilde{D}$. Then $\tilde{A}$ is invertible. Recall the matrix $\rho(\tilde{A})$ in Notation 3.4. We get

$$\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix} = \begin{pmatrix} I & \tilde{B}\tilde{A}^t \\ 0 & I \end{pmatrix} \rho(\tilde{A}).$$

Hence $\begin{pmatrix} I & \tilde{B}\tilde{A}^t \\ 0 & I \end{pmatrix} = \tilde{g}(\rho(\tilde{A}))^{-1} \in SO(n,n|\mathbb{Z})$. By (1) the matrix $I^t(\tilde{B}\tilde{A}^t) = \tilde{B}\tilde{A}^t$ is skew-symmetric. So we get

$$\tilde{g} = \nu(\tilde{B}\tilde{A}^t)\rho(\tilde{A}), \quad g = \tilde{g}g' = \nu(\tilde{B}\tilde{A}^t)\rho(\tilde{A})g'.$$

Notice that $g', \tilde{B}\tilde{A}^t$ and $\tilde{A}$ do not depend on θ. This finishes the proof of Proposition 4.1. $\square$

Remark 4.3. (1) We would like to point out that the argument right after (52) of [14] is not quite complete. When $n = 2$ the fact that W transforms the integral lattice $\wedge^{ev}(D^*)$ into itself does not imply that $\tilde{W}$ has integral entries. In other words, (7) above may not give integral matrices when $n = 2$. So the case $n = 2$ in [14] has to be dealt separately, and it does follow from [9]. In our situation we don't need to separate the case $n = 2$ since g' obviously has integral entries.

(2) Given g' explicitly, one can also check directly that $g' \in SO(n,n|\mathbb{Z})$ and $\theta' = g'\theta$: straight-forward calculations show that g' satisfies (1) and $\theta' = g'\theta$. Then g', $\tilde{g} \in O(n,n|\mathbb{Z})$ and hence we still have $I = \tilde{A}^t\tilde{D}$. Thus $det(g') = det(\tilde{g}^{-1}) = 1$. Therefore $g' \in SO(n,n|\mathbb{Z})$.

Proof of Theorem 1.1. We may think of A_θ as the universal C^*-algebra generated by unitaries $\{U_{x,\theta}\}_{x\in\mathbb{Z}}$ satisfying the relation $U_{x,\theta}U_{y,\theta} = e(x \cdot \theta y)U_{y,\theta}U_{x,\theta}$. For any $R \in GL(n|\mathbb{Z})$ and $\theta \in \mathcal{T}_n$ there is a natural isomorphism $A^\infty_\theta \to A^\infty_{\rho(R)\theta} = A^\infty_{R\theta R^t}$ given by $U_{x,\theta} \mapsto U_{(R^{-1})^t x,\rho(R)\theta}$. Under this isomorphism $\delta_{X,\theta}$ becomes $\delta_{RX,\rho(R)\theta}$ for any $X \in L^*$. Similarly, for any $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$ and $\theta \in \mathcal{T}_n$ there is a natural isomorphism $A^\infty_\theta \to A^\infty_{\mu(N)\theta} = A^\infty_{\theta+N}$ given by $U_{x,\theta} \mapsto U_{x,\mu(N)\theta}$, under which $\delta_{X,\theta}$ becomes $\delta_{X,\mu(N)\theta}$ for any $X \in L^*$. Now Theorem 1.1 follows from Lemma 3.5 and Propositions 4.1 and 2.2. $\square$

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