ZETA FUNCTIONS OF TOTALLY RAMIFIED p-COVERS OF THE PROJECTIVE LINE

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Abstract. In this paper we prove that there exists a Zariski dense open subset $\mathcal{U}$ defined over the rationals $\mathbb{Q}$ in the space of all one-variable rational functions with prescribed ℓ poles with fixed orders, such that for every geometric point f in $\mathcal{U}(\overline{\mathbb{Q}})$, the L-function of the exponential sum of f at a prime p has Newton polygon approaching the Hodge polygon as p approaches infinity. As an application to algebraic geometry, we prove that the p-adic Newton polygon of the zeta function of a p-cover of the projective line totally ramified at arbitrary ℓ points with prescribed orders has an asymptotic generic lower bound.

1. Introduction

This paper investigates the asymptotics of the zeta functions of p-covers of the projective line which are totally (wildly) ramified at arbitrary ℓ points. Our approach is via Dwork's method on one-variable exponential sums.

Throughout this paper we fix positive integers $\ell, d_1, \ldots, d_\ell$, and let $d := \sum_{j=1}^{\ell} d_j + \ell - 2$. For simplicity we assume $d \geq 2$ if $\ell = 1$. Let $P_1 = \infty$, $P_2 = 0$, $P_3, \ldots, P_\ell$ be fixed poles in $\overline{\mathbb{Q}}$ of orders $d_1, \ldots, d_\ell$, respectively. Let f be a one-variable function over $\overline{\mathbb{Q}}$ with these prescribed ℓ poles. It can be written in a unique form of partial fractions (remark: we have assumed f has vanishing constant term, which does not affect the p-adic Newton polygons of f) [13, Introduction]):

$$f = \sum_{i=1}^{d_1} a_{1,i} x^i + \sum_{j=2}^{\ell} \sum_{i=1}^{d_j} a_{ji} (x - P_j)^{-i} \tag{1}$$

with $a_{ji} \in \overline{\mathbb{Q}}$. Let $\mathbb{A}$ be the space of a_{ji}'s with $\prod_{j=1}^{\ell} a_{j,d_j} \neq 0$. It is an affine $(\sum_{j=1}^{\ell} d_j)$-space over $\mathbb{Q}$. Let the Hodge polygon of $\mathbb{A}$, denoted by $\mathrm{HP}(\mathbb{A})$, be the lower convex graph of the piecewise-linear function defined on the interval $[0, d]$ passing through the two endpoints $(0,0)$ and $(d, d/2)$ and assuming every slope in the list below of (horizontal) length 1:

$$\overbrace{0, \ldots, 0}^{\ell-1}; \overbrace{1, \ldots, 1}^{\ell-1}; \overbrace{\frac{1}{d_1}, \cdots, \frac{d_1 - 1}{d_1}}^{d_1-1}; \overbrace{\frac{1}{d_2}, \cdots, \frac{d_2 - 1}{d_2}}^{d_2-1}; \ldots\ldots; \overbrace{\frac{1}{d_\ell}, \cdots, \frac{d_\ell - 1}{d_\ell}}^{d_\ell-1}.$$

A non-smooth point on a polygon (as the graph of a piece-wise linear function) is called a vertex. We remark that the classical and geometrical 'Hodge polygon'

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for any curve (including Artin-Schreier curve as a special case) is the one with end points $(0,0)$ and $(d, d/2)$ and one vertex at $(d/2, 0)$. So the Hodge polygon in our paper is different from the classical Hodge polygon. We anticipate a p-adic arithmetic interpretation of our Hodge polygon, but it remains an open question.

In [13] it is shown that in the case $\ell = 1$ there is a Zariski dense open subset $\mathcal{U}$ defined over $\mathbb{Q}$ such that every geometric closed point f in $\mathcal{U}(\overline{\mathbb{Q}})$ has p-adic Newton polygon approaching the Hodge polygon as p approaches ∞. Wan has proposed conjectures regarding multivariable exponential sums, including the above as a special case (see [10, Conjecture 1.15]). This series of study traces back at least to Katz [4, Introduction], where Katz proposed to study exponential sums in families instead of examining one at a time. He systematically studied families of multivariable Kloosterman sum in [4].

Let $\mathbb{Q}_f$ be the extension field of $\mathbb{Q}$ generated by coefficients and poles $P_1, \ldots, P_\ell$ of f. For every prime number p we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ once and for all. This fixes a place $\mathcal{P}$ in $\mathbb{Q}_f$ lying over p of residue degree a for some positive integer a. As usual, we let $E(x) = \exp(\sum_{i=0}^{\infty} x^{p^i}/p^i)$ be the p-adic Artin-Hasse exponential function. Let γ be a root of the p-adic $\log E(x)$ with $\mathrm{ord}_p(\gamma) = \frac{1}{p-1}$. Then $E(\gamma)$ is a primitive p-th root of unity and we set $\zeta_p := E(\gamma)$. Let $\mathbb{F}_p$ be the prime field of p elements. Let $\mathbb{F}_q$ be a finite field of p^a elements. For $k \geq 1$, let $\psi_k : \mathbb{F}_{q^k} \to \mathbb{Q}(\zeta_p)^\times$ be a nontrivial additive character of $\mathbb{F}_{q^k}$. Henceforth we fix $\psi_k(\cdot) = \zeta_p^{\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(\cdot)}$. Let $\prod_{j=1}^{\ell} d_j$, and all poles and leading coefficients a_{j,d_j} of f be p-adic units. Let all coefficients $a_{j,i}$ of f are p-adically integral. (These are satisfied when p is large enough.) Let $S_k(f \bmod \mathcal{P}) = \sum_x \psi_k(f(x) \bmod \mathcal{P})$ where the sum ranges over all x in $\mathbb{F}_{q^k} \backslash \{\overline{P}_1, \ldots, \overline{P}_\ell\}$ (where $\overline{P}_j$ are reductions of $P_j \bmod \mathcal{P}$). The L-function of f at p is defined as

$$L(f \bmod \mathcal{P}; T) \quad = \quad \exp(\sum_{k=1}^{\infty} S_k(f \bmod \mathcal{P}) T^k/k).$$

This function lies in $\mathbb{Z}[\zeta_p][T]$ of degree d. It is independent of the choice of $\mathcal{P}$ (that is, the embedding of $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$) for p large enough, but we remark that its Newton polygon is independent of the choice of $\mathcal{P}$ for all p (see [15, Section 1]). One notes immediately that for every prime p (coprime to the leading coefficients, the poles and their orders) we have a map $\mathrm{NP}_p(\cdot)$ which sends every p-adic integral point f of $\mathbb{A}(\overline{\mathbb{Z}_p})$ to the Newton polygon $\mathrm{NP}_p(f)$ of the L-function of exponential sums of f at p. Given any $f \in \mathbb{A}(\overline{\mathbb{Q}})$, we have for p large enough that $f \in \mathbb{A}(\overline{\mathbb{Z}_p})$ and hence we obtain the Newton polygon $\mathrm{NP}_p(f)$ of f at p. Presently it is known that $\mathrm{NP}_p(f)$ lies over $\mathrm{HP}(\mathbb{A})$ for every p. These two polygons do not always coincide. (See [15, Introduction].) Some investigation on first slopes suggests the behavior is exceptional if p is small (see [7, Introduction]). There has been intensive investigation on how the (Archimedean) distance between $\mathrm{NP}_p(f)$ and $\mathrm{HP}(\mathbb{A})$ on the real plane $\mathbb{R}^2$ varies when p approaches infinity. Inspired by Wan's conjecture [10, Conjecture 1.15] (proved in [13] for the one-variable polynomial case), we believe that "almost all" points f in $\mathbb{A}(\overline{\mathbb{Q}})$ satisfies $\lim_{p\to\infty} \mathrm{NP}_p(f) = \mathrm{HP}(\mathbb{A})$. Our main result is the following.

Theorem 1.1. *Let $\mathbb{A}$ be the coefficients space $\{a_{ji}\}$ of the f's as in (1). There is a Zariski dense open subset $\mathcal{U}$ defined over $\mathbb{Q}$ in $\mathbb{A}$ such that for every geometric* *closed point f in $\mathcal{U}(\overline{\mathbb{Q}})$ one has $f \in \mathcal{U}(\overline{\mathbb{Z}_p})$ for p large enough (only depending on f), and*

$$\lim_{p\to\infty} \mathrm{NP}_p(f) = \mathrm{HP}(\mathbb{A}).$$

The two polygons $\mathrm{NP}_p(f)$ and $\mathrm{HP}(\mathbb{A})$ coincide if and only if $p \equiv 1 \bmod \mathrm{lcm}(d_j)$ (see [15]). The case $\ell = 1$ is known from [13, Theorem 1.1]. For $p \not\equiv 1 \bmod \mathrm{lcm}(d_j)$, the point $f = x^{d_1} + \sum_{j=1}^{\ell}(x - P_j)^{-d_j}$ does not lie in $\mathcal{U}$. This means $\mathcal{U}$ is always a proper subset of $\mathbb{A}$.

For any $\overline{f} \in \mathbb{A}(\mathbb{F}_q)$, where q is a p-power, and the (generalized) Artin-Schreier curve $C_{\overline{f}} : y^p - y = \overline{f}$, let $\mathrm{NP}(C_{\overline{f}}; \mathbb{F}_q)$ be the usual p-adic Newton polygon of the numerator of the zeta function of $C_{\overline{f}}/\mathbb{F}_q$. If it is shrunk by a factor of $1/(p-1)$ vertically and horizontally, we denote it by $\frac{\mathrm{NP}(C_{\overline{f}};\mathbb{F}_q)}{p-1}$.

Corollary 1.2. *Let notations be as in Theorem 1.1 and above. For any $\overline{f} \in \mathbb{A}(\mathbb{F}_q)$ we have $\frac{\mathrm{NP}(\overline{f};\mathbb{F}_q)}{p-1}$ lies over $\mathrm{HP}(\mathbb{A})$ with the same endpoints, and they coincide if and only if $p \equiv 1 \bmod (\mathrm{lcm}(d_j))$. Moreover, there is a Zariski dense open subset $\mathcal{U}$ defined over $\mathbb{Q}$ in $\mathbb{A}$ such that for every geometric closed point f in $\mathcal{U}(\overline{\mathbb{Q}})$ one has $f \in \mathcal{U}(\overline{\mathbb{Z}_p})$ for p large enough (only depending on f), and*

$$\lim_{p\to\infty} \frac{\mathrm{NP}(C_{\overline{f}}; \mathbb{F}_q)}{p-1} = \mathrm{HP}(\mathbb{A}).$$

Proof. This follows from the theorem above and a similar argument as the proof of Corollary 1.3 in [15], which we shall omit here. ☐

Remark 1.3. (1) The result in Theorem 1.1 does not depend on where those ℓ poles are (as long as they are distinct).

(2) By Deuring-Shafarevic formula (see for instance [3, Corollary 1.5]), one knows that $\mathrm{NP}_p(f)$ always has slope-0 segment precisely of horizontal length $\ell - 1$. By symmetry it also has slope-1 segment of the same length. See Remark 1.4 of [15].

Plan of the paper is as follows: section 2 introduces sheaves of (infinite dimensional) φ-modules over some affinoid algebra arising from one-variable exponential sums. We consider two Frobenius maps α_1 and α_a. Section 4 is the main technical part, where major combinatorics of this paper is done. After working out several combinatorial observations we are able to reduce our problem to an analog of the one-variable polynomial case as that in [13]. Now back to Section 3 we improve the key lemma 3.5 of [13] to make the generic Fredholm polynomial straightforward to compute. Section 5 uses p-adic Banach theory to give a new transformation theorem from α_1 to α_a for any $a \geq 1$. This approach is very different from [9] or [14]. It shreds some new light on p-adic approximations of L-functions of exponential sums and we believe that it will find more application in the future. Finally at the end of section 5 we prove our main result Theorem 1.1.

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2. Sheaves of φ-modules over affinoid algebra

The purpose of this section is to generalize the trace formula (see [14, Section 2]) for an exponential sum to that for families of exponential sums. See [2] for fundamentals in rigid geometry and see [1] for an excellent setup for rigid cohomology related to p-adic Dwork theory.

Let $\mathcal{O}_1 := \mathbb{Z}_p[\zeta_p]$ and $\Omega_1 := \mathbb{Q}_p(\zeta_p)$. Fix a positive integer a. Let Ω_a be the unramified extension of Ω_1 of degree a and $\mathcal{O}_a$ its ring of integers. Let $\hat{P}_1,\ldots,\hat{P}_\ell$ in $\mathcal{O}_a^\times$ be Teichmüller lifts of $\overline{P}_1,\ldots,\overline{P}_\ell$ in $\mathbb{F}_{p^a}$. Similarly let $A_{j,i}$ be that of $\overline{a}_{j,i}$ and let $\vec{A}$ denote the sequence of $A_{j,i}$ (we remark that for most part of the paper $\vec{A}$ will be treated as a variable). Let τ be the lift of Frobenius to Ω_a which fixes Ω_1. Then $\tau(A_{ji}) = A_{ji}^p$. Let $1\leq j\leq \ell$. Pick a root γ^{1/d_j} of γ in $\overline{\mathbb{Q}_p}$ (or in $\overline{\mathbb{Z}}_p$, all the same) for the rest of the paper, and denote $\Omega_1' := \Omega_1(\gamma^{1/d_1},\ldots,\gamma^{1/d_j})$. Let $\mathcal{O}_1'$ be its ring of integers. Let $\Omega_a' := \Omega_a\Omega_1'$ and let $\mathcal{O}_a'$ be its ring of integers. Then the affinoid algebra $\mathcal{O}_a'\langle\vec{A}\rangle$ (with $\vec{A}$ as variables) forms a Banach algebra over $\mathcal{O}_a'$ under the supremum norm.

Let $0<r<1$ and $r\in|\Omega_a'|_p$. Let $\mathbf{A}_r$ be the affinoid with ℓ deleted discs centering at $\hat{P}_1,\ldots,\hat{P}_\ell$ each of radius r on the rigid projective line $\mathbf{P}^1$ over Ω_a' (as defined in [15]). The topology on $\mathbf{A}_r$ is given by the fundamental system of strict neighborhood $\mathbf{A}_{r'}$ with $r\leq r'<1$ and $r'\in|\Omega_a'|_p$. Let $\mathbf{A}$ be $\mathbf{A}_r$ for some unspecified r sufficiently close to 1^- (the precise bound on the size of r was discussed in [15, Section 2]). Let $\mathcal{H}(\Omega_a')$ be the ring of rigid analytic functions on $\mathbf{A}$ over Ω_a'. Then it is a p-adic Banach space over Ω_a'. It consists of functions in one variable X of the form $\xi = \sum_{i=0}^\infty c_{1,i}X^i + \sum_{j=2}^\ell\sum_{i=1}^\infty c_{j,i}(X-\hat{P}_j)^{-i}$ where $c_{j,i}\in\Omega_a'$ and $\forall j\geq 1, \lim_{i\to\infty}\frac{|c_{j,i}|_p}{r^i}=0$. Its norm is defined as $||\xi|| = \max_j(\sup_i\frac{|c_{j,i}|_p}{r^i})$. (See [15, Section 2.1].) Let $\mathcal{H}(\Omega_a'\langle\vec{A}\rangle) := \mathcal{H}(\Omega_a')\hat{\otimes}_{\Omega_a'}\Omega_a'\langle\vec{A}\rangle$ where $\hat{\otimes}$ means p-adic completion after the tensor product. It is a p-adic Banach modules over $\Omega_a'\langle\vec{A}\rangle$ with the natural norm on the tensor product of two Banach modules defined by the followings. For any $\sum v\otimes w\in\mathcal{H}(\Omega_a')\otimes\Omega_a\langle\vec{A}\rangle$ let $||\sum v\otimes w|| = \inf(\max_i(||v_i||\cdot||w_i||))$, where the inf ranges over all representatives $\sum_i v_i\otimes w_i$ with $\sum v\otimes w = \sum_i v_i\otimes w_i$. From the p-adic Mittag-Leffler decomposition theorem derived in [15, Section 2.1], we can generalize it to the decomposition of $\Omega_a'\langle\vec{A}\rangle$ as a Banach $\Omega_a'\langle\vec{A}\rangle$-module. Write $X_1 = X$ or $X_j = (X-\hat{P}_j)^{-1}$ for $2\leq j\leq\ell$. Let $Z_j = \gamma^{1/d_j}X_j$. Note that $\vec{b}_{\rm w} = \{1, Z_1^i,\ldots,Z_\ell^i\}_{i\geq 1}$ is a formal basis of the Banach $\Omega_a'\langle\vec{A}\rangle$-module $\mathcal{H}(\Omega_a'\langle\vec{A}\rangle)$, that is, every v in $\mathcal{H}(\Omega_a'\langle\vec{A}\rangle)$ can be written uniquely as an infinite sum of $c_{j,i}'Z_j^i$'s with $c_{j,i}'\in\Omega_a'\langle\vec{A}\rangle$ and $\frac{|c_{j,i}'|_p}{r'^i}\to 0$ as $i\to\infty$, where $r' = r\ p^{-\frac{1}{(p-1)d_j}}$. The Banach module $\mathcal{H}(\Omega_a'\langle\vec{A}\rangle)$ is orthonormalizable (even though $\vec{b}_{\rm w}$ is not its orthonormal basis).

In this paper we extend the τ-action so that it acts on $\gamma^{\frac{1}{d_J}}$ trivially for any J. Below we begin to construct the Frobenius operator α_1 on $\mathcal{H}(\Omega_a'\langle\vec{A}\rangle)$. Recall the p-adic Artin-Hasse exponential function $E(X)$. Take expansion of $E(\gamma X)$ at X one gets $E(\gamma X) = \sum_{m=0}^\infty\lambda_m X^m$ for some $\lambda_m\in\mathcal{O}_1$. Clearly ${\rm ord}_p\lambda_m\geq\frac{m}{p-1}$ for all $m\geq 0$. In particular, for $0\leq m\leq p-1$ the equality holds and $\lambda_m = \frac{\gamma^m}{m!}$. Let

$F_j(X_j) := \prod_{i=1}^{d_j} E(\gamma A_{j,i} X_j^i)$. Then

$$F_j(X_j) = \sum_{n=0}^{\infty} F_{j,n}(A_{j,1}, \ldots, A_{j,d_j}) X_j^n,$$

where $F_{j,n} := 0$ for $n < 0$ and for $n \geq 0$

$$F_{j,n} \quad := \quad \sum \lambda_{m_1} \cdots \lambda_{m_{d_j}} A_{j,1}^{m_1} \cdots A_{j,d_j}^{m_{d_j}}, \tag{2}$$

where the sum ranges over all $m_1, \ldots, m_{d_j} \geq 0$ and $\sum_{k=1}^{d_j} k m_k = n$. It is clear that $F_{j,n}$ lies in $\mathcal{O}_1[A_{j,1}, \ldots, A_{j,d_j}]$. One observes that $F_j(X_j) \in \mathcal{O}_1\langle A_{j,1}, \ldots, A_{j,d_j}\rangle\langle X_j\rangle$, the affinoid algebra in one variable X_j (actually it lies in $\mathcal{O}_1[A_{j,1}, \ldots, A_{j,d_j}]\langle X_j\rangle$). Taking product over $j = 1, \ldots, \ell$, we have that $F(X) := \prod_{j=1}^{\ell} F_j(X_j)$ lies in $\mathcal{H}(\mathcal{O}_a\langle \vec{A}\rangle)$. Let τ_*^{a-1} be the push-forward map of τ^{a-1}, that is, for any function ξ, $\tau_*^{a-1}(\xi) = \tau^{a-1} \circ \xi \circ \tau$. For example, $\tau_*^{a-1}(B/(X - \hat{P}^p)) = \tau^{a-1}(B)/(X - \hat{P})$ for any $B \in \mathcal{O}_1\langle \vec{A}\rangle$ and $\hat{P}$ a Teichmüller lift of some $\overline{P}$. Let U_p be the Dwork operator and let $F(X)$ denote the multiplication map by $F(X)$, as defined in [15, Section 2]. Let $\alpha_1 := \tau_*^{a-1} \circ U_p \circ F(X)$ denote the composition map. Then α_1 is a τ^{a-1}-linear endomorphism of $\mathcal{H}(\Omega'_a\langle \vec{A}\rangle)$ as a Banach $\Omega'_a\langle \vec{A}\rangle$-module.

Let S be the affinoid over Ω'_a with affinoid algebra $\Omega'_a\langle \vec{A}\rangle$. If $\mathcal{L}$ is a sheaf of p-adic Banach $\Omega'_a\langle \vec{A}\rangle$-module (with formal basis) and α_1 is the Frobenius map which is τ^{a-1}-linear with respect to $\Omega'_a\langle \vec{A}\rangle$, then we call the pair $(\mathcal{L}, \alpha_1)$ a *sheaf of φ-module of infinite rank*. Note that the pair $(\mathcal{H}(\Omega'_a\langle \vec{A}\rangle), \alpha_1)$ can be considered as sections of a sheaf of $\Omega'_a\langle \vec{A}\rangle$-module of infinite rank on $\mathbf{A}$. This is intimately related to Wan's *nuclear σ-module of infinite rank* (see [11]) if replacing his σ by our τ^{a-1}. Wan has defined L-functions of nuclear σ-modules and he also showed that it is p-adic meromorphic on the closed unit disc (see Wan's papers [11, 12] which proved Dwork's conjecture). Finally we define $\alpha_a := \alpha_1^a$.

Recall that α_1 is a τ^{a-1}-linear (with respect to $\Omega'_a\langle \vec{A}\rangle$) completely continuous endomorphism on the p-adic Banach module $\mathcal{H}(\Omega'_a\langle \vec{A}\rangle)$ over $\Omega'_a\langle \vec{A}\rangle$. Let $1 \leq J_1, J \leq \ell$. Write $(\alpha_1 Z_J^i)_{\hat{P}_{J_1}} = \sum_{n=0}^{\infty} (\tau^{a-1} C_{J_1,J}^{n,i}) Z_{J_1}^n$ for some $C_{J_1,J}^{n,i}$ in $\Omega'_a\langle \vec{A}\rangle$. The matrix of α_1, consisting of all these $\tau^{a-1} C_{J_1,J}^{\star,\star}$'s, is a nuclear matrix (see section 5). This matrix is the subject of the next section. Below we extend Dwork, Monsky and Reich's trace formula to families of one-variable exponential sums.

Theorem 2.1. *Let $\overline{f} = \sum_{i=1}^{d_1} \overline{a}_{1,i} x^i + \sum_{j=2}^{\ell} \sum_{i=1}^{d_i} \overline{a}_{j,i} (x - P_j)^{-i} \in \mathbb{A}(\mathbb{F}_q)$ and let $\hat{f}$ be its Teichmüller lift with coefficient $\overline{a}_{ji}$ being lifted to A_{ji}. Let $\mathcal{H}(\Omega'_a\langle \vec{A}\rangle)_r$ be the Banach module $\mathcal{H}(\Omega'_a\langle \vec{A}\rangle)$ for some suitably chosen $0 < r < 1$ with $r \in |\Omega'_a|_p$ close enough to 1^-. Then*

$$L(\overline{f}/\mathbb{F}_q; T) \quad = \quad \frac{\det(1 - T\alpha_a | \mathcal{H}(\Omega'_a\langle \vec{A}\rangle))}{\det(1 - Tq\alpha_a | \mathcal{H}(\Omega'_a\langle \vec{A}\rangle))}$$

lies in $\mathcal{O}_a\langle \vec{A}\rangle[T]$ as a polynomial of degree d in T. Its Teichmüller specialization of $\vec{A}$ in $\mathcal{O}_a$ lies in $\mathbb{Z}[\zeta_p][T]$.

Proof. The proof is similar to that of [14, Lemma 2.7]. Let $\mathcal{H}^\dagger := \bigcup_{0<r<1} \mathcal{H}(\Omega'_a\langle \vec{A}\rangle)_r$. Then it is the Monsky-Washnitzer dagger space. Then α_a is a completely continuous endomorphism on $\mathcal{H}^\dagger$ and the determinant $\det(1 - T\alpha_a | \mathcal{H}^\dagger) = \det(1 -$

$T\alpha_a|\mathcal{H}(\Omega'_a\langle\vec{A}\rangle)_r)$ for any r within suitable range in $(0,1)$ is independent of r. Finally one knows that the coefficients are all integral so lies in $\mathcal{O}_a$ and coefficient of T^m vanishes for all $m>d$. We omit details of the proof. $\square$

3. Explicit approximation of the Frobenius matrix

This section uses some standard techniques in p-adic approximation and it is very technical. The readers are recommended to skip it at first reading and continue at the next section.

3.1. **The nuclear matrix.** Let notation be as in the previous section. Assign $\phi(1)=0$. Let $\phi(Z_j^n)=\frac{n}{d_j}$ for $j\leq 2$ or $\frac{n-1}{d_j}$ for $j\geq 3$. Order the elements in $\vec{b}_{\rm w}$ as $e_1,e_2,\cdots$ such that $\phi(e_1)\leq\phi(e_2)\leq\cdots$. Consider the infinite matrix representing the endomorphism α_1 of the $\Omega'_a\langle\vec{A}\rangle$-module $\mathcal{H}(\Omega'_a\langle\vec{A}\rangle)$ with respect to the basis $\vec{b}_{\rm w}$. This matrix can be written as $\tau^{a-1}\mathbf{M}$, where each entry is $\tau^{a-1}C_{J_1,J}^{n,i}$ for $1\leq J_1,J\leq\ell$.

Our goal of this section is to collect delicate information about entries of the matrix $\mathbf{M}$. Recall the is polynomial F_{J,n_J} in $\mathcal{O}_1[\vec{A}]$ as in (2), which we have already built up some satisfying knowledge. Below we will express $C_{J_1,J}^{\star,\star}$ as a polynomial expression in these $F_{J_1,n_{J_1}}$'s. In this paper the *formal expansion* of $C_{J_1,J}^{\star,\star}$ will always mean the formal sum in $\mathcal{O}'_a[\vec{A}]$ by the composition of (3) and the formula in Lemma 3.1.

For $n,i\geq 1$, and if $J=1$ or $J_1=1$ then for $i\geq 0$ or for $n\geq 0$ respectively one has

$$C_{J_1,J}^{n,i}=\begin{cases}\gamma^{\frac{i}{d_J}-\frac{n}{d_{J_1}}}H_{J_1,J}^{np,i} & J_1=1,2\\ \gamma^{\frac{i}{d_J}-\frac{n}{d_{J_1}}}\sum_{m=n}^{np}C^{n,m}\hat{P}_{J_1}^{np-m}H_{J_1,J}^{m,i} & J_1\geq 3\end{cases}\tag{3}$$

where $C^{\star,\star}\in\mathbb{Z}_p$ is defined in [15, Lemma 3.1] and $H_{J_1,J}^{\star,\star}\in\mathcal{O}_a\langle\vec{A}\rangle$ is formulated in Lemma 3.1 below. Indeed, we recall that $C^{n,m}$ is actually a rational integer and it only depends on n,m and p.

Lemma 3.1. *Let $\vec{n}:=(n_1,\ldots,n_\ell)\in\mathbb{Z}_{\geq 0}^\ell$.*

(1) For $i,n\geq 0$, then $H_{1,J}^{n,i}$ is equal to

$$\sum\left(F_{1,n_1}\cdot\left(\sum_{\substack{J\neq 1\\ 0\leq m_J\leq n_J}}F_{J,m_J}\binom{n_J+i-1}{m_J+i-1}\hat{P}_J^{n_J-m_J}\right)\cdot\prod_{j\neq 1,J}\left(\sum_{m_j=0}^{n_j}F_{j,m_j}\binom{n_j-1}{m_j-1}\hat{P}_j^{n_j-m_j}\right)\right),$$

where the sum ranges over all $\vec{n}\in\mathbb{Z}_{\geq 0}^\ell$ such that $n=n_1\pm i-\sum_{j=2}^\ell n_j$ and the $+$ or $-$ depends on $J=1$ or $J\neq 1$, respectively.

(2) For $J_1, J \neq 1$, one has that $H^{n,i}_{J_1,J}$ is equal to

$$\sum \left(F_{J_1,n_{J_1}} \cdot \left(\sum_{\substack{J\neq J_1\\ m_J\geq 0}} F_{J,m_J}(-1)^{m_J+i}\binom{n_J+m_J+i-1}{m_J+i-1}(\hat{P}_J-\hat{P}_{J_1})^{-(n_J+m_J+i)} \right) \cdot \left(\sum_{m_1=n_1}^{\infty} F_{1,m_1}\binom{m_1}{n_1}\hat{P}_{J_1}^{m_1-n_1} \right) \prod_{j\neq 1,J_1,J} \left(\sum_{m_j=0}^{\infty} F_{j,m_j}(-1)^{m_j}\binom{n_j+m_j-1}{m_j-1}(\hat{P}_j-\hat{P}_{J_1})^{-(n_j+m_j)} \right) \right)$$

where the sum ranges over all $\vec{n} \in \mathbb{Z}^{\ell}_{\geq 0}$ such that $n = n_{J_1} + i - \sum_{j\neq J_1} n_j$ if $J = J_1$ and $n = n_{J_1} - \sum_{j\neq J_1} n_j$ if $J \neq J_1$.

(3) For $J_1 \neq 1$ and $J = 1$ we have that $H^{n,i}_{J_1,J}$ is equal to

$$\sum \left(F_{J_1,n_{J_1}} \cdot \left(\sum_{m_1=n_1-i} F_{1,m_1}\binom{m_1+i}{n_1}\hat{P}_{J_1}^{m_1+i-n_1} \right) \cdot \prod_{j\neq J_1,1} \left(\sum_{m_j=0}^{\infty} F_{j,m_j}(-1)^{m_j}\binom{n_j+m_j-1}{m_j-1}(\hat{P}_j-\hat{P}_{J_1})^{-(n_j+m_j)} \right) \right)$$

where the sum ranges over all $\vec{n} \in \mathbb{Z}^{\ell}_{\geq 0}$ such that $n = n_{J_1} - \sum_{j\neq J_1} n_j$.

Proof. We shall use "$\overset{\hat{P}_j}{=}$" to mean expansion at $\hat{P}_j$. Clearly for any J_1 one has $F_{J_1}(X_{J_1})X^i_{J_1} \overset{\hat{P}_{J_1}}{=} \sum_{n=0}^{\infty} F_{J_1,n}X^{n+i}$.

For $J \geq 2$ one has the expansion at $\hat{P}_1 = \infty$:

$$\begin{aligned} F_J(X_J)X^i_J &= \sum_{m=0}^{\infty} F_{J,m}(X^{-1}(1-\hat{P}_J X^{-1})^{-1})^{m+i} \\ &\overset{\hat{P}_1}{=} \sum_{m=0}^{\infty} F_{J,m} \sum_{k=m+i}^{\infty} \binom{k-1}{m+i-1}\hat{P}_J^{k-(m+i)}X^{-k} \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n} F_{J,m}\binom{n+i-1}{m+i-1}\hat{P}_J^{n-m})X^{-n-i}. \end{aligned}$$

For $J_1 \neq 1$ and $J \neq 1, J_1$, its expansion at $\hat{P}_{J_1}$ is:

$$\begin{aligned} F_J(X_J)X^i_J &= \sum_{m=0}^{\infty} F_{J,m}(X^{-1}_{J_1} - (\hat{P}_J - \hat{P}_{J_1}))^{-(m+i)} \\ &\overset{\hat{P}_{J_1}}{=} \sum_{m=0}^{\infty} F_{J,m}(-1)^{m+i}\sum_{n=0}^{\infty}\binom{n+m+i-1}{m+i-1}(\hat{P}_J-\hat{P}_{J_1})^{-(n+m+i)}X^{-n}_{J_1} \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} F_{J,m}(-1)^{m+i}\binom{n+m+i-1}{m+i-1}(\hat{P}_J-\hat{P}_{J_1})^{-(n+m+i)})X^{-n}_{J_1}. \end{aligned}$$

For $J_1 \neq 1$ and $J = 1$ then one has

$$F_J(X)X_J^i \overset{\hat{P}_{J_1}}{=} \sum_{n=0}^{\infty}\left(\sum_{m=n-i}^{\infty} F_{J,m}\binom{m+i}{n}\hat{P}_{J_1}^{m+i-n}\right)X_{J_1}^{-n}.$$

By $F(X)X_J^i = (F_J(X_J)X_J^i)\cdot\prod_{j\neq J}F_j(X_j)$, and Key Computational Lemma of [15], one can compute and obtain $(F(X)X_J^i)_{\hat{P}_{J_1}}$ for the case $J_1 = 1$ or $J_1 \neq 1$ presented respectively in the two formulas in our assertion. This proves the lemma. □

Remark 3.2. If we are dealing with the case of unique pole at ∞ then one sees easily that $C_{1,1}^{\star,\star}$ lies in $\mathcal{O}_1'[\vec{A}]$. This greatly reduces the complexity of situation.

The following results were presented in [15]. See Section 3 and in particular, Theorem 3.7 of [15] for a proof. We shall use t_{J_1} to denote the lower bound in Lemma 3.3 c).

Lemma 3.3. *Let notation be as above.*

(a) For all J and n_J we have $\mathrm{ord}_p(F_{J,n_J}) \geq \frac{\lceil \frac{n_J}{d_J}\rceil}{p-1} \geq \frac{n_J}{d_J(p-1)}$.

(b) For all J_1, J, and all n, i we have $\mathrm{ord}_p(H_{J_1,J}^{n,i}) \geq \frac{n-i}{d_{J_1}(p-1)}$.

(c) For any J_1 and any n we have $\mathrm{ord}_p(C_{J_1,\star}^{n,\star}) \geq \frac{n}{d_{J_1}}$ *or* $\frac{n-1}{d_{J_1}}$ *depending on* $J_1 = 1, 2$ *or* $3 \leq J_1 \leq \ell$. *Moreover,* $\mathrm{ord}_p(C_{J_1,J_1}^{n,i}) \geq \lceil\frac{np-i}{d_{J_1}}\rceil/(p-1)$ *or* $\lceil\frac{(n-1)p-(i-1)}{d_{J_1}}\rceil/(p-1)$ *depending on* $J_1 = 1, 2$ *or* $3 \leq J_1 \leq \ell$*, respectively.*

3.2. **Approximation by truncation.** From the previous subsection one has noticed an unpleasant feature of $C_{J_1,J}^{\star,\star}$ for the purpose of approximation by $F_{J_1,n_{J_1}}$'s. First, in the sum for (3) when $J_1 \geq 3$, the range of m is too 'large'. Second, $H_{J_1,J}^{\star,\star}$ of Lemma 3.1 is generally an infinite sum of $F_{J_1,n_{J_1}}$'s. In this subsection we will define an approximation in terms of truncated finite sum of $F_{J_1,n_{J_1}}$'s. Below we prove two lemmas which will be used for approximation in Lemma 4.3.

For any integer $0 < t \leq p$, let ${}^tC_{J_1,J}^{n,i}$ be the same as $C_{J_1,J}^{n,i}$ except for $J_1 \geq 3$ its sum ranges over all m in the sub-interval $[(n-1)p+1, (n-1)p+t]$.

Lemma 3.4. *Let $3 \leq J_1 \leq \ell$, $1 \leq J \leq \ell$. Let $n \leq d_{J_1}$ and $i \leq d_J$.*

(1) For p large enough, one has

$$\mathrm{ord}_p(C_{J_1,J}^{n,i} - {}^pC_{J_1,J}^{n,i}) \quad > \quad \frac{n-1}{d_{J_1}} + \frac{d}{p-1}. \tag{4}$$

(2) There is a constant $\beta > 0$ depending only on d such that for $t \geq \beta$ one has

$$\mathrm{ord}_p({}^pC_{J_1,J}^{n,i} - {}^tC_{J_1,J}^{n,i}) \quad > \quad \frac{n-1}{d_{J_1}} + \frac{d}{p-1}. \tag{5}$$

Proof. (1) By [15, Lemma 3.1], one knows that for any $m \leq (n-1)p$ one has $\mathrm{ord}_p(C^{n,m}) \geq 1$ and hence $\mathrm{ord}_p(C_{J_1,J}^{n,i}) \geq 1 + (\frac{i}{d_J} - \frac{n}{d_{J_1}})\frac{1}{p-1}$. For $n \leq d_{J_1}$ and for p large enough one has $1 + (\frac{i}{d_J} - \frac{n}{d_{J_1}})\frac{1}{p-1} > \frac{n-1}{d_{J_1}} + \frac{d}{p-1}$. Combining these two inequalities, one concludes.

(2) We may assume $J_1 \geq 3$. Then for any $1 \leq v \leq p$, by Lemma 3.3,

$$\mathrm{ord}_p(H_{J_1,J}^{(n-1)p+v,i}) \quad \geq \quad \frac{(n-1)p+v-i}{d_{J_1}(p-1)} > (\frac{n}{d_{J_1}} - \frac{i}{d_J})\frac{1}{p-1} + \frac{n-1}{d_{J_1}} + \frac{d}{p-1},$$

if $v \geq \beta$ for some $\beta > 0$ only depending on d. Therefore,

$$\mathrm{ord}_p({}^pC^{n,i}_{J_1,J} - {}^tC^{n,i}_{J_1,J}) \quad > \quad \frac{n-1}{d_{J_1}} + \frac{d}{p-1}.$$

This finishes our proof. □

Fix β for the rest of the paper. We will truncate the infinite expansion of $H^{\star,\star}_{J_1,J}$. Let $w > 0$ be any integer. For $J_1 = 1, 2$ let ${}^wH^{np,i}_{J_1,J}$ be the sub-sum in $H^{np,i}_{J_1,J}$ where $\vec{n} = (n_1, \ldots, n_\ell)$ are such that $n_{J_1} - np$ and n_j lie the interval $[-w, w]$ for $j \neq J_1$. Similarly, for $J_1 \geq 3$ let ${}^wH^{m,i}_{J_1,J}$ be the sub-sum of $H^{m,i}_{J_1,J}$ where $\vec{n}$ ranges over the finite set of vectors $(n_1, \ldots, n_\ell)$ such that $n_{J_1} - (n-1)p$ and n_j lie in the interval $[-w, w]$ for $j \neq J_1$. Consider ${}^\beta C^{n,i}_{J_1,J}$ as a polynomial expression in $H^{\star,\star}_{J_1,J}$'s, then we set ${}^wK^{n,i}_{J_1,J} := {}^\beta C^{n,i}_{J_1,J}({}^wH^{n,i}_{J_1,J})$.

Lemma 3.5. *There is a constant α depending only on d such that*

$$\mathrm{ord}_p({}^\beta C^{n,i}_{J_1,J} - {}^\alpha K^{n,i}_{J_1,J}) \quad > \quad \frac{n-1}{d_{J_1}} + \frac{d}{p-1}. \tag{6}$$

Proof. This part is similar to Lemma 3.4 2), so we omit its proof. □

3.3. **Minimal weight terms.** The *weight* of a monomial (with nonzero coefficient) $(\prod_{j=1}^{\ell}\prod_{i=1}^{d_j} A_{j,i}^{k_{j,i}})$ in $\mathcal{O}'_a[\vec{A}]$ is defined as $\sum_{j=1}^{\ell}\sum_{i=1}^{d_j} ik_{j,i}$. For example, the weight of $A^a_{1,2}A^b_{1,3}$ is equal to $2a + 3b$. We will later utilize the simple observation that every monomial in F_{J,n_J} is of weight n_J.

We call those entries with $J_1 = J$ the diagonal one (or blocks). As we have seen in Lemma 3.1, the off-diagonal entries are less manageable while the diagonal entries behave well in principle. For any integer $0 < t \leq p$, let ${}^t\mathbf{M} := ({}^tC^{n,i}_{J_1,J})$ with respect to the basis arranged in the same order as that for $\mathbf{M}$. Consider the diagonal blocks, consisting of ${}^pC^{\star,\star}_{J,J}$'s. Despite ${}^pC^{\star,\star}_{J,J}$ lives in $\mathcal{O}'_a\langle\vec{A}\rangle$, its minimal weight terms live in $\hat{P}^{\mathbb{Z}}_J\gamma^{\frac{i-n}{d_J}}\mathcal{O}_1[\vec{A}_J]$.

Lemma 3.6. *Let $p > d_j$ for all j. The minimal weight monomials of ${}^pC^{n,i}_{J,J}$ (with $J = 1, 2$) live in the term $\gamma^{\frac{i-n}{d_1}}F_{1,np-i}$ where $d_1 > n, i \geq 0$ unless $n = 0$ and $i > 0$. For $J \geq 3$ and $n \geq 2$, the minimal weight monomials of ${}^pC^{n,i}_{J,J}$ live in the term*

$$\gamma^{\frac{i-n}{d_J}}C^{n,(n-1)p+1}\hat{P}^{p-1}_J F_{J,(n-1)p-(i-1)}$$

where $d_J > n, i \geq 1$.

Proof. This follows from Lemma 3.1. We omit its proof. □

Given a $k \times k$ matrix $M := (m_{ij})_{1 \leq i,j \leq k}$ with a given formal expansion of $m_{ij} \in \mathcal{O}'_a\langle\vec{A}\rangle$, the *formal expansion* of $\det M$ means the formal expansion as $\sum_{\sigma \in S_k}\mathrm{sgn}(\sigma)\prod_{n=1}^{k} m_{ij}$ where the product is expanded according to the given formal expansion m_{ij}. For example, if $m_{ij} = C^{\star,\star}_{J_1,J}$ then its formal expansion is given by composition of (3) and formulas in Lemma 3.1.

Lemma 3.7. *Let notation be as above and let $p > d_j$ for all j. Then in the formal expansion of* $\det({}^p\mathbf{M})^{[k]}$ *in $\mathcal{O}'_a\langle\vec{A}\rangle$, all minimal weight terms are from $\prod_{J=1}^{\ell}\det {}^pC^{n,i}_{J,J}$ (with $n, i \geq 1$ in a suitable range for $J = 1, 2$ and with $n, i \geq 2$ for $J \geq 3$) of the diagonal blocks.*

Proof. We will show that picking an arbitrary entry on the diagonal block, every off-diagonal entry on the same row has strictly higher minimal weight among its monomials.

Let $\vec{A}_J$ stand for the vector $(A_{J,1},\ldots,A_{J,d_J})$. As we have noticed earlier the polynomial F_{J,n_J} in $\mathcal{O}_1[\vec{A}_J]$ has every monomial of equal weight n_J for any J. For simplicity we assume $n,i\geq 1$ here. Using data from Lemma 3.1, we find all minimal weight monomials in $H^{\star,\star}_{J_1,J}$'s illustrated below by an arrow: $H^{np,i}_{1,1}\to F_{1,np-i}, H^{np,i}_{1,J\geq 2}\to F_{1,np+i}, H^{np,i}_{2,2}\to F_{2,np-i}, H^{np,i}_{2,J\neq 2}\to F_{2,np}$. One also notes that for $J_1\geq 3$ one has that $H^{(n-1)p+1,i}_{J_1\geq 3,J}\to F_{J_1,(n-1)p-(i-1)}$ if $J_1=J$, and $H^{(n-1)p+1,i}_{J_1\geq 3,J}\to F_{J_1,(n-1)p+1}$ if $J_1\neq J$. One notices from (3) and the above that the minimal weight monomials of ${}^pC^{n,i}_{J_1,J}$ live in $H^{np,i}_{J_1,J}$ if $J_1=1,2$ and in $H^{(n-1)p+1,i}_{J_1,J}$ if $3\leq J_1\leq \ell$.

Recall that for $J=1$ the range for i is $i\geq 0$. In all other cases the range is $i\geq 1$. From the above we conclude our claim in the beginning of the proof. Consequently, all minimal weight monomials in the formal expansion of the determinant $\det\mathbf{M}^{[k]}$ come from the diagonal blocks. By Lemma 3.6, $C^{0,i}_{1,1}$ and $C^{1,i}_{J,J}$ (with $J\geq 3$) both have their minimal weight equal to 0 if $i=0$ and >0 if $i>0$. Then it is not hard to conclude that the minimal weight monomials of $\det(C^{n,i}_{1,1})_{n,i\geq 0}$ (resp. $\det(C^{n,i}_{J,J})_{n,i\geq 1}$) are from $\det(C^{n,i}_{1,1})_{n,i\geq 1}$ (resp. $\det(C^{n,i}_{J,J})_{n,i\geq 2}$). □

For $1\leq J\leq \ell$, let $D^{[k]}_J := \det(F_{J,ip-j})_{1\leq i,j\leq k}\in\mathcal{O}_1[A_1,\ldots,A_d]$.

Proposition 3.8. *Let $p>d_j$ for all j. The minimal weight monomials of $\det(({}^pC^{i,j}_{J,J})_{1\leq i,j\leq k}$ for $J=1,2$ (resp. $\det(({}^pC^{i,j}_{J,J})_{2\leq i,j\leq k}$ for $J\geq 3$) lie in $D^{[k]}_J$ (resp. $D^{[k-1]}_J$). Every monomial of $D^{[k]}_J$ (resp. $D^{[k-1]}_J$) corresponds to a monomial in the formal expansion of $\det(({}^pC^{i,j}_{J,J})_{1\leq i,j\leq k})$ for $J=1,2$ (resp. $\det(({}^pC^{i,j}_{J,J})_{2\leq i,j\leq k}$ for $J\geq 3$) by the same permutation $\sigma\in S_k$ in the natural way.*

Proof. It follows from Lemmas 3.6 and 3.7 above. □

3.4. **Local at each pole.** For ease of notation, we drop the subindex J for the rest of this subsection. One should understand that d, A_i, F_{ip-j}, D_n stand for $d_J, A_{J,i}, F_{J,ip-j}, D^{[n]}_J$, respectively. Let $1\leq n\leq d-1$ and let S_n be the permutation group. Let $D_n := \det(F_{ip-j})_{1\leq i,j\leq n}\in\mathcal{O}_1[A_1,\ldots,A_d]$. Then we have the formal expansion of $D^{[n]}$:

$$D^{[n]} = \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\sum\prod_{i=1}^n g_{\sigma,i},$$

where the second $\sum$ runs over all terms $g_{\sigma,i}$ of the polynomial $F_{ip-\sigma(i)}$ in $\mathcal{O}_1[A_1,\ldots,A_d]$.

Proposition 3.9. *Let $1\leq n\leq d$. Then there is a unique monomial in the above formal expansion of $D^{[n]}$ with highest lexicographic order (according to $A_d,\ldots,A_1$). Moreover, the p-adic order of this monomial (with coefficient) is minimal among the p-adic orders of all monomials in the above formal expansion.*

Remark 3.10. We shall fix the unique σ_0 found in the proposition for the rest of the paper. The minimal p-adic order of this monomial (with coefficient) is equal to (8) while every row achieve its minimal order in Lemma 3.3c). We shall use this fact later.

Proof. Denote by r the least non-negative residue of $p \mod d$. Recall the n by n matrix $\mathbf{r}_n := \{r_{ij}\}_{1\le i,j\le n}$ where $r_{ij} := d\lceil\frac{ri-j}{d}\rceil - (ri-j)$. The properties of this matrix can be found in [13, Lemma 3.1]. Let $\prod_{i=1}^n h_{\sigma,i}$ be a highest-lexicographic-order-monomial in the formal expansion of $D^{[n]}$. Then $h_{\sigma,i}$ must be the highest-lexicographic-order-monomial in $F_{ip-\sigma(i)}$, which is easily seen to be $c_{ip-\sigma(i)}A_d^{\frac{ip-\sigma(i)}{d}}$ or $c_{ip-\sigma(i)}A_d^{\lfloor\frac{ip-\sigma(i)}{d}\rfloor}A_{d-r_{i,\sigma(i)}}$ depending on $r_{i,\sigma(i)}=0$ or not, where $c_{ip-\sigma(i)}\in\Omega_1$. We show first that

$$\sigma(i)=k \text{ for any } 1\le i,k\le n \text{ with } r_{ik}=0. \tag{7}$$

Suppose that (7) does not hold. Pick a pair (i,k) among the pairs failing (7) such that $|\sigma(i)-k|$ is minimal. Say $\sigma(j)=k$. Define another permutation $\sigma'\in S_n$ by $\sigma'(i)=k$ and $\sigma'(j)=\sigma(i)$ while $\sigma'(s)=\sigma(s)$ for all other s. Denote by $h_{\sigma',t}$ the highest-lexicographic-order-monomial in $F_{tp-\sigma'(t)}$. Then it is easy to see that the lexicographic order of $\prod_{t=1}^n h_{\sigma',t}$ is strictly higher than that of $\prod_{t=1}^n h_{\sigma,t}$, which is a contradiction. Therefore (7) holds.

Notice that for permutations $\sigma''\in S_n$ satisfying (7), the degree of A_d in $\prod_{t=1}^n h_{\sigma'',t}$ does not depend on the choice of σ'', where $h_{\sigma'',t}$ is the highest-lexicographic-order-monomial in $F_{tp-\sigma''(t)}$. Then the proof of [13, Lemma 3.2] shows that there exists a unique $\sigma_0\in S_n$ such that $\prod_{t=1}^n h_{\sigma_0,t}$ has highest lexicographic order among the corresponding monomials for all $\sigma''\in S_n$ satisfying (7). In fact, σ_0 is exactly the permutation in [13, Lemma 3.2]. By the above discussion, this monomial $\prod_{t=1}^n h_{\sigma_0,t}$ also has the unique highest lexicographic order in the formal expansion of $D^{[n]}$.

Next we show that the p-adic order of $\prod_{i=1}^n h_{\sigma_0,i}$ is minimal (among the p-adic orders of the monomials in the formal expansion of $D^{[n]}$). Let $\prod_{i=1}^n g_{\sigma,i}$ be an arbitrary monomial in the formal expansion of $D^{[n]}$. Then clearly $\mathrm{ord}_p(g_{\sigma,i})\ge\lceil\frac{ip-\sigma(i)}{d}\rceil=\frac{pi-\sigma(i)+r_{i,\sigma(i)}}{d}$ for all $1\le i\le n$. Since $h_{\sigma_0,i}$ is the highest-lexicographic-order-monomial in $F_{ip-\sigma_0(i)}$, one sees easily that $\mathrm{ord}_p(h_{\sigma_0,i})=\frac{pi-\sigma_0(i)+r_{i,\sigma_0(i)}}{d}$ for all $1\le i\le n$. From (7) it is easy to see that $r_{i,j}-r_{i,\sigma_0(i)}\ge j-\sigma_0(i)$ for all $1\le i,j\le n$. It follows that $\mathrm{ord}_p(\prod_{i=1}^n g_{\sigma,i})\ge\mathrm{ord}_p(\prod_{i=1}^n h_{\sigma_0,i})$. $\square$

Remark 3.11. In the proof of Proposition 3.9 we have noticed that the σ_0 is exactly the permutation in [13, Lemma 3.2]. Therefore, one can always take $t_0=0$, that is, $f_n^0(\vec{A})\ne 0$ in [13, Lemma 3.5].

4. Newton polygon of α_1

Recall that $\mathrm{HP}(\mathbb{A})$ lives on the real plane over the interval $[0,d]$. Because of Remark 1.3, one only has to consider the part of $\mathrm{NP}_p(f)$ with slope <1, that is, to consider the part of $\mathrm{NP}_p(f)$ over the interval $[0,d-\ell]$. This part is our focus of this section. Suppose for some $1\le k\le d-\ell$, the point (k,c_0) is a vertex on $\mathrm{HP}(\mathbb{A})$. Then one notices that

$$c_0=\sum_{J=1}^{2}\sum_{i=1}^{k_J} i/d_J+\sum_{J=3}^{\ell}\sum_{i=1}^{k_J}(i-1)/d_J$$

for a sequence of nonnegative integers $k_1,\ldots,k_\ell$ such that $k_1+\ldots+k_\ell=k$. This sequence is unique because (k,c_0) is a vertex. From now on we fix such a k.

For our purpose we also fix the residue classes of $p \bmod d_J$ for all J. Let $r_{J,ij}$ be the least nonnegative residue of $-(ip-j) \bmod d_J$. Let σ_0 be the permutation in S_k which is the union of those permutations found in Proposition 3.9 locally at each pole P_J. Let s_{J_1} be the rational number defined by

$$s_{J_1} \quad := \quad \frac{(p-1)k_{J_1}(k_{J_1}\pm 1)/2}{d_{J_1}} + \frac{\sum_{i=1}^{k_{J_1}} r_{J_1,i,\sigma_0(i)}}{d_{J_1}} \tag{8}$$

where $+$ and $-$ is taken according to $J_1 = 1,2$ or $J_1 \geq 3$. Let $s_0 := s_1 + \cdots + s_\ell$. Clearly $s_0 - c_0(p-1) < k \leq d - \ell$.

Let α and β be the integers chosen in Lemmas 3.4 and 3.5 (they depend only on d). Let $\mathbb{Q}' := \mathbb{Q}(\gamma^{1/d_1}, \ldots, \gamma^{1/d_\ell})$.

Lemma 4.1. *For any $J_1 = 1,2$, $1 \leq J \leq \ell$ and for any n, i in their range, for p large enough, there is a polynomial $G_{J_1,J}^{n,i}$ in $\mathbb{Q}'(\vec{P})[\vec{A}]$ such that*

$$ {}^{\alpha}K_{J_1,J}^{n,i} = \gamma^{(p-1)n/d_{J_1}} U_{J_1,n} G_{J_1,J}^{n,i} \bmod \gamma^{(p-1)n/d_{J_1}+d+1}.$$

For the case $J_1 \geq 3$, one has a similar $G_{J_1,J}^{n,i}$ such that

$$ {}^{\alpha}K_{J_1,J}^{n,i} = \gamma^{(p-1)(n-1)/d_{J_1}} U_{J_1,n} G_{J_1,J}^{n,i} \bmod \gamma^{(p-1)(n-1)/d_{J_1}+d+1},$$

where $U_{J_1,n}$ is a p-adic unit depending only on the the row index (J_1, n).

Proof. We use the same technique as [13], so we only outline our proof here for the case $J_1 = J = 1$. Let $n_j \in [-\alpha, \alpha]$ for $j \neq J_1$. Let $n_{J_1} = np + \sum_{j\neq J_1} n_j - i$. For any $\vec{n} = (n_1, \ldots, n_\ell)$ in this range, we have

$$F_{j,n_j} \equiv \gamma^{\frac{n_j}{d_j}} Q_j \bmod \gamma^{\frac{n_j}{d_j}+d+1}$$

and

$$F_{J_1,n_{J_1}} = \gamma^{\frac{n_{J_1}}{d_{J_1}}} V_{J_1} Q_{J_1} \bmod \gamma^{\frac{n_{J_1}}{d_{J_1}}+d+1},$$

where Q_j's and Q_{J_1} are in $\mathbb{Q}'[\vec{A}]$ independent of p and V_{J_1} is some p-adic unit depending only on the row index J_1. Now let $\vec{n}$ be in the range for ${}^{\alpha}K_{J_1,J}^{n,i}$ such that n_j's vary in $[-\alpha, \alpha]$ and

$$\frac{i}{d_J} - \frac{n}{d_{J_1}} + \frac{np + \sum_{j\neq J_1} n_j - i}{d_{J_1}} + \sum_{j\neq J_1} \frac{n_j}{d_j} \geq (p-1)n/d_{J_1}.$$

Then by the formula of Lemma 3.1 (1), and for p large enough,

$$ {}^{\alpha}K_{J_1,J}^{n,i} = \gamma^{\frac{i}{d_J} - \frac{n}{d_{J_1}}} \, {}^{\alpha}H_{J_1,J}^{np,i} \equiv \gamma^{\frac{n(p-1)}{d_{J_1}}} W G_{J_1,J}^{n,i} \bmod \gamma^{(p-1)n/d_{J_1}+d+1},$$

where W is a suitable p-adic unit. The rest of the cases are similar. $\square$

Proposition 4.2. *Let notation be as in Lemma 4.1. Let $\mathbf{K} := ({}^{\alpha}K_{J_1,J}^{n,i})$. For p large enough, there are a polynomial Y_k in $\mathbb{Q}'(\vec{P})[\vec{A}]$ and some p-adic unit U such that*

$$\det \mathbf{K}^{[k]} \equiv \gamma^{c_0(p-1)} U Y_k \bmod \gamma^{c_0(p-1)+d+1}.$$

Proof. By Lemmas 4.1 and 3.3(c) we have

$$\det \mathbf{K}^{[k]} \equiv \gamma^{c_0(p-1)} U \det \mathbf{G}^{[k]} \quad \bmod \gamma^{c_0(p-1)+d+1},$$

where $\mathbf{G}^{[k]}$ is the matrix we obtain via replacing ${}^{\alpha}K_{J_1,J}^{n,i}$ by $G_{J_1,J}^{n,i}$ in $\mathbf{K}^{[k]}$, and U is the product of the $U_{J_1,n}$'s for the pairs (J_1, n) whose corresponding row appears in $\mathbf{M}^{[k]}$. Now just set $Y_k = \det \mathbf{G}^{[k]}$. $\square$

Lemma 4.3. *Let $1 \le k \le d-\ell$. (1) For p large enough one has*

$$\operatorname{ord}_p(\det \mathbf{M}^{[k]} - \det {}^p\mathbf{M}^{[k]}) \quad > \quad \frac{s_0}{p-1}. \tag{9}$$

(2) Let α and β be the integers chosen in Lemmas 3.4 and 3.5 (they depend only on d). Then

$$\operatorname{ord}_p(\det {}^p\mathbf{M}^{[k]} - \det \mathbf{K}^{[k]}) \quad > \quad \frac{s_0}{p-1}. \tag{10}$$

Proof. (1) Note that $d \ge k > s_0 - c_0(p-1)$. Note that in ${}^p\mathbf{M}^{[k]}$ the row minimal p-adic order is the same as that for $C'^{n,i}_{J_1,J}$ in Lemma 3.3 (c). By Lemma 3.4, for p large enough one has

$$\operatorname{ord}_p(\det \mathbf{M}^{[k]} - \det {}^p\mathbf{M}^{[k]}) \quad > \quad c_0 + \frac{d}{p-1} \ge \frac{s_0}{p-1}. \tag{11}$$

(2) By Lemma 3.5, one knows that $\operatorname{ord}_p(C'^{n,i}_{J_1,J} - K^{n,i}_{J_1,J}) > \frac{n-1}{d_{J_1}} + \frac{d}{p-1}$. Thus

$$\operatorname{ord}_p(\det {}^p\mathbf{M}^{[k]} - \det \mathbf{K}^{[k]}) \quad > \quad c_0 + \frac{d}{p-1} \ge \frac{s_0}{p-1}, \tag{12}$$

since $s_0 - c_0(p-1) < k$. □

In any formal expansion we group the terms with same p-adic orders together and then write this in increasing order. For any number t in $\mathbb{Q}$ if a term can be written as $\gamma^t u$ for some u with $\operatorname{ord}_p u = 0$, then u is called the γ^t-coefficient of this term. Let $\mathbf{M}(\hat{f})$ denote the specialization of $\mathbf{M}$ at variables $\vec{A}$ by assigning $\vec{A}$ as the Teichmüller lifts of coefficients of $f \bmod \mathcal{P}$ (see [14, Section 1] for more details).

Proposition 4.4. *Let $1 \le k \le d-\ell$. Let $(k, c_0) \in \mathbb{R}^2$ be a vertex of the slope < 1 part of $\mathrm{HP}(\mathbb{A})$, where $1 \le k \le d-\ell$. There is a Zariski dense open subset $\mathcal{U}_k$ defined over $\mathbb{Q}$ in $\mathbb{A}$ such that if $f \in \mathcal{U}_k(\overline{\mathbb{Q}})$ and if $\mathcal{P}$ is a prime ideal in the ring of integers of $\mathbb{Q}(f)$ lying over p, one has $\lim_{p\to\infty} \operatorname{ord}_p \det(\mathbf{M}(\hat{f})^{[k]}) = c_0$.*

Proof. Without loss of generality, we fix the residues of p as above. Consider the γ-expansions of $\det \mathbf{M}^{[k]}, \det {}^p\mathbf{M}^{[k]}$, and $\det \mathbf{K}^{[k]}$. By Lemma 4.3, their γ^{s_0}-coefficients are the same. Proposition 4.2 implies that for p large enough there is a polynomial G in $\mathbb{Q}(\vec{P})[\vec{A}]$ such that the γ^{s_0}-coefficient is congruent to $U\,G \bmod \gamma$ for some p-adic unit U. Moreover, from the proofs of Lemma 4.1 and Proposition 4.2, one observes easily that the monomials of G are a subset of all monomials in the formal expansion of $\det {}^p\mathbf{M}^{[k]}$ (with all $\gamma^{\mathbb{Q}}$-factors squeezed out from its coefficients at appropriate places).

We claim that the γ^{s_0}-coefficient in $\det {}^p\mathbf{M}^{[k]}$ is nonzero because it has a unique monomial (in variable $\vec{A}$) among all monomials of minimal weight in its formal expansion. We first look locally at an arbitrary pole P_J where $1 \le J \le \ell$. By Proposition 3.9 there is a unique local monomial among all terms in $\det D_J^{[k_J]}$ for $J = 1, 2$ and $\det D_J^{[k_J-1]}$ for $J \ge 3$. This local monomial corresponds to a permutation $\sigma_{J,0} \in S_{k_J}$. Note that the composition of these σ_{J,k_J}'s for all J is equal to σ_0 defined in the beginning of the section. Then the unique monomial we desire is precisely the product of these local monomials (see Lemma 3.7 and Proposition 3.8). By the remark in last paragraph, it is not hard to see that $G \neq 0$.

Let $\gamma^{>s_0}$ denote all those terms with p-adic order $> \frac{s_0}{p-1}$. Recall from Lemma 4.3 and Proposition 4.2 that one has the p-adic unit U (as in the above paragraphs) and some polynomials G'_m and G' (in $\mathbb{Q}_p(\vec{\hat{P}})[\vec{A}]$) such that

$$\det(\mathbf{M}^{[k]}) = \sum_{c_0\leq m<s_0} \gamma^m U\, G'_m + \gamma^{s_0} U\, G' + \gamma^{>s_0}$$

and $G' \equiv G \bmod \gamma$ for the polynomial G (same G as in above paragraphs) in $\mathbb{Q}(\vec{P})[\vec{A}]$ independent of p. If $G(f) \not\equiv 0 \bmod \mathcal{P}$ (the specialization of G at f over $\mathbb{Q}(\vec{P})$) then $\mathrm{ord}_p(G'(\hat{f})) = 0$. For $m < s_0$ one has $\mathrm{ord}_p G'_m(\hat{f}) = 0$ or ≥ 1. Thus if $G(f) \neq 0$ then for p large enough one has $c_0 \leq \mathrm{ord}_p(\det \mathbf{M}(\hat{f})^{[k]}) \leq \frac{s_0}{p-1}$. But we already know from the beginning of this section that $0 \leq \frac{s_0}{p-1} - c_0 \leq \frac{d-\ell}{p-1}$ and hence by simple calculus one has that $\lim_{p\to\infty} \mathrm{ord}_p(\det \mathbf{M}(\hat{f})^{[k]}) = c_0$.

Last, taking the norm of G from $\mathbb{Q}(\vec{P})[\vec{A}]$ to $\mathbb{Q}[\vec{A}]$ with the automorphism acting on $\vec{A}$ trivially, one gets a polynomial g in $\mathbb{Q}[\vec{A}]$. Let $\mathcal{V}$ be the complement of the variety defined by $g = 0$ in $\mathbb{A}$. It is Zariski dense in $\mathbb{A}$ because $g \neq 0$. $\square$

5. A TRANSFORMATION THEOREM FROM NEWTON POLYGONS OF α_1 TO α_a

We refer the reader to [8, 5] for basic facts about Serre's theory of completely continuous maps and Fredholm determinants. Let $\mathbb{C}_p$ be the p-adic completion of $\overline{\mathbb{Q}_p}$. For any $\mathbb{C}_p$-Banach spaces E and F that admit orthonormal bases, denote by $\mathscr{C}(E, F)$ the set of completely continuous $\mathbb{C}_p$-linear maps from E to F. We say that a matrix M over $\mathbb{C}_p$ is *nuclear* if there exist a $\mathbb{C}_p$-Banach space E and a $u \in \mathscr{C}(E, E)$ such that M is the matrix of u with respect to some orthonormal basis of E. If $M = (m_{ij})_{i,j\geq 1}$ is a matrix over $\mathbb{C}_p$, then M is nuclear if and only if $\lim_{i\to\infty}(\inf_{j\geq 1} \mathrm{ord}_p m_{i,j}) = +\infty$. Recall $\mathrm{ord}_q(\cdot) = \mathrm{ord}_p(\cdot)/a$ for $q = p^a$.

Lemma 5.1. *Let $\vec{M} = (M_0, M_1, \cdots, M_{a-1})$ be an $a-$tuple of nuclear matrices over $\mathbb{C}_p$. Set*

$$\vec{M}_{[a]} := \begin{pmatrix} 0 & & \cdots & 0 & M_{a-1} \\ M_0 & 0 & & & 0 \\ 0 & M_1 & 0 & & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & M_{a-2} & 0 \end{pmatrix}.$$

Then $\det(1 - (M_{a-1}\cdots M_1 M_0)T^a) = \det(1 - \vec{M}_{[a]}T)$.

Lemma 5.1 follows directly from

Lemma 5.2. *Let $\{E_i\}_{i\in\mathbb{Z}/a\mathbb{Z}}$ be a family of Banach spaces over $\mathbb{C}_p$ that admit orthonormal basis. Set $E = E_0 \oplus E_1 \oplus \cdots \oplus E_{a-1}$ equipped with the supremum norm, that is for $v = (v_0, \ldots, v_{a-1})$ in E one has $||v|| = \max_{i=0}^{a-1} ||v_i||$, where $||\cdot||$ are the norms on E and E_i's, respectively. Let $u_i \in \mathscr{C}(E_i, E_{i+1})$ and set $u \in \mathscr{C}(E, E)$ such that $u|_{E_i} = u_i$. Then*

$$\det(1 - (u_{a-1}\cdots u_1 u_0)T^a) = \det(1 - uT).$$

Proof. By [8, page 77, Corollaire 3] we have $\det(1-uT) = \exp(-\sum_{s=1}^{\infty} \mathrm{Tr}(u^s)T^s/s)$. Notice that for any $s \in \mathbb{Z}_{\geq 1}$, the trace $\mathrm{Tr}((u_{i+a-1}\cdots u_{i+1}u_i)^s)$ is independent of $i \in \mathbb{Z}/a\mathbb{Z}$. Clearly $\mathrm{Tr}(u^s) = 0$ unless $a|s$. Thus

$$\begin{aligned}
\det(1-uT) &= \exp(-\sum_{s=1}^{\infty} \mathrm{Tr}(u^s)T^s/s) = \exp(-\sum_{s=1}^{\infty} \mathrm{Tr}(u^{as})T^{as}/(as)) \\
&= \exp(-\sum_{s=1}^{\infty} \sum_{i\in\mathbb{Z}/a\mathbb{Z}} \mathrm{Tr}((u_{i+a-1}\cdots u_{i+1}u_i)^s))T^{as}/(as)) \\
&= \exp(-\sum_{s=1}^{\infty} \mathrm{Tr}((u_{a-1}\cdots u_1u_0)^s))T^{as}/s) \\
&= \det(1-(u_{a-1}\cdots u_1u_0)T^a).
\end{aligned}$$

This concludes our proof. ☐

Remark 5.3. Lemmas 5.1 and 5.2 still hold when $\mathbb{C}_p$ is replaced by any field K equipped with a nontrivial complete non-Archimedean valuation. But we shall not need this more general fact in the present paper.

For any nuclear matrix $M = (m_{ij})_{i,j\geq 1}$ and $k \in \mathbb{Z}_{\geq 1}$, denote by $M^{[k]}$ the $k\times k$ submatrix of M consisting of its first k rows and columns.

Proposition 5.4. *Let $M = (m_{ij})_{i,j\geq 1}$ be a nuclear matrix over $\mathbb{C}_p$ and let $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Fix $k \in \mathbb{Z}_{\geq 1}$ and denote by C_k the coefficient of T^k in $\det(1-(M^{g^{a-1}}\cdots M^gM)T)$. Denote by $\mathscr{A}$ the set of $k\times k$ submatrices of M contained in the first k rows of M, and denote by $\mathscr{B}$ the set of all other $k\times k$ submatrices of M. Set $t_{\mathscr{A}} = \inf_{W\in\mathscr{A}} \mathrm{ord}_p \det W$ and $t_{\mathscr{B}} = \inf_{W\in\mathscr{B}} \mathrm{ord}_p \det W$. Consider the following conditions:*

(i) $2\mathrm{ord}_p \det M^{[k]} < t_{\mathscr{A}} + t_{\mathscr{B}}$;
(ii) $2\mathrm{ord}_q C_k < t_{\mathscr{A}} + t_{\mathscr{B}}$ *and* $t_{\mathscr{A}} < t_{\mathscr{B}}$;
(iii) $\mathrm{ord}_q C_k = \mathrm{ord}_p \det M^{[k]}$.

Then (i) $\Longleftrightarrow$ (ii)$\Longrightarrow$(iii).

Proof. Notice that $\mathrm{ord}_p \det M^{[k]} \geq t_{\mathscr{A}}$. So (i) is equivalent to

$$\min(\frac{t_{\mathscr{A}}+t_{\mathscr{B}}}{2}, t_{\mathscr{B}}) > \mathrm{ord}_p \det M^{[k]}. \tag{13}$$

It suffices to show that (ii)$\Rightarrow$(13)$\Rightarrow$(iii). Let $\vec{M} := (M, M^g, \cdots, M^{g^{a-1}})$. Then we have $\vec{M}_{[a]}$ in Lemma 5.1 and $\det(1-\vec{M}_{[a]}T) = \det(1-(M^{g^{a-1}}\cdots M^gM)T^a)$. Thus C_k is the coefficient of T^{ak} in $\det(1-\vec{M}_{[a]}T)$, which is the infinite sum of $(-1)^{ak}\det N$ for N running over all principal $ak\times ak$ submatrices of $\vec{M}_{[a]}$. Let N be such a matrix, and let N_s be the intersection of N and M^{g^s} as submatrices of $\vec{M}_{[a]}$ for all $0 \leq s \leq a-1$. It is easy to see that $\det N = (-1)^{(a-1)k}\prod_{0\leq s\leq a-1}\det N_s$ or 0 depending on whether every N_s is a $k\times k$ matrix or not. So we may assume that every N_s is a $k\times k$ matrix. Think of N_s as a submatrix of M^{g^s} from now on. Let $X = \{s : 0 \leq s \leq a-1 \text{ and } (N_s)^{g^{-s}} \in \mathscr{A}\setminus\{M^{[k]}\}\}$ and $Y = \{s : 0 \leq s \leq a-1 \text{ and } (N_s)^{g^{-s}} \in \mathscr{B}\}$. We shall think of the families $\{M^{g^s}\}_{0\leq s\leq a-1}$ and $\{N_s\}_{0\leq s\leq a-1}$ as parameterized by $\mathbb{Z}/a\mathbb{Z}$. Then X and Y are disjoint subsets of $\mathbb{Z}/a\mathbb{Z}$. Since N is principal, the set of the columns of N_s as a subset in $\mathbb{Z}_{\geq 1}$ is exactly the same as the set of the rows of N_{s-1}. Consequently, if $s \in X$, then $s-1 \in Y$. Let $Y' = \{s-1 : s \in X\}$ and $Z = (\mathbb{Z}/a\mathbb{Z}) \setminus (X \cup Y)$. Then $\mathbb{Z}/a\mathbb{Z}$ is the disjoint union of $X \cup Y'$, $Y \setminus Y'$ and Z. If $s \in X$, then $\mathrm{ord}_p(\det N_s \cdot \det N_{s-1}) \geq t_{\mathscr{A}} + t_{\mathscr{B}}$. If $s \in Y \setminus Y'$, then $\mathrm{ord}_p \det N_s \geq t_{\mathscr{B}}$. If $s \in Z$, then $\mathrm{ord}_p \det N_s = \mathrm{ord}_p \det M^{[k]}$. Therefore

$$\mathrm{ord}_q \det N \geq \min(\frac{t_{\mathscr{A}} + t_{\mathscr{B}}}{2}, t_{\mathscr{B}}, \mathrm{ord}_p \det M^{[k]}), \tag{14}$$

and hence

$$\mathrm{ord}_q C_k \geq \min(\frac{t_{\mathscr{A}} + t_{\mathscr{B}}}{2}, t_{\mathscr{B}}, \mathrm{ord}_p \det M^{[k]}). \tag{15}$$

(13)$\Rightarrow$(iii): Clearly there is a unique N with $X = Y = \emptyset$, i.e. $(N_s)^{g^{-s}} = M^{[k]}$ for all $0 \leq s \leq a-1$. Denote it by $\mathcal{N}$. We have $\mathrm{ord}_q \det \mathcal{N} = \mathrm{ord}_p \det M^{[k]}$. If $N \neq \mathcal{N}$, then X or $Y \setminus Y'$ is nonempty and hence from (13) and the derivation of (14) we see that $\mathrm{ord}_q \det N > \mathrm{ord}_p \det M^{[k]}$. Now (iii) follows immediately.

(ii)$\Rightarrow$(13): (13) follows directly from (ii) and (15). $\square$

Theorem 5.5. *Let M, g, k and C_k be as in Proposition 5.4. Let $h_1 \leq h_2 \leq \cdots$ be a non-decreasing sequence in $\mathbb{R}$ satisfying $h_i \leq \inf_{j\geq 1} \mathrm{ord}_p m_{ij}$ for all $i \geq 1$. Consider the following conditions:*

(i) $\mathrm{ord}_p \det M^{[k]} < \sum_{1\leq i\leq k} h_i + \frac{h_{k+1}-h_k}{2}$;

(ii) $\mathrm{ord}_q C_k < \sum_{1\leq i\leq k} h_i + \frac{h_{k+1}-h_k}{2}$;

(iii) $\mathrm{ord}_q C_k = \mathrm{ord}_p \det M^{[k]}$.

Then (i) $\Longleftrightarrow$ (ii)$\Longrightarrow$(iii).

Proof. Let $t_{\mathscr{A}}$ and $t_{\mathscr{B}}$ be as in Proposition 5.4. Then $\sum_{1\leq i\leq k} h_i + \frac{h_{k+1}-h_k}{2} \leq \min(\frac{t_{\mathscr{A}}+t_{\mathscr{B}}}{2}, t_{\mathscr{B}})$. So (i) follows from (ii) and (15). Thus Theorem 5.5 follows from Proposition 5.4. $\square$

Remark 5.6. Theorem 5.5 is a Wan-type theorem in relating the Newton polygon to its tight lower bound Hodge polygon: In [9, Theorem 8], Wan showed that the Newton polygon for α_1 (more precisely, the Fredholm determinant of the nuclear matrix representing α_1 with respect to the specific basis) coincides with the Hodge one if and only if the Newton polygon for α_a does. Our result in Theorem 5.5 generalizes it and says that the Newton polygon for α_1 is close to the Hodge one if and only if the Newton polygon for α_a is.

Proof of Theorem 1.1. For any vertex $(k, c_0) \in \mathbb{R}^2$ (but not the right end point) of the slope < 1 part of $\mathrm{HP}(\mathbb{A})$, where $1 \leq k \leq d - \ell$, let $\mathcal{U}_k$ be the Zariski dense open subset in Proposition 4.4. Let $f \in \mathcal{U}_k(\overline{\mathbb{Q}})$. Then $\lim_{p\to\infty} \mathrm{ord}_p \det(\mathbf{M}(\hat{f})^{[k]}) = c_0$. Recall $\phi(\cdot)$ from the beginning of section 3.1. Say the coefficients of $f \mod \mathcal{P}$ lie in $\mathbb{F}_{p^a}$. Set $M := \mathbf{M}(\hat{f})$ and $h_i := \phi(e_i)$ for all $i \geq 1$ in Theorem 5.5. Notice that $\sum_{1\leq i\leq k} h_i = c_0$. Since (k, c_0) is a vertex of $\mathrm{HP}(\mathbb{A})$, we have $h_{k+1} > h_k$. In particular, when p is large enough, we have $\mathrm{ord}_p \det M^{[k]} < c_0 + \frac{h_{k+1}-h_k}{2}$. Combining this with Lemma 3.3(c), one observes that the hypotheses of Theorem 5.5 are satisfied. Recall the maps α_1 and α_a defined in Lemma 2.9 and section 2.5 of [15]. These maps are not the same as the maps defined in section 2 of this article, but are the specialization of those maps in section 2 at the Teichmüller lifts of coefficients of $f \mod \mathcal{P}$. Then $M^{\tau^{-1}}$ and $M^{\tau^{-1}} \cdots M^{\tau^{-(a-1)}} M^{\tau^{-a}}$ are the matrices of α_1 and α_a (over Ω'_a) with respect to the formal basis $\vec{b}_{\mathrm{w}} = \{1, Z_1^i, \cdots, Z_\ell^i\}_{i\geq 1}$ of $\mathcal{H}$ respectively. Notice that $M^{\tau^a} = M$. By Theorem 5.5 one has $\lim_{p\to\infty} \mathrm{ord}_q C_k = c_0$, where C_k is the coefficient of T^k in $\det(1-(M^{\tau^{a-1}}\cdots M^\tau M)T) = \det(1-(M^{\tau^{-1}}\cdots M^{\tau^{-(a-1)}} M^{\tau^{-a}})T) = \det_{\Omega'_a}(1-\alpha_a T)$. Set $\mathcal{U}$ to be the intersection of $\mathcal{U}_k$ for all such vertices (k, c_0). Then for any $f \in \mathcal{U}(\overline{\mathbb{Q}})$, we have $\lim_{p\to\infty} \mathrm{NP}_q(\det_{\Omega'_a}(1-\alpha_a T) \mod T^{d-\ell+1}) = \mathrm{HP}(\mathbb{A})$. Now Theorem 1.1 follows from Remark 1.3 and the fact that the slope < 1 part of $\mathrm{NP}_p(f)$ coincides with $\mathrm{NP}_q(\det_{\Omega'_a}(1-\alpha_a T) \mod T^{d-\ell+1})$ (see [15, Proposition 2.10]). □

Remark 5.7. (1) Our main result Theorem 1.1 is related but not included in a conjecture of Daqing Wan (see [10, Conjectures 1.12 and 1.14]).

(2) This paper is concerned with the space of all one-variable rational function with fixed poles on the projective line. One naturally wonders if there is a multi-variable generalization of Theorem 1.1. We do not know the answer.

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