BERNOULLICITY OF LOPSIDED PRINCIPAL ALGEBRAIC ACTIONS

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Abstract. We show that the principal algebraic actions of countably infinite groups associated to lopsided elements in the integral group ring satisfying some orderability condition are Bernoulli.

1. Introduction

Actions of countably infinite groups Γ on compact metrizable abelian groups X via continuous automorphisms attracted much attention since the beginning of ergodic theory. There is a natural one-to-one correspondence between such actions and the countable left modules over the integral group ring $\mathbb{Z}\Gamma$ of Γ, whence the name *algebraic actions* for such actions. The algebraic actions automatically preserve the normalized Haar measure μ_X of X [6], thus can be studied as a probability-measure-preserving action $\Gamma \curvearrowright (X, \mu_X)$. In this article we are concerned with the Bernoullicity of $\Gamma \curvearrowright (X, \mu_X)$ for algebraic actions.

For algebraic actions $\Gamma \curvearrowright X$ of $\mathbb{Z}$, the ergodicity, CPE (completely positive entropy), and Bernoullicity of $\Gamma \curvearrowright (X, \mu_X)$ are all equivalent, as shown in a series of papers culminating in [10, 13].

For algebraic actions of $\mathbb{Z}^d$ with $d > 1$, ergodicity and CPE are no longer equivalent, as any ergodic algebraic action of $\mathbb{Z}$ can be treated as an ergodic algebraic action of $\mathbb{Z}^d$ with zero entropy via composing it with the projection $\mathbb{Z}^d \to \mathbb{Z}$ to the first coordinate. Rudolph and Schmidt showed that CPE and Bernoullicity are still equivalent for algebraic actions of $\mathbb{Z}^d$ [19].

Not much is known about Bernoullicity of algebraic actions of general countably infinite amenable groups, even though one has the Ornstein-Weiss theory for Bernoulli actions of such groups [15]. For instance, it is unknown whether essentially free CPE and Bernoullicity are equivalent for algebraic actions of such groups.

For algebraic actions of general countably infinite (possibly non-amenable) groups, very little is known. Despite that much progress on Bernoulli actions was made in the last decade such as the entropy theory for actions of sofic groups [1], the extension of Sinai's theorem about Bernoulli factors [21], and the isomorphism of Bernoulli

Date: July 31, 2022.
2010 *Mathematics Subject Classification.* Primary 37A15, 37A35; Secondary 37B10, 37A20.
Key words and phrases. Algebraic action, Bernoulli action, almost topological conjugation.

actions with equal base entropy [2, 22], some of the key results the Ornstein-Weiss theory do not hold anymore [3]: for example, a result of Popa says that for any countably infinite group Γ with property (T) and any infinite compact metrizable abelian group K, the algebraic action $\Gamma \curvearrowright K^\Gamma/K$ (the quotient group of K^Γ by the closed subgroup of constant points) is not Bernoulli [17, 18], in particular, essentially free factors of Bernoulli actions may fail to be Bernoulli. On the affirmative side, Ornstein and Weiss observed that the algebraic action $\mathbb{F}_2 \curvearrowright (\mathbb{Z}/2\mathbb{Z})^{\mathbb{F}_2}/(\mathbb{Z}/2\mathbb{Z})$ of the free group $\mathbb{F}_2$ with 2 generators is Bernoulli [15]. This was extended to the algebraic action $\Gamma \curvearrowright K^\Gamma/K$ for any free product Γ of finitely many countably infinite amenable groups and any nontrivial compact metrizable abelian group K by Meesschaert, Raum, and Vaes [12].

Recently Lind and Schmidt established the Bernoullicity for an interesting class of algebraic actions [11]. For any f in the integral group ring $\mathbb{Z}\Gamma$ (Section 2.1) one has the induced algebraic Γ-action on the Pontryagin dual X_f of the left $\mathbb{Z}\Gamma$-module $\mathbb{Z}\Gamma/(\mathbb{Z}\Gamma f)$, called the *principal algebraic action* associated to f (Section 2.2). When $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1_{\mathbb{R}}(\Gamma)$ (equivalently when $\Gamma \curvearrowright X_f$ is expansive [4]), for the finite set $S = \{0, \dots, \|f\|_1 - 1\}$ one has a natural continuous surjective Γ-equivariant map $\phi_f : S^\Gamma \to X_f$ (see (4)), which can be thought of as a symbolic cover of $\Gamma \curvearrowright X_f$. When Γ is equipped with a group homomorphism $[\cdot] : \Gamma \to \mathbb{Z}$, a, b are distinct elements of Γ with $[a] = [b] = 1$ (for example, when $\Gamma = \mathbb{F}_2$ with the 2 generators a, b), and $f = M - a - b \in \mathbb{Z}\Gamma$ for some integer $M \geq 3$, Lind and Schmidt showed that the restriction of ϕ_f to $\{0, \dots, M-1\}^\Gamma$ sends the product measure ν_M^Γ to μ_{X_f} and is injective on a conull set, where ν_M is the uniform probability measure on $\{0, \dots, M-1\}$, thus provides an isomorphism between $\Gamma \curvearrowright (\{0, \dots, M-1\}^\Gamma, \nu^\Gamma)$ and $\Gamma \curvearrowright (X_f, \mu_{X_f})$ [11, Theorem 7.1]. A nice feature of this isomorphism is that it is explicit and continuous on the shift space $\{0, \dots, M-1\}^\Gamma$. Lind and Schmidt conjecture that their result holds more generally when $f = M - \sum_{s \in I} f_s s \in \mathbb{Z}\Gamma$, where $I \subseteq \Gamma$ is finite and $[s] \geq 1$ and $f_s > 0$ for every $s \in I$ and $M > \sum_{s \in I} f_s$ [11, Conjecture 8.1].

Hayes extended the factor part of the Lind-Schmidt result to a more general situation [7]. The element $f = \sum_{s \in \Gamma} f_s s \in \mathbb{Z}\Gamma$ is called *lopsided* if there is some $s_0 \in \Gamma$ such that $|f_{s_0}| > \sum_{s \in \Gamma \setminus \{s_0\}} |f_s|$. We say a lopsided f is *positively lopsided* if furthermore there is some right-invariant partial order $\leq$ on Γ such that $s_0 < t$ for every $t \in \Gamma \setminus \{s_0\}$ with $f_t \neq 0$ (in the presence of a homomorphism $[\cdot] : \Gamma \to \mathbb{Z}$ as in [11], one can use the right-invariant partial order given by $s < t$ when $[s] < [t]$). When $f \in \mathbb{Z}\Gamma$ is positively lopsided, Hayes showed that the restriction of ϕ_f to $\{0, \dots, |f_{s_0}| - 1\}^\Gamma$ sends the product measure $\nu^\Gamma_{|f_{s_0}|}$ to μ_{X_f}, thus $\Gamma \curvearrowright (X_f, \mu_{X_f})$ is a factor of a Bernoulli action [7, Corollary 5.2], and conjectures that ϕ_f is an isomorphism between $\Gamma \curvearrowright (\{0, \dots, |f_{s_0}| - 1\}^\Gamma, \nu^\Gamma_{|f_{s_0}|})$ and $\Gamma \curvearrowright (X_f, \mu_{X_f})$ [7, Conjecture 1]. If furthermore Γ is amenable and torsion-free, then using [15] he concludes that $\Gamma \curvearrowright (X_f, \mu_{X_f})$ is Bernoulli [7, Theorem 1.2], though it's still not clear whether the restriction of ϕ_f to $\{0, \dots, |f_{s_0}|-1\}^\Gamma$ is an isomorphism of measure spaces.

In this work we consider not only algebraic actions associated to elements in $\mathbb{Z}\Gamma$, but also algebraic actions associated to square matrices over $\mathbb{Z}\Gamma$, as some new phenomenon shows up. For any $n \in \mathbb{N}$ and $f \in M_n(\mathbb{Z}\Gamma)$, we have the generalized principal algebraic action $\Gamma \curvearrowright X_f$ (see (3)). When $f \in M_n(\mathbb{Z}\Gamma)$ is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$, one still has the continuous Γ-equivariant map $\phi_f : S^\Gamma \to X_f$ for any nonempty finite subset S of $\mathbb{Z}^n$ (see (4)). It turns out that there are two ways to extend lopsidedness to matrices: row lopsidedness and column lopsidedness (Definitions 3.1 and 3.4). When $f \in M_n(\mathbb{Z}\Gamma)$ is row or column lopsided, there is a finite symbol set $S_f \subseteq \mathbb{Z}^n$ (Notation 3.2) which is of the form $\prod_{j=1}^n S_j$ for some $S_1, \dots, S_n \subseteq \mathbb{Z}$ and plays the role of $\{0, \dots, |f_{s_0}| - 1\}$ for lopsided $f \in \mathbb{Z}\Gamma$.

It turns out that Hayes' result holds whenever $f \in M_n(\mathbb{Z}\Gamma)$ is either positively row or column lopsided (Proposition 5.1), while we only know that injectivity holds when f is positively row lopsided:

Theorem 1.1. *Let Γ be a countably infinite group. Let $f \in M_n(\mathbb{Z}\Gamma)$ be positively row lopsided. Let ν be a probability measure on S_f such that its marginal distribution on S_j is the uniform probability measure of S_j for all $j = 1, \dots, n$. Then the set $\{y \in S_f^\Gamma : |\phi_f^{-1}(\phi_f(y)) \cap S_f^\Gamma| > 1\}$ has ν^Γ measure 0. Thus ϕ_f is a conjugation between $\Gamma \curvearrowright (S_f^\Gamma, \nu^\Gamma)$ and $\Gamma \curvearrowright (X_f, (\phi_f)_*\nu^\Gamma)$.*

We do not know whether Theorem 1.1 still holds when $f \in M_n(\mathbb{Z}\Gamma)$ is positively column lopsided.

Given continuous actions of Γ on compact metrizable spaces Y and Z, and Γ-invariant Borel probability measures μ and μ' on Y and Z respectively, one says that $\Gamma \curvearrowright (Y, \mu)$ and $\Gamma \curvearrowright (Z, \mu')$ are *almost topologically conjugate* if there are residual Γ-invariant Borel sets $Y' \subseteq Y$ and $Z' \subseteq Z$ with $\mu(Y') = \mu'(Z') = 1$ and a bimeasurable, bicontinuous Γ-equivariant isomorphism $(Y', \mu) \to (Z', \mu')$. Combining Theorem 1.1 and the result of Hayes, we have the following corollaries.

Corollary 1.2. *Let Γ be a countably infinite group. Let $f \in M_n(\mathbb{Z}\Gamma)$ be positively row lopsided. Let ν be the uniform probability measure on S_f. Then ϕ_f gives rise to an almost topological conjugation between $\Gamma \curvearrowright (S_f^\Gamma, \nu^\Gamma)$ and $\Gamma \curvearrowright (X_f, \mu_{X_f})$.*

The case $n = 1$ of Corollary 1.2 proves the conjectures of Lind-Schmidt and Hayes.

Corollary 1.3. *Let Γ be a countably infinite group. Let $f \in M_n(\mathbb{Z}\Gamma)$ and $h \in M_m(\mathbb{Z}\Gamma)$ be positively row lopsided such that $|S_f| = |S_h|$. Then $\Gamma \curvearrowright (X_f, \mu_{X_f})$ and $\Gamma \curvearrowright (X_h, \mu_{X_h})$ are almost topologically conjugate.*

Corollary 1.2 leaves open the question whether $\Gamma \curvearrowright (X_f, \mu_{X_f})$ is Bernoulli for any torsion-free countably infinite group Γ and any $f \in \mathbb{Z}\Gamma$ invertible in $\ell^1_{\mathbb{R}}(\Gamma)$.

We remark that Hayes also proved his result for some $f \in \mathbb{Z}\Gamma$ with a formal ℓ^2 inverse [7, Corollary 5.2], and conjectures that ϕ_f is also injective on a conull set in this situation [7, Conjecture 1]. Our method does not apply to such case.

This paper is organized as follows. We recall some basic facts about group algebras, algebraic actions, and right-invariant partial orders in Section 2. The notions of row or column lopsided matrices are introduced in Section 3. We prove Theorem 1.1 in Section 4. A proof of Hayes' result is included in Section 5.

Acknowledgements. H.L. was partially supported by NSF grant DMS-1900746. We are grateful to the referee for helpful comments.

2. Preliminaries

In this section we recall some basic facts and set up some notations. Throughout this paper, Γ will be a countably infinite group with identity element e_Γ. For $n \in \mathbb{N}$, we write $[n]$ for $\{1, \dots, n\}$.

2.1. **Group algebras.**

We refer the reader to [16] for general information about group rings.

The *integral group ring* $\mathbb{Z}\Gamma$ of Γ is the set of all finitely supported functions $f : \Gamma \to \mathbb{Z}$. We shall write f as $\sum_{s\in\Gamma} f_s s$, where $f_s \in \mathbb{Z}$ for all $s \in \Gamma$ and $f_s = 0$ for all except finitely many $s \in \Gamma$. The set $\{s \in \Gamma : f_s \neq 0\}$ is denoted by $\mathrm{supp}(f)$. The addition and multiplication of $\mathbb{Z}\Gamma$ are given by

$$\sum_{s\in\Gamma} f_s s + \sum_{s\in\Gamma} g_s s = \sum_{s\in\Gamma} (f_s + g_s)s,$$

and

$$(\sum_{s\in\Gamma} f_s s)\big(\sum_{s\in\Gamma} g_s s\big) = \sum_{t\in\Gamma}\big(\sum_{s\in\Gamma} f_s g_{s^{-1}t}\big)t = \sum_{t\in\Gamma}\big(\sum_{s\in\Gamma} f_{ts^{-1}} g_s\big)t. \tag{1}$$

There is also an involution $f \mapsto f^*$ on $\mathbb{Z}\Gamma$ given by

$$(\sum_{s\in\Gamma} f_s s)^* = \sum_{s\in\Gamma} f_{s^{-1}} s. \tag{2}$$

One has $(f + g)^* = f^* + g^*$ and $(fg)^* = g^* f^*$ for all $f, g \in \mathbb{Z}\Gamma$.

We also have the Banach space $\ell^\infty_{\mathbb{R}}(\Gamma)$ of all bounded functions $\Gamma \to \mathbb{R}$ with the canonical norm $\|\cdot\|_\infty$ and the Banach space $\ell^1_{\mathbb{R}}(\Gamma)$ of all absolutely summable functions $\Gamma \to \mathbb{R}$ with the canonical norm $\|\cdot\|_1$. We shall also write the elements of $\ell^\infty_{\mathbb{R}}(\Gamma)$ and $\ell^1_{\mathbb{R}}(\Gamma)$ formally as $\sum_{s\in\Gamma} f_s s$ with $f_s \in \mathbb{R}$ for all $s \in \Gamma$. Then

$$\big\|\sum_{s\in\Gamma} f_s s\big\|_\infty = \sup_{s\in\Gamma} |f_s|$$

for $f \in \ell^\infty_{\mathbb{R}}(\Gamma)$, and

$$\big\|\sum_{s\in\Gamma} f_s s\big\|_1 = \sum_{s\in\Gamma} |f_s|$$

for $f \in \ell^1_{\mathbb{R}}(\Gamma)$. Note that $\ell^1_{\mathbb{R}}(\Gamma)$ is a $*$-algebra with multiplication and $*$-operations given by (1) and (2) respectively. It is a Banach $*$-algebra in the sense that $\|fg\|_1 \leq \|f\|_1\|g\|_1$ and $\|f^*\|_1 = \|f\|_1$ for all $f, g \in \ell^1_{\mathbb{R}}(\Gamma)$.

For $n \in \mathbb{N}$, we shall write elements $f \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$ as $f = (f^{(km)})_{k,m\in[n]}$. The $*$-operation extends to $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ by $(f^*)^{(km)} = (f^{(mk)})^*$ for all $f \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$ and $k, m \in [n]$. Then we still have $(fg)^* = g^*f^*$ for all $f, g \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$.

On $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ we have the norms $\|\cdot\|_{\infty,1}$ and $\|\cdot\|_{1,\infty}$ given by

$$\|f\|_{\infty,1} := \max_{k\in[n]} \sum_{m\in[n]} \|f^{(km)}\|_1, \text{ and } \|f\|_{1,\infty} := \max_{m\in[n]} \sum_{k\in[n]} \|f^{(km)}\|_1$$

for $f = (f^{(km)})_{k,m\in[n]} \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$. These two norms are equivalent, and are related via $\|f^*\|_{\infty,1} = \|f\|_{1,\infty}$ for all $f \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$. We have $\|fg\|_{\infty,1} \leq \|f\|_{\infty,1} \cdot \|g\|_{\infty,1}$ and $\|fg\|_{1,\infty} \leq \|f\|_{1,\infty} \cdot \|g\|_{1,\infty}$ for all $f, g \in M_n(\ell^1_{\mathbb{R}}(\Gamma))$. Then $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ is a unital Banach algebra under either of these two norms.

For any $f = (f^{(km)})_{k,m\in[n]} \in M_n(\mathbb{Z}\Gamma)$, we put

$$\mathrm{supp}(f) = \{(s, k, m) : k, m \in [n], s \in \mathrm{supp}(f^{(km)})\} \subseteq \Gamma \times [n]^2.$$

2.2. **Algebraic actions.** We refer the reader to [8, 20] for general information about algebraic actions.

For any countable abelian group $\mathcal{M}$, denote by $\widehat{\mathcal{M}}$ the Pontryagin dual of $\mathcal{M}$, i.e. the set of all group homomorphisms $\mathcal{M} \to \mathbb{R}/\mathbb{Z}$. Under the pointwise addition and the topology of pointwise convergence, $\widehat{\mathcal{M}}$ is a compact metrizable abelian group. Up to isomorphism, every compact metrizable abelian group arises this way. We shall denote the pairing $\widehat{\mathcal{M}} \times \mathcal{M} \to \mathbb{R}/\mathbb{Z}$ by $\langle x, a\rangle = x(a)$.

Let $\mathcal{M}$ be a countable left $\mathbb{Z}\Gamma$-module. Then Γ has an induced action on $\widehat{\mathcal{M}}$ via continuous automorphisms determined by

$$\langle sx, sa\rangle = \langle x, a\rangle$$

for all $x \in \widehat{\mathcal{M}}$, $a \in \mathcal{M}$, and $s \in \Gamma$. As an example, for $\mathcal{M} = \mathbb{Z}\Gamma$, we have $\widehat{\mathbb{Z}\Gamma} = \widehat{\bigoplus_{s\in\Gamma} \mathbb{Z}} = \prod_{s\in\Gamma} \mathbb{R}/\mathbb{Z} = (\mathbb{R}/\mathbb{Z})^\Gamma$, and the pairing $\widehat{\mathbb{Z}\Gamma} \times \mathbb{Z}\Gamma \to \mathbb{R}/\mathbb{Z}$ is given by

$$\langle x, g\rangle = \sum_{s\in\Gamma} x_s g_s = (xg^*)_{e_\Gamma},$$

where the product $xg^* \in (\mathbb{R}/\mathbb{Z})^\Gamma$ for $x \in (\mathbb{R}/\mathbb{Z})^\Gamma$ and $g^* \in \mathbb{Z}\Gamma$ is defined using (1). The induced Γ-action on $(\mathbb{R}/\mathbb{Z})^\Gamma$ is the left-shift action given by $(sx)_t = x_{s^{-1}t}$ for all $x \in (\mathbb{R}/\mathbb{Z})^\Gamma$ and $s, t \in \Gamma$. More generally, for any $n \in \mathbb{N}$ and $\mathcal{M} = (\mathbb{Z}\Gamma)^n$, we have $\widehat{(\mathbb{Z}\Gamma)^n} = ((\mathbb{R}/\mathbb{Z})^\Gamma)^n$, and the pairing $\widehat{(\mathbb{Z}\Gamma)^n} \times (\mathbb{Z}\Gamma)^n \to \mathbb{R}/\mathbb{Z}$ is given by

$$\langle x, g\rangle = \sum_{k\in[n], s\in\Gamma} x_{s,k} g_{s,k} = (xg^*)_{e_\Gamma},$$

where we write $x \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^n$ and $g \in (\mathbb{Z}\Gamma)^n$ as row vectors so that g^* is a column vector. For any left $\mathbb{Z}\Gamma$-submodule $\mathcal{J}$ of $(\mathbb{Z}\Gamma)^n$ and $\mathcal{M} = (\mathbb{Z}\Gamma)^n/\mathcal{J}$, the dual $\widehat{\mathcal{M}}$ is the Γ-invariant closed subgroup of $\widehat{(\mathbb{Z}\Gamma)^n}$ consisting of elements annihilating $\mathcal{J}$ under the pairing $\widehat{(\mathbb{Z}\Gamma)^n} \times (\mathbb{Z}\Gamma)^n \to \mathbb{R}/\mathbb{Z}$, i.e.

$$\widehat{(\mathbb{Z}\Gamma)^n/\mathcal{J}} = \{x \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^n : \langle x, g\rangle = 0_{\mathbb{R}/\mathbb{Z}} \text{ for all } g \in \mathcal{J}\}.$$

In particular, for any $f \in M_n(\mathbb{Z}\Gamma)$, we have the corresponding *generalized principal algebraic action* $\Gamma \curvearrowright X_f := \widehat{(\mathbb{Z}\Gamma)^n/((\mathbb{Z}\Gamma)^n f)}$. Note that for any $x \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^n$, one has $(xg^*)_{e_\Gamma} = \langle x, g\rangle = 0_{\mathbb{R}/\mathbb{Z}}$ for all $g \in (\mathbb{Z}\Gamma)^n f$ exactly when $xf^* = 0$. Thus

$$X_f = \{x \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^n : xf^* = 0\}. \tag{3}$$

Let $f \in M_n(\mathbb{Z}\Gamma)$ be invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$. Since $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ is a $*$-algebra, f^* is also invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ with $(f^*)^{-1} = (f^{-1})^*$. Denote by π the quotient map $(\mathbb{R}^\Gamma)^n = \mathbb{R}^{\Gamma\times[n]} \to \mathbb{R}^{\Gamma\times[n]}/\mathbb{Z}^{\Gamma\times[n]} = (\mathbb{R}/\mathbb{Z})^{\Gamma\times[n]} = ((\mathbb{R}/\mathbb{Z})^\Gamma)^n$. For any $y \in (\mathbb{Z}^n)^\Gamma \cap (\ell^\infty_{\mathbb{R}}(\Gamma))^n$, we have $y(f^*)^{-1} \in (\ell^\infty_{\mathbb{R}}(\Gamma))^n$, and from (3) one sees easily that $\pi(y(f^*)^{-1}) \in X_f$. This defines a map $\phi_f : (\mathbb{Z}^n)^\Gamma \cap (\ell^\infty_{\mathbb{R}}(\Gamma))^n \to X_f$ by

$$\phi_f(y) = \pi(y(f^*)^{-1}). \tag{4}$$

This is called the *homoclinic map* since $\phi_f((\mathbb{Z}\Gamma)^n)$ is exactly the group of homoclinic points of X_f, i.e. elements x of X_f satisfying $sx \to 0_{X_f}$ as $\Gamma \ni s \to \infty$. For any nonempty finite set $S \subseteq \mathbb{Z}^n$, the restriction of ϕ_f to S^Γ is a continuous Γ-equivariant map $S^\Gamma \to X_f$.

2.3. **Right-invariant partial order.** A (partial) order $\leq$ on Γ is called *right-invariant* if for any $s, t, \gamma \in \Gamma$ one has $s \leq t$ if and only if $s\gamma \leq t\gamma$. We refer the reader to [5, 9, 14] for general information about groups equipped with right-invariant (partial) orders.

Given a right-invariant partial order $\leq$ on Γ, the set $P_\leq := \{s \in \Gamma : e_\Gamma < s\}$ of positive elements is a semigroup contained in $\Gamma \setminus \{e_\Gamma\}$. Conversely, given any semigroup P contained in $\Gamma \setminus \{e_\Gamma\}$, we have the right-invariant partial order $\leq_P$ on Γ defined by $s \leq_P t$ if and only if $ts^{-1} \in P \cup \{e_\Gamma\}$. It is easily checked that this gives us a 1-1 correspondence between right-invariant partial orders on Γ and semigroups contained in $\Gamma \setminus \{e_\Gamma\}$. A right-invariant partial order $\leq$ is an order on Γ exactly when $\Gamma = P_\leq \cup \{e_\Gamma\} \cup P_\leq^{-1}$.

3. Lopsided matrices

Definition 3.1. Let $n \in \mathbb{N}$. We say $f \in M_n(\mathbb{Z}\Gamma)$ is *row lopsided* if f is of the form $M - g$ with $M, g \in M_n(\mathbb{Z}\Gamma)$ such that $\mathrm{supp}(M) \cap \mathrm{supp}(g) = \emptyset$, $M = \mathrm{diag}(M_1 s_1, \dots, M_n s_n)$ for some $s_1, \dots, s_n \in \Gamma$, $M_1, \dots, M_n \in \mathbb{Z}$, and $g = (g^{(km)})_{k,m\in[n]}$

satisfy

$$|M_k| > \sum_{m\in[n]} \|g^{(km)}\|_1 \tag{5}$$

for each $k \in [n]$. Similarly, we say f is *column lopsided* if instead of (5) we have

$$|M_k| > \sum_{m\in[n]} \|g^{(mk)}\|_1$$

for each $k \in [n]$.

Notation 3.2. Let $f \in M_n(\mathbb{Z}\Gamma)$ be row or column lopsided. Using the notation in Definition 3.1 we put

$$S_f := \prod_{k\in[n]} \{0, \dots, |M_k| - 1\} \subseteq \mathbb{Z}^n.$$

Lemma 3.3. *If $f \in M_n(\mathbb{Z}\Gamma)$ is either row lopsided or column lopsided, then it is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$.*

Proof. Let $f \in M_n(\mathbb{Z}\Gamma)$ be column lopsided. We shall use the notation in Definition 3.1. We have $f = M - g = (I_n - gM^{-1})M$ with $\|gM^{-1}\|_{1,\infty} \leq \max_{k\in[n]} \frac{|M_k|-1}{|M_k|} < 1$. Thus f is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ with

$$f^{-1} = M^{-1}(I_n - gM^{-1})^{-1} = \sum_{l=0}^{\infty} M^{-1}(gM^{-1})^l.$$

Similarly, if $f \in M_n(\mathbb{Z}\Gamma)$ is row lopsided, then using $\|\cdot\|_{\infty,1}$ one can show that f is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$. Another way to see this is that if f is row lopsided, then f^* is column lopsided, so f^* is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$, whence f is invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ with $f^{-1} = ((f^*)^{-1})^*$. □

Definition 3.4. We say a row (resp. column) lopsided $f \in M_n(\mathbb{Z}\Gamma)$ is *positively row (resp. column) lopsided* if, in the notation of Definition 3.1, there is a right-invariant partial order $\leq$ on Γ such that $s_m < t$ for all $t \in \bigcup_{k\in[n]} \mathrm{supp}(g^{(km)})$ and $m \in [n]$, equivalently, there is a semigroup P contained in $\Gamma \setminus \{e_\Gamma\}$ such that $P \supseteq \bigcup_{k,m\in[n]} \mathrm{supp}(g^{(km)}s_m^{-1})$.

Example 3.5. Let $a, b \in \Gamma$ such that the subsemigroup of Γ generated by a and b does not contain e_Γ. Put

$$f = \begin{bmatrix} 7 - 2a^3b^2 & 3a + b^7a \\ 5b^2a^4 & -10 - 3a^8 \end{bmatrix}, \qquad h = \begin{bmatrix} 8a^{-1} - 2a^3b^2a^{-1} & 3ab + b^7ab \\ 5b^2a^3 & -10b - 3a^8b \end{bmatrix}$$

in $M_2(\mathbb{Z}\Gamma)$. Then f is positively row lopsided but not column lopsided, while h is both positively row and column lopsided. We have $S_f = \{0, \dots, 6\} \times \{0, \dots, 9\} \subseteq \mathbb{Z}^2$ and $S_h = \{0, \dots, 7\} \times \{0, \dots, 9\} \subseteq \mathbb{Z}^2$.

4. Bernoullicity

In this section we prove Theorem 1.1. Before moving on to the more complicated situation, it might be helpful for the reader's motivation to have a brief outline of the proof for the following example.

Example 4.1. Let $f = M - f_a a - f_b b \in \mathbb{Z}\Gamma$ such that a and b are distinct elements in some semigroup P contained in $\Gamma \setminus \{e_\Gamma\}$, and $0 < f_a \leq f_b$, $f_a + f_b < M$. Let ν be the uniform probability measure on $S_f = \{0, \dots, M-1\}$. For any subset E of Γ, denote by ν^E the product measure on S_f^E with base measure ν, and by π_E the restriction map $S_f^\Gamma \to S_f^E$. We fix an integer $N \geq M\|(f^*)^{-1}\|_1$, set $V = \{-N, -N+1, \cdots, N-1, N\}^\Gamma \backslash \{0\}$ and

$$Z = \{(y, c) \in S_f^\Gamma \times V \colon y + cf^* \in S_f^\Gamma\}.$$

Denote by φ the projection $S_f^\Gamma \times V \to S_f^\Gamma$. It is easily checked that $\varphi(Z)$ is exactly the set of $y \in S_f^\Gamma$ with $|\phi_f^{-1}(\phi_f(y))| > 1$ (see Lemma 4.3). Then Theorem 1.1 for this f amounts to $\nu^\Gamma(\varphi(Z)) = 0$, which we prove now.

For every $(j, s) \in [N] \times \Gamma$, we put

$$Z_{j,s}^+ = \{(y, c) \in Z : \|c\|_\infty = c_s = j\} \text{ and } Z_{j,s}^- = \{(y, c) \in Z : \|c\|_\infty = -c_s = j\}.$$

Then $\varphi(Z_{j,s}^\dagger)$ is a closed subset of S_f^Γ for every $(j, s) \in [N] \times \Gamma$ and $\dagger \in \{+, -\}$, and

$$Z = \bigcup_{(j,s) \in [N] \times \Gamma, \dagger \in \{+,-\}} Z_{j,s}^\dagger.$$

Denote by $\overline{M-1}$ the element in S_f^Γ taking value $M-1$ at every $s \in \Gamma$. We note that for any $(y, c) \in S_f^\Gamma \times V$ and $(j, s) \in [N] \times \Gamma$, one has $(y, c) \in Z_{j,s}^+$ if and only if $(\overline{M-1} - y, -c) \in Z_{j,s}^-$. Thus we only need to show that $\nu^\Gamma\big(\varphi\big(\bigcup_{(j,s) \in [N] \times \Gamma} Z_{j,s}^+\big)\big) = 0$.

For every $(j, s, i) \in [N] \times \Gamma \times S_f$ and $A \subseteq \{a, b\}$, we set

$$Z_{j,s}^+(A, i) = \big\{(y, c) \in Z_{j,s}^+ : c_{st} = j \text{ for } t \in A, c_{st} \neq j \text{ for } t \in \{a, b\} \backslash A \text{ and } y_s = i\big\}.$$

It is easily checked that $Z_{j,s}^+(\emptyset, i) = \emptyset$ for every $i \in S_f$, whence

$$Z_{j,s}^+ = \bigcup_{i=0}^{M-1} H_{j,s}(i),$$

where $H_{j,s}(i) := Z_{j,s}^+(\{a\}, i) \cup Z_{j,s}^+(\{b\}, i) \cup Z_{j,s}^+(\{a, b\}, i)$. For every $(j, s) \in [N] \times \Gamma$, we note the following three facts.

Fact 1:

$$H_{j,s}(i) = \begin{cases} Z_{j,s}^+(\{a\}, i) \cup Z_{j,s}^+(\{b\}, i) \cup Z_{j,s}^+(\{a, b\}, i), & \text{if } 0 \leq i \leq f_a - 1; \\ Z_{j,s}^+(\{b\}, i) \cup Z_{j,s}^+(\{a, b\}, i), & \text{if } f_a \leq i \leq f_b - 1; \\ Z_{j,s}^+(\{a, b\}, i), & \text{if } f_b \leq i \leq f_a + f_b - 1; \\ \emptyset, & \text{if } f_a + f_b \leq i \leq M - 1. \end{cases} \tag{6}$$

In fact, for every $A \subseteq \{a, b\}$, if there exists $(y, c) \in Z^+_{j,s}(A, i)$, one has $(y+cf^*)_s \in S_f$, then

$$M - 1 \geq y_s + Mc_s - f_a c_{sa} - f_b c_{sb} = i + Mj - \big(\sum_{t\in A} f_t j + \sum_{t\in\{a,b\}\setminus A} f_t c_{st}\big)$$
$$\geq i + Mj - \big(\sum_{t\in A} f_t j + \sum_{t\in\{a,b\}\setminus A} f_t (j-1)\big).$$

Thus if $Z^+_{j,s}(A, i)$ is not empty, then $i \leq \sum_{t\in A} f_t - 1 - (M - f_a - f_b)(j-1) \leq \sum_{t\in A} f_t - 1$ and (6) follows.

Fact 2: Put $\bar{P} = P \cup \{e_\Gamma\}$. For every $C \subseteq \{a, b\}$, denote by $Y_{j,s,C}$ the set of $y \in S_f^{sP}$ satisfying $y \in \pi_{st\bar{P}}\varphi(Z^+_{j,st}) \times S_f^{sP\setminus st\bar{P}}$ for every $t \in C$ and $y \notin \pi_{sa\bar{P}}\varphi(Z^+_{j,st}) \times S_f^{sP\setminus sa\bar{P}}$ for every $t \in \{a, b\}\setminus C$. It is clear that the family $\{Y_{j,s,C} : C \subseteq \{a, b\}\}$ is a finite Borel partition of S_f^{sP}, and for every $A \subseteq \{a, b\}$ and $i \in S_f$ one has

$$\pi_{sP}\varphi(Z^+_{j,s}(A, i)) \subseteq \bigcup_{A\subseteq C\subseteq\{a,b\}} Y_{j,s,C}.$$

Fact 3: For each $t \in \{a, b\}$, one has $\nu^{sP}(Y_{j,s,\{t\}} \cup Y_{j,s,\{a,b\}}) \leq \nu^{st\bar{P}}(\pi_{st\bar{P}}\varphi(Z^+_{j,st}))$, since $Y_{j,s,\{t\}} \cup Y_{j,s,\{a,b\}} \subseteq S_f^{sP\setminus st\bar{P}} \times \pi_{st\bar{P}}\varphi(Z^+_{j,st})$.

By Facts 1–3, for every $(j, s) \in [N] \times \Gamma$ we have

$$\begin{aligned}
\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^+_{j,s})) &= \nu^{s\bar{P}}\Big(\pi_{s\bar{P}}\varphi\big(\bigcup_{i=0}^{M-1} H_{j,s}(i)\big)\Big) \leq \sum_{i=0}^{M-1} \nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(H_{j,s}(i))) \\
&= \sum_{i=0}^{M-1} \nu(\{i\}) \cdot \nu^{sP}(\pi_{sP}\varphi(H_{j,s}(i))) \\
&\overset{\text{Fact 1}}{=} \frac{1}{M}\sum_{i=0}^{f_a-1} \nu^{sP}\Big(\pi_{sP}\varphi\big(Z^+_{j,s}(\{a\}, i) \cup Z^+_{j,s}(\{b\}, i) \cup Z^+_{j,s}(\{a, b\}, i)\big)\Big) \qquad (7)\\
&+ \frac{1}{M}\sum_{i=f_a}^{f_b-1} \nu^{sP}\Big(\pi_{sP}\varphi(Z^+_{j,s}(\{b\}, i) \cup Z^+_{j,s}(\{a, b\}, i))\Big) \\
&+ \frac{1}{M}\sum_{i=f_b}^{f_a+f_b-1} \nu^{sP}\Big(\pi_{sP}\varphi(Z^+_{j,s}(\{a, b\}, i))\Big) \\
&\overset{\text{Fact 2}}{\leq} \frac{1}{M}\sum_{i=0}^{f_a-1} \nu^{sP}(Y_{j,s,\{a\}} \cup Y_{j,s,\{b\}} \cup Y_{j,s,\{a,b\}})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{M}\sum_{i=f_a}^{f_b-1}\nu^{sP}(Y_{j,s,\{b\}}\cup Y_{j,s,\{a,b\}})+\frac{1}{M}\sum_{i=f_b}^{f_a+f_b-1}\nu^{sP}(Y_{j,s,\{a,b\}})\\
&=\frac{1}{M}f_a\cdot\nu^{sP}(Y_{j,s,\{a\}}\cup Y_{j,s,\{a,b\}})+\frac{1}{M}f_b\cdot\nu^{sP}(Y_{j,s,\{b\}}\cup Y_{j,s,\{a,b\}})\\
&\overset{\text{Fact 3}}{\leq}\frac{1}{M}f_a\cdot\nu^{sa\bar{P}}(\pi_{sa\bar{P}}\varphi(Z^+_{j,sa}))+\frac{1}{M}f_b\cdot\nu^{sb\bar{P}}(\pi_{sb\bar{P}}\varphi(Z^+_{j,sb})).
\end{aligned}
$$

For every $j\in[N]$, if we set $p_j=\sup_{s\in\Gamma}\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^+_{j,s}))\geq 0$, by (7) we have $p_j\leq\frac{f_a+f_b}{M}p_j\leq\frac{M-1}{M}p_j$, thus $p_j=0$. Therefore $\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^+_{j,s}))=0$ for every $(j,s)\in[N]\times\Gamma$. This implies $\nu^{\Gamma}(\varphi(Z^+_{j,s}))=0$ for every $(j,s)\in[N]\times\Gamma$.

Now we consider the general situation, where we have to handle the complications caused by the different entries of f and different signs of the coefficients of the entries of f. We start with the following lemma, which will allow us to reduce the study of $\Gamma\curvearrowright X_f$ for row or column lopsided f to another matrix of better shape.

Lemma 4.2. *Let $f\in M_n(\mathbb{Z}\Gamma)$ be invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$ and let $u\in M_n(\mathbb{Z}\Gamma)$ be invertible in $M_n(\mathbb{Z}\Gamma)$. Let S be a nonempty finite subset of $\mathbb{Z}^n$ and put $Y=S^\Gamma$. Then there is a Γ-equivariant isomorphism $\overline{\Phi}:X_f\to X_{fu^{-1}}$ of compact abelian groups such that the diagram*

$$
\begin{tikzcd}
 & Y \arrow[dl, "\phi_f"'] \arrow[dr, "\phi_{fu^{-1}}"] & \\
X_f \arrow[rr, "\overline{\Phi}"] & & X_{fu^{-1}}
\end{tikzcd}
\tag{8}
$$

commutes.

Proof. We have a left $\mathbb{Z}\Gamma$-module isomorphism $\Phi:(\mathbb{Z}\Gamma)^n\to(\mathbb{Z}\Gamma)^n$ sending a to au. Note that $\Phi((\mathbb{Z}\Gamma)^nfu^{-1})=(\mathbb{Z}\Gamma)^nf$. Thus Φ induces a left $\mathbb{Z}\Gamma$-module isomorphism $\Phi':(\mathbb{Z}\Gamma)^n/(\mathbb{Z}\Gamma)^nfu^{-1}\to(\mathbb{Z}\Gamma)^n/(\mathbb{Z}\Gamma)^nf$ sending $a+(\mathbb{Z}\Gamma)^nfu^{-1}$ to $au+(\mathbb{Z}\Gamma)^nf$. At the dual level, it induces an isomorphism $\overline{\Phi}:X_f\to X_{fu^{-1}}$ of compact abelian groups sending x to xu^*, commuting with the Γ-action. Now clearly the diagram (8) commutes. $\square$

Let $f\in M_n(\mathbb{Z}\Gamma)$ be positively row lopsided. We use the notation in Definitions 3.1 and 3.4. Put $u=\mathrm{diag}(\mathrm{sgn}(M_1)s_1,\dots,\mathrm{sgn}(M_n)s_n)\in M_n(\mathbb{Z}\Gamma)$. Note that u is invertible in $M_n(\mathbb{Z}\Gamma)$ with $u^{-1}=u^*$. Then $fu^{-1}=\mathrm{diag}(|M_1|,\dots,|M_n|)-gu^{-1}$ with $|M_k|>\sum_{m\in[n]}\|(gu^{-1})^{(km)}\|_1$ for all $k\in[n]$, and $P\supseteq\bigcup_{k,m\in[n]}\mathrm{supp}((gu^{-1})^{(km)})$. Thus fu^{-1} is positively row lopsided, and $S_f=S_{fu^{-1}}$. By Lemma 4.2 we know that $\phi_f^{-1}(\phi_f(y))\cap S_f^\Gamma=\phi_{fu^{-1}}^{-1}(\phi_{fu^{-1}}(y))\cap S_{fu^{-1}}^\Gamma$ for all $y\in S_f^\Gamma=S_{fu^{-1}}^\Gamma$. Thus Theorem 1.1 holds for f if and only if it holds for fu^{-1}. Therefore we may replace f by fu^{-1}, and assume that $f=M-g$ such that $M=\mathrm{diag}(M_1,\dots,M_n)$, $M_k>\sum_{m\in[n]}\|g^{(km)}\|_1$ for all $k\in[n]$, and $P\supseteq\bigcup_{k,m\in[n]}\mathrm{supp}(g^{(km)})$.

Put

$$Y = S_f^\Gamma, \bar{P} = P \cup \{e_\Gamma\}, \bar{M} = \max_{k\in[n]} M_k, A = \mathrm{supp}(g) \subseteq \Gamma \times [n]^2, L_k = \sum_{m\in[n]} \|g^{(km)}\|_1$$

for $k \in [n]$. Let ν be a probability measure on S_f such that $(\psi_k)_*\nu$ is the uniform probability measure on $\{0, 1, \ldots, M_k - 1\}$ for every $k \in [n]$ where $\psi_k : S_f \to \{0, 1, \ldots, M_k - 1\}$ is the projection. For any subset E of Γ, denote by ν^E the product measure on S_f^E with base measure ν, and by π_E the restriction map $S_f^\Gamma \to S_f^E$. Note that for any closed subset Y' of Y and any $t, s \in \Gamma$, we have

$$\nu^{ts\bar{P}}(\pi_{ts\bar{P}} tY') = \nu^{s\bar{P}}(\pi_{s\bar{P}} Y'). \tag{9}$$

Fix an integer $N \geq \bar{M}\|(f^*)^{-1}\|_{1,\infty}$. Denote by V the set of nonzero elements in $W = (\{-N, -N + 1, \ldots, N\}^\Gamma)^{[n]} \subseteq (\mathbb{Z}^n)^\Gamma \cap (\ell^\infty_\mathbb{R}(\Gamma))^n$, and set

$$Z = \{(y, c) \in Y \times V : y + cf^* \in Y\}.$$

Denote by φ the projection $Y \times V \to Y$. For $z = (z_1, \ldots, z_n) \in (\ell^\infty_\mathbb{R}(\Gamma))^n$, put

$$\|z\|_\infty = \max_{k\in[n]} \|z_k\|_\infty.$$

Lemma 4.3. *We have*

$$\{y \in Y : |\phi_f^{-1}(\phi_f(y)) \cap Y| > 1\} = \varphi(Z).$$

Proof. Let $y, y' \in Y$ be distinct such that $\phi_f(y) = \phi_f(y')$. Then $c := y'(f^*)^{-1} - y(f^*)^{-1}$ is nonzero and is in $(\mathbb{Z}^n)^\Gamma$. Note that

$$\|c\|_\infty \leq \|y' - y\|_\infty \|(f^*)^{-1}\|_{1,\infty} \leq \bar{M}\|(f^*)^{-1}\|_{1,\infty} \leq N.$$

Thus $c \in W$, whence $c \in V$. Since $y' = y + cf^* \in Y$, we have $(y, c) \in Z$, thus $\{y \in Y : |\phi_f^{-1}(\phi_f(y)) \cap Y| > 1\} \subseteq \varphi(Z)$.

Conversely, let $y \in \varphi(Z)$. Say, $(y, c) \in Z$. Then $y + cf^* \in Y$ is not equal to y, and $\phi_f(y) = \phi_f(y + cf^*)$. Therefore $\varphi(Z) \subseteq \{y \in Y : |\phi_f^{-1}(\phi_f(y)) \cap Y| > 1\}$ as desired. $\square$

Note that V and Z are F_σ-subsets of W and $Y \times W$ respectively. Thus $\varphi(Z)$ is an F_σ-subset of Y. For each $(j, s, k) \in [N] \times \Gamma \times [n]$, put

$$Z^+_{j,s,k} = \{(y, c) \in Z : \|c\|_\infty = c_{s,k} = j\} \text{ and } Z^-_{j,s,k} = \{(y, c) \in Z : \|c\|_\infty = -c_{s,k} = j\}.$$

Then $\varphi(Z^\dagger_{j,s,k})$ is a closed subset of Y for each $(j, s, k) \in [N]\times\Gamma\times[n]$ and $\dagger \in \{+, -\}$, and

$$Z = \bigcup_{(j,s,k)\in[N]\times\Gamma\times[n],\dagger\in\{+,-\}} Z^\dagger_{j,s,k}. \tag{10}$$

For any $(j,k)\in[N]\times[n]$, $\dagger\in\{+,-\}$, and $s,t\in\Gamma$, note that $(ty,tc)\in Z^\dagger_{j,ts,k}$ for every $(y,c)\in Z^\dagger_{j,s,k}$. It follows that $\varphi(Z^\dagger_{j,ts,k})=t\varphi(Z^\dagger_{j,s,k})$, whence $\nu^{ts\bar{P}}(\pi_{ts\bar{P}}\varphi(Z^\dagger_{j,ts,k}))=\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^\dagger_{j,s,k}))$ by (9). For each $j\in[N]$, put

$$p_j:=\max_{k\in[n],\dagger\in\{+,-\}}\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^\dagger_{j,s,k}))\geq 0$$

for all $s\in\Gamma$.

The key fact for the proof of Theorem 1.1 is the following lemma.

Lemma 4.4. *For each $j\in[N]$ we have $p_j\leq\frac{\bar{M}-1}{\bar{M}}p_j$.*

Let us show first how to derive Theorem 1.1 from Lemma 4.4.

Proof of Theorem 1.1. From Lemma 4.4 we obtain $p_j=0$ for all $j\in[N]$. For any $(j,s,k)\in[N]\times\Gamma\times[n]$ and $\dagger\in\{+,-\}$, since $\nu^\Gamma(\varphi(Z^\dagger_{j,s,k}))\leq p_j$, we conclude that $\nu^\Gamma(\varphi(Z^\dagger_{j,s,k}))=0$. From (10) we get $\nu^\Gamma(\varphi(Z))=0$. Then Theorem 1.1 follows from Lemma 4.3. $\square$

The rest of this section is devoted to the proof of Lemma 4.4.

Fix $(j,s,k)\in[N]\times\Gamma\times[n]$. Put

$$A_k=\{(a,m):(a,k,m)\in A\}.$$

Then $L_k=\sum_{(a,m)\in A_k}|g_a^{(km)}|$.

Let $B_k\subseteq A_k$. Put $L_{B_k}=\sum_{(a,m)\in B_k}|g_a^{(km)}|$. For $B_k=\emptyset$, we set $L_{B_k}=0$. For $\dagger\in\{+,-\}$, put

$$\begin{aligned}Z^\dagger_{j,s,k,B_k}=\{(y,c)\in Z^\dagger_{j,s,k}:c_{sa,m}=\dagger\mathrm{sgn}(g_a^{(km)})j\text{ for all }(a,m)\in B_k,\\ c_{sa,m}\neq\dagger\mathrm{sgn}(g_a^{(km)})j\text{ for all }(a,m)\in A_k\setminus B_k\},\end{aligned}$$

and

$$Z^\dagger_{j,s,k,B_k,i}=\{(y,c)\in Z^\dagger_{j,s,k,B_k}:y_{s,k}=i\}$$

for each $0\leq i\leq M_k-1$. Then for each $\dagger\in\{+,-\}$ we have

$$Z^\dagger_{j,s,k}=\bigsqcup_{0\leq i\leq M_k-1}\ \bigsqcup_{B_k\subseteq A_k}Z^\dagger_{j,s,k,B_k,i}.\tag{11}$$

Lemma 4.5. *Let $0\leq i\leq M_k-1$ and $B_k\subseteq A_k$. The following hold:*

(1) If $Z^+_{j,s,k,B_k,i}$ is nonempty, then $0\leq i\leq L_{B_k}-1$.
(2) If $Z^-_{j,s,k,B_k,i}$ is nonempty, then $M_k-L_{B_k}\leq i\leq M_k-1$.

Proof. For any $(y,c)\in Z$ we have

$$(cf^*)_{s,k}=c_{s,k}M_k-\sum_{(a,m)\in A_k}c_{sa,m}(g^*)^{(mk)}_{a^{-1}}\tag{12}$$

$$
\begin{aligned}
&= c_{s,k}M_k - \sum_{(a,m)\in A_k} c_{sa,m} g_a^{(km)} \\
&= c_{s,k}M_k - \sum_{(a,m)\in B_k} c_{sa,m} g_a^{(km)} - \sum_{(a,m)\in A_k\setminus B_k} c_{sa,m} g_a^{(km)}.
\end{aligned}
$$

(1). Let $(y,c)\in Z^+_{j,s,k,B_k}$. We have

$$
\begin{aligned}
(cf^*)_{s,k} &\overset{(12)}{=} jM_k - \sum_{(a,m)\in B_k} j|g_a^{(km)}| - \sum_{(a,m)\in A_k\setminus B_k} c_{sa,m} g_a^{(km)} \\
&\geq jM_k - \sum_{(a,m)\in B_k} j|g_a^{(km)}| - \sum_{(a,m)\in A_k\setminus B_k} (j-1)|g_a^{(km)}| \\
&= j(M_k - L_k) + L_{A_k\setminus B_k} \\
&\geq M_k - L_k + L_{A_k\setminus B_k} = M_k - L_{B_k}.
\end{aligned}
$$

Since $y + cf^* \in Y$, we have $y_{s,k} + (cf^*)_{s,k} \leq M_k - 1$, whence

$$0 \leq y_{s,k} \leq L_{B_k} - 1.$$

Therefore $Z^+_{j,s,k,B_k,i} = \emptyset$ unless $0 \leq i \leq L_{B_k} - 1$.

(2). Let $(y,c)\in Z^-_{j,s,k,B_k}$. We have

$$
\begin{aligned}
(cf^*)_{s,k} &\overset{(12)}{=} -jM_k + \sum_{(a,m)\in B_k} j|g_a^{(km)}| - \sum_{(a,m)\in A_k\setminus B_k} c_{sa,m} g_a^{(km)} \\
&\leq -jM_k + \sum_{(a,m)\in B_k} j|g_a^{(km)}| + \sum_{(a,m)\in A_k\setminus B_k} (j-1)|g_a^{(km)}| \\
&= -j(M_k - L_k) - L_{A_k\setminus B_k} \\
&\leq -(M_k - L_k) - L_{A_k\setminus B_k} = -M_k + L_{B_k}.
\end{aligned}
$$

Since $y + cf^* \in Y$, we have $y_{s,k} + (cf^*)_{s,k} \geq 0$, whence

$$M_k - L_{B_k} \leq y_{s,k} \leq M_k - 1.$$

Therefore $Z^-_{j,s,k,B_k,i} = \emptyset$ unless $M_k - L_{B_k} \leq i \leq M_k - 1$. □

For $\dagger \in \{+,-\}$ we introduce two $\mathbb{R}$-valued Borel functions on S_f^{sP} as follows:

$$u_k^\dagger := \sum_{0\leq i\leq M_k-1} \chi_{\pi_{sP}\varphi(\bigcup_{B_k\subseteq A_k} Z^\dagger_{j,s,k,B_k,i})}$$

and

$$h_k^\dagger := \sum_{(a,m)\in A_k} |g_a^{(km)}| \chi_{\pi_{sa\bar{P}}\varphi(Z^{\dagger\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}},$$

where χ_U denotes the characteristic function of a set $U \subseteq S_f^{sP}$.

Lemma 4.6. *We have*

$$\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^+_{j,s,k})) \leq \frac{1}{M_k}\nu^{sP}(u^+_k), \tag{13}$$

and

$$\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^-_{j,s,k})) \leq \frac{1}{M_k}\nu^{sP}(u^-_k). \tag{14}$$

Proof. We prove (13) first. Since $(\psi_k)_*\nu$ is the uniform probability measure on $\{0,\dots,M_k-1\}$, we have $(\psi_k)_*\nu(\{i\}) = \frac{1}{M_k}$ for every $0 \leq i \leq M_k-1$. Thus

$$\begin{aligned}
\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^+_{j,s,k})) &\overset{(11)}{=} \nu^{s\bar{P}}\big(\pi_{s\bar{P}}\varphi\big(\bigcup_{0\leq i\leq M_k-1}\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i}\big)\big)\\
&= \sum_{0\leq i\leq M_k-1}\nu^{s\bar{P}}\big(\pi_{s\bar{P}}\varphi\big(\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i}\big)\big)\\
&\leq \sum_{0\leq i\leq M_k-1}((\psi_k)_*\nu(\{i\}))\cdot\nu^{sP}\big(\pi_{sP}\varphi\big(\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i}\big)\big)\\
&= \frac{1}{M_k}\sum_{0\leq i\leq M_k-1}\nu^{sP}\big(\pi_{sP}\varphi\big(\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i}\big)\big)\\
&= \frac{1}{M_k}\nu^{sP}(u^+_k).
\end{aligned}$$

The inequality (14) is proved in the same way replacing $+$ everywhere by $-$. $\square$

Lemma 4.7. *We have*

$$u^+_k \leq h^+_k, \tag{15}$$

and

$$u^-_k \leq h^-_k. \tag{16}$$

Proof. We prove (15) first. For each $C_k \subseteq A_k$, denote by Y^+_{s,C_k} the set of $y \in S_f^{sP}$ satisfying $y \in \pi_{sa\bar{P}}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}$ for every $(a,m) \in C_k$ and $y \notin \pi_{sa\bar{P}}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}$ for every $(a,m) \in A_k \setminus C_k$. Then the family $\{Y^+_{s,C_k} : C_k \subseteq A_k\}$ is a finite Borel partition of S_f^{sP}.

We have

$$\begin{aligned}
\sum_{C_k\subseteq A_k} L_{C_k}\chi_{Y^+_{s,C_k}} &= \sum_{C_k\subseteq A_k}\sum_{(a,m)\in C_k}|g_a^{(km)}|\chi_{Y^+_{s,C_k}}\\
&= \sum_{(a,m)\in A_k}|g_a^{(km)}|\sum_{(a,m)\in C_k\subseteq A_k}\chi_{Y^+_{s,C_k}}\\
&= \sum_{(a,m)\in A_k}|g_a^{(km)}|\chi_{\pi_{sa\bar{P}}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m})\times S_f^{sP\setminus sa\bar{P}}} = h^+_k.
\end{aligned} \tag{17}$$

For any $0 \leq i \leq M_k - 1$ and $B_k \subseteq A_k$, note that $Z^+_{j,s,k,B_k,i} \subseteq \bigcap_{(a,m)\in B_k} Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}$, whence

$$\begin{aligned}\pi_{sP}\varphi(Z^+_{j,s,k,B_k,i}) &\subseteq \bigcap_{(a,m)\in B_k} \pi_{sP}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m})\\ &\subseteq \bigcap_{(a,m)\in B_k} \pi_{sa\bar{P}}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}\\ &= \bigcup_{B_k\subseteq C_k\subseteq A_k} Y^+_{s,C_k}.\end{aligned}$$

If $Z^+_{j,s,k,B_k,i} \neq \emptyset$ and $B_k \subseteq C_k \subseteq A_k$, then by Lemma 4.5 we have

$$0 \leq i \leq L_{B_k} - 1 \leq L_{C_k} - 1.$$

Therefore for each $0 \leq i \leq M_k - 1$ we have

$$\bigcup_{B_k\subseteq A_k} \pi_{sP}\varphi(Z^+_{j,s,k,B_k,i}) \subseteq \bigcup_{C_k\subseteq A_k, i\leq L_{C_k}-1} Y^+_{s,C_k},$$

and hence

$$\chi_{\pi_{sP}\varphi(\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i})} = \chi_{\bigcup_{B_k\subseteq A_k} \pi_{sP}\varphi(Z^+_{j,s,k,B_k,i})} \leq \sum_{C_k\subseteq A_k, i\leq L_{C_k}-1} \chi_{Y^+_{s,C_k}}.$$

Now we have

$$\begin{aligned}u_k^+ &= \sum_{0\leq i\leq M_k-1} \chi_{\pi_{sP}\varphi(\bigcup_{B_k\subseteq A_k} Z^+_{j,s,k,B_k,i})}\\ &\leq \sum_{0\leq i\leq M_k-1} \sum_{C_k\subseteq A_k, i\leq L_{C_k}-1} \chi_{Y^+_{s,C_k}}\\ &= \sum_{C_k\subseteq A_k} \sum_{0\leq i\leq L_{C_k}-1} \chi_{Y^+_{s,C_k}}\\ &= \sum_{C_k\subseteq A_k} L_{C_k}\chi_{Y^+_{s,C_k}} \overset{(17)}{=} h_k^+.\end{aligned}$$

This proves (15).

Next we prove (16). For each $C_k \subseteq A_k$, denote by Y^-_{s,C_k} the set of $y \in S_f^{sP}$ satisfying $y \in \pi_{sa\bar{P}}\varphi(Z^{-\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}$ for every $(a,m) \in C_k$ and $y \notin \pi_{sa\bar{P}}\varphi(Z^{-\mathrm{sgn}(g_a^{(km)})}_{j,sa,m}) \times S_f^{sP\setminus sa\bar{P}}$ for every $(a,m) \in A_k \setminus C_k$. Then the family $\{Y^-_{s,C_k} : C_k \subseteq A_k\}$ is a finite Borel partition of S_f^{sP}.

We have

$$\sum_{C_k\subseteq A_k} L_{C_k}\chi_{Y^-_{s,C_k}} = \sum_{C_k\subseteq A_k} \sum_{(a,m)\in C_k} |g_a^{(km)}|\chi_{Y^-_{s,C_k}} \tag{18}$$

$$
\begin{aligned}
&= \sum_{(a,m)\in A_k} |g_a^{(km)}| \sum_{(a,m)\in C_k\subseteq A_k} \chi_{Y^-_{s,C_k}} \\
&= \sum_{(a,m)\in A_k} |g_a^{(km)}| \chi_{\pi_{sa\bar{P}}\varphi(Z_{j,sa,m}^{-\mathrm{sgn}(g_a^{(km)})})\times S_f^{sP\setminus sa\bar{P}}} = h_k^-.
\end{aligned}
$$

For any $0 \leq i \leq M_k - 1$ and $B_k \subseteq A_k$, note that $Z^-_{j,s,k,B_k,i} \subseteq \bigcap_{(a,m)\in B_k} Z_{j,sa,m}^{-\mathrm{sgn}(g_a^{(km)})}$, whence

$$
\begin{aligned}
\pi_{sP}\varphi(Z^-_{j,s,k,B_k,i}) &\subseteq \bigcap_{(a,m)\in B_k} \pi_{sP}\varphi(Z_{j,sa,m}^{-\mathrm{sgn}(g_a^{(km)})}) \\
&\subseteq \bigcap_{(a,m)\in B_k} \pi_{sa\bar{P}}\varphi(Z_{j,sa,m}^{-\mathrm{sgn}(g_a^{(km)})}) \times S_f^{sP\setminus sa\bar{P}} \\
&= \bigcup_{B_k\subseteq C_k\subseteq A_k} Y^-_{s,C_k}.
\end{aligned}
$$

If $Z^-_{j,s,k,B_k,i} \neq \emptyset$ and $B_k \subseteq C_k \subseteq A_k$, then by Lemma 4.5 we have

$$M_k - 1 \geq i \geq M_k - L_{B_k} \geq M_k - L_{C_k}.$$

Therefore for each $0 \leq i \leq M_k - 1$ we have

$$\bigcup_{B_k\subseteq A_k} \pi_{sP}\varphi(Z^-_{j,s,k,B_k,i}) \subseteq \bigcup_{C_k\subseteq A_k, i\geq M_k - L_{C_k}} Y^-_{s,C_k},$$

and hence

$$\chi_{\pi_{sP}\varphi(\bigcup_{B_k\subseteq A_k} Z^-_{j,s,k,B_k,i})} = \chi_{\bigcup_{B_k\subseteq A_k} \pi_{sP}\varphi(Z^-_{j,s,k,B_k,i})} \leq \sum_{C_k\subseteq A_k, i\geq M_k - L_{C_k}} \chi_{Y^-_{s,C_k}}.$$

Now we have

$$
\begin{aligned}
u_k^- &= \sum_{0\leq i\leq M_k-1} \chi_{\pi_{sP}\varphi(\bigcup_{B_k\subseteq A_k} Z^-_{j,s,k,B_k,i})} \\
&\leq \sum_{0\leq i\leq M_k-1} \sum_{C_k\subseteq A_k, i\geq M_k - L_{C_k}} \chi_{Y^-_{s,C_k}} \\
&= \sum_{C_k\subseteq A_k} \sum_{M_k-1\geq i\geq M_k - L_{C_k}} \chi_{Y^-_{s,C_k}} \\
&= \sum_{C_k\subseteq A_k} L_{C_k} \chi_{Y^-_{s,C_k}} \overset{(18)}{=} h_k^-.
\end{aligned}
$$

This finishes the proof of (16). □

We are ready to prove Lemma 4.4.

Proof of Lemma 4.4. For any $k \in [n]$, we have

$$\begin{aligned}
\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^{+}_{j,s,k})) &\overset{(13)}{\leq} \frac{1}{M_k}\nu^{sP}(u_k^{+}) \\
&\overset{(15)}{\leq} \frac{1}{M_k}\nu^{sP}(h_k^{+}) \\
&= \frac{1}{M_k}\sum_{(a,m)\in A_k} |g_a^{(km)}|\nu^{sa\bar{P}}(\pi_{sa\bar{P}}\varphi(Z^{\mathrm{sgn}(g_a^{(km)})}_{j,sa,m})) \\
&\leq \frac{1}{M_k}\sum_{(a,m)\in A_k} |g_a^{(km)}|p_j \\
&= \frac{L_k}{M_k}p_j \leq \frac{M_k-1}{M_k}p_j \leq \frac{\bar{M}-1}{\bar{M}}p_j,
\end{aligned}$$

and

$$\begin{aligned}
\nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^{-}_{j,s,k})) &\overset{(14)}{\leq} \frac{1}{M_k}\nu^{sP}(u_k^{-}) \\
&\overset{(16)}{\leq} \frac{1}{M_k}\nu^{sP}(h_k^{-}) \\
&= \frac{1}{M_k}\sum_{(a,m)\in A_k} |g_a^{(km)}|\nu^{sa\bar{P}}(\pi_{sa\bar{P}}\varphi(Z^{-\mathrm{sgn}(g_a^{(km)})}_{j,sa,m})) \\
&\leq \frac{1}{M_k}\sum_{(a,m)\in A_k} |g_a^{(km)}|p_j \\
&= \frac{L_k}{M_k}p_j \leq \frac{M_k-1}{M_k}p_j \leq \frac{\bar{M}-1}{\bar{M}}p_j.
\end{aligned}$$

Therefore

$$p_j = \max_{k\in[n],\dagger\in\{+,-\}} \nu^{s\bar{P}}(\pi_{s\bar{P}}\varphi(Z^{\dagger}_{j,s,k})) \leq \frac{\bar{M}-1}{\bar{M}}p_j.$$

□

5. Haar measure

The following result is due to Hayes [7, Corollary 5.2]. Though Hayes only treated the case $n = 1$, his argument there works for any n. For convenience of the reader, we give a proof here.

Proposition 5.1. *Let $f \in M_n(\mathbb{Z}\Gamma)$ be either positively row lopsided or positively column lopsided. Let ν be the uniform probability measure on S_f. Then $(\phi_f)_*\nu^\Gamma = \mu_{X_f}$.*

Lemma 5.2. *Let $f \in M_n(\mathbb{Z}\Gamma)$ be invertible in $M_n(\ell^1_{\mathbb{R}}(\Gamma))$. Let $M_1, \dots, M_n$ be positive integers. Let ν be a probability measure on $S = \prod_{k\in[n]}\{0, 1, \dots, M_k - 1\} \subseteq \mathbb{Z}^n$. Put $\mu = (\phi_f)_*\nu^\Gamma$, as a measure on $((\mathbb{R}/\mathbb{Z})^\Gamma)^n$. For any $h \in (\mathbb{Z}\Gamma)^n$, we have*

$$\widehat{\mu}(h) = \prod_{s\in\Gamma} \widehat{\nu}((hf^{-1})_s).$$

Proof. For any row vector $z \in \mathbb{R}^n$, write z^{t} for the transpose column vector of z. For any $y \in S^\Gamma$, we have

$$\begin{aligned}\exp(-2\pi i \langle \phi_f(y), h\rangle) &= \exp(-2\pi i(\phi_f(y)h^*)_{e_\Gamma}) = \exp(-2\pi i(\pi(y(f^*)^{-1})h^*)_{e_\Gamma}) \\ &= \exp(-2\pi i(y(f^*)^{-1}h^*)_{e_\Gamma}) = \exp(-2\pi i(y(hf^{-1})^*)_{e_\Gamma}) \\ &= \exp\big(-2\pi i \sum_{s\in\Gamma} y_s((hf^{-1})_s)^{\mathrm{t}}\big).\end{aligned}$$

Now

$$\begin{aligned}\widehat{\mu}(h) &= \int_{((\mathbb{R}/\mathbb{Z})^\Gamma)^n} \exp(-2\pi i \langle x, h\rangle)\, d\mu(x) = \int_{S^\Gamma} \exp(-2\pi i \langle \phi_f(y), h\rangle)\, d\nu^\Gamma(y) \\ &= \int_{S^\Gamma} \exp\big(-2\pi i \sum_{s\in\Gamma} y_s((hf^{-1})_s)^{\mathrm{t}}\big)\, d\nu^\Gamma(y) \\ &= \prod_{s\in\Gamma}\int_S \exp(-2\pi i y_s((hf^{-1})_s)^{\mathrm{t}})\, d\nu(y_s) = \prod_{s\in\Gamma} \widehat{\nu}((hf^{-1})_s).\end{aligned}$$

□

Let $f \in M_n(\mathbb{Z}\Gamma)$ be either positively row lopsided or positively column lopsided. We use the notation in Definitions 3.1 and 3.4. Using Lemma 4.2 and arguing as in the paragraph after it, we may assume that $f = M - g$ such that $M = \mathrm{diag}(M_1, \dots, M_n)$, $M_k > \sum_{m\in[n]} \|g^{(km)}\|_1$ (resp. $M_k > \sum_{m\in[n]} \|g^{(mk)}\|_1$) for all $k \in [n]$ when f is positively row (resp. column) lopsided, and $P \supseteq \bigcup_{m,k\in[n]} \mathrm{supp}(g^{(mk)})$.

Lemma 5.3. *Let $h \in (\mathbb{Z}\Gamma)^n \setminus (\mathbb{Z}\Gamma)^n f$. Then there are some $(s, k) \in \Gamma \times [n]$ and $1 \le j \le M_k - 1$ such that $(hf^{-1})_{s,k} - j/M_k \in \mathbb{Z}$.*

Proof. We consider first the case f is positively row lopsided. Let x be the unique element in $[-1/2, 1/2)^{\Gamma\times[n]}$ satisfying that $hf^{-1} - x \in \mathbb{Z}^{\Gamma\times[n]}$. Since $hf^{-1} \in (\ell^1_{\mathbb{R}}(\Gamma))^n$, we have $|(hf^{-1})_{s,k}| < 1/2$ for all except finitely many $(s, k) \in \Gamma \times [n]$. Whenever $|(hf^{-1})_{s,k}| < 1/2$, we have $(hf^{-1})_{s,k} = x_{s,k}$. Thus $hf^{-1} - x \in (\mathbb{Z}\Gamma)^n$. Put $y = hf^{-1} - x$. Then $yf = h - xf$, whence $xf \in (\mathbb{Z}\Gamma)^n$. Moreover, $xf \neq 0$ as $h \notin (\mathbb{Z}\Gamma)^n f$. Thus we can find some $(s_0, k_0) \in \Gamma \times [n]$ such that $(xf)_{s_0,k_0} \neq 0$ and $s_0 \notin tP$ for all $(t, m) \in \Gamma \times [n]$ with $(xf)_{t,m} \neq 0$. From

$$f^{-1} = (M(I_n - M^{-1}g))^{-1} = \sum_{l=0}^{\infty}(M^{-1}g)^l M^{-1} = M^{-1} + \sum_{l=1}^{\infty}(M^{-1}g)^l M^{-1} \tag{19}$$

it is easily checked that $x_{s_0,k_0} = (xff^{-1})_{s_0,k_0} = \frac{1}{M_{k_0}}(xf)_{s_0,k_0} \neq 0$. Since $xf \in (\mathbb{Z}\Gamma)^n$ and $x \in [\frac{-1}{2}, \frac{1}{2})^{\Gamma\times[n]}$, we conclude that $M_{k_0}x_{s_0,k_0}$ is a nonzero integer with absolute value at most $M_{k_0}/2$. Then there is a unique integer $1 \leq j \leq M_{k_0}-1$ with $M_{k_0}x_{s_0,k_0} - j \in M_{k_0}\mathbb{Z}$. Since $(hf^{-1})_{s_0,k_0} - x_{s_0,k_0} \in \mathbb{Z}$, it follows that $(hf^{-1})_{s_0,k_0} - j/M_{k_0} \in \mathbb{Z}$.

The proof for the case of positively column lopsided f is similar, replacing (19) by

$$f^{-1} = ((I_n - gM^{-1})M)^{-1} = M^{-1}\sum_{l=0}^{\infty}(gM^{-1})^l = M^{-1} + M^{-1}\sum_{l=1}^{\infty}(gM^{-1})^l.$$

$\square$

Proof of Proposition 5.1. For $h \in (\mathbb{Z}\Gamma)^n f$, we have $(hf^{-1})_s \in \mathbb{Z}^n$ and thus $\widehat{\nu}((hf^{-1})_s) = 1$ for every $s \in \Gamma$, whence $\widehat{(\phi_f)_*\nu^\Gamma}(h) = 1$ by Lemma 5.2.

Let $h \in (\mathbb{Z}\Gamma)^n \setminus (\mathbb{Z}\Gamma)^n f$. By Lemma 5.3 there are some $(s,k) \in \Gamma\times[n]$ and $1 \leq j \leq M_k - 1$ such that $(hf^{-1})_{s,k} - j/M_k \in \mathbb{Z}$. Since ν is the uniform probability measure on S_f, we have $\widehat{\nu}((hf^{-1})_s) = 0$. By Lemma 5.2 we conclude that $\widehat{(\phi_f)_*\nu^\Gamma}(h) = 0$.

We have shown that $\widehat{(\phi_f)_*\nu^\Gamma}(h) = \widehat{\mu_{X_f}}(h)$ for all $h \in (\mathbb{Z}\Gamma)^n$. Therefore $(\phi_f)_*\nu^\Gamma = \mu_{X_f}$. $\square$

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