ON GROMOV-HAUSDORFF CONVERGENCE FOR OPERATOR METRIC SPACES

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Abstract. We introduce an analogue for Lip-normed operator systems of the second author's order-unit quantum Gromov-Hausdorff distance and prove that it is equal to the first author's complete distance. This enables us to consolidate the basic theory of what might be called operator Gromov-Hausdorff convergence. In particular we establish a completeness theorem and deduce continuity in quantum tori, Berezin-Toeplitz quantizations, and θ-deformations from work of the second author. We show that approximability by Lip-normed matrix algebras is equivalent to 1-exactness of the underlying operator space and, by applying a result of Junge and Pisier, that for $n \geq 7$ the set of isometry classes of n-dimensional Lip-normed operator systems is nonseparable. We also treat the question of generic complete order structure.

1. Introduction

In [24] Marc Rieffel introduced a notion of quantum Gromov-Hausdorff distance for compact quantum metric spaces, which are order-unit spaces equipped with a kind of generalized Lipschitz seminorm called a Lip-norm. One of the principal motivations was to build an analytic framework for explaining the kinds of convergence of spaces in string theory that involve changes of topology (see [24] for a discussion). In addition to Rieffel's analogues of the Gromov completeness and compactness theorems, there have been developed various convergence and continuity results which apply for instance to θ-deformations [15], quantum tori [24, 13], and Berezin quantization [22] (see also [15]).

Given the C^*-algebraic nature of the examples of primary interest and the fact that unital C^*-algebras are not determined by their order-unit structure, Rieffel posed the problem of how to develop a version of quantum Gromov-Hausdorff distance which would incorporate algebraic or matricial information so as to be able to fully distinguish the underlying noncommutative topology. Two different methods for doing this have been independently proposed by the present two authors. Working in the setting of Lip-normed operator systems (or what might be more suggestively dubbed "compact operator metric spaces" in accord with Rieffel's terminology), the first author defined a matricial version of quantum Gromov-Hausdorff distance called *complete distance* which formally elaborates on Rieffel's definition so as to bring the matrix state spaces, and hence the complete order structure, into the picture [10]. The second author meanwhile devised a strategy for quantizing Gromov-Hausdorff distance which operates entirely at the "function" level, in the spirit of noncommutative geometry. This versatile approach was implemented in both the order-unit and C^*-algebraic contexts under the terminology *order-unit* (resp.

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C^*-algebraic) quantum Gromov-Hausdorff distance [15, 14] and affords many technical advantages, as the continuity results in [15, 14] illustrate.

The immediate aim of the present paper is to show that these two approaches become naturally reconciled in the framework of operator systems, in parallel with what happens in the order-unit case [15]. More precisely, we introduce an operator system version of order-unit quantum Gromov-Hausdorff distance called "operator Gromov-Hausdorff distance" and prove that it coincides with complete distance. In fact the methods for treating the order-unit situation can be transferred to the matricial order framework, and so our main task here is to supply the necessary operator-system-theoretic input, including material on amalgamated sums. As a consequence of the equivalence of the two perspectives we can speak unambiguously of "operator Gromov-Hausdorff convergence" and the "operator Gromov-Hausdorff topology".

Exploiting the viewpoint of operator Gromov-Hausdorff distance we establish a completeness theorem and infer continuity in quantum tori, Berezin-Toeplitz quantizations, and θ-deformations by comparison with the second author's distance $\mathrm{dist}_{\mathrm{nu}}$ from [14]. Frédéric Latrémolière's result on quantum Gromov-Hausdorff approximation of quantum tori by finite-dimensional ones [13] is also observed to be valid at the operator level.

Furthermore, we show that certain problems particularly pertinent to our operator version of quantum metric theory can be resolved at or reduced to the purely "topological" level of the operator space structure. More specifically, we prove that a Lip-normed operator system is a limit of Lip-normed matrix algebras if and only if it is 1-exact as an operator space, and, by invoking a result of Marius Junge and Gilles Pisier, that for $n \geq 7$ the set OM_n of (isometry classes of) n-dimensional Lip-normed operator systems is nonseparable. We thereby obtain answers to some questions about complete distance that were left open in [10].

The organization of the paper is as follows. In Section 2 we define and discuss amalgamated sums of operator spaces and systems. Operator Gromov-Hausdorff distance is introduced in Section 3, and amalgamated sums of operator systems are used here for showing that we obtain a metric on the set of isometry classes of Lip-normed operator systems. We also show the Lipschitz equivalence of operator Gromov-Hausdorff distance and complete distance in Section 3. Section 4 treats the completeness theorem. In Section 5 we record the continuity results which follow from comparison with $\mathrm{dist}_{\mathrm{nu}}$ as defined for Lip-normed unital C^*-algebras. We also give here our characterization of matrix approximability for the operator Gromov-Hausdorff topology in terms of 1-exactness, as well as a similar characterization for $\mathrm{dist}_{\mathrm{nu}}$. In the latter case, however, quasidiagonality must be added to 1-exactness, and so we deduce that the operator Gromov-Hausdorff topology is strictly weaker than the $\mathrm{dist}_{\mathrm{nu}}$ topology on the set $C^*\mathrm{M}$ of isometry classes of Lip-normed unital C^*-algebras. We furthermore show that matrix approximability for C^*-algebraic quantum Gromov-Hausdorff distance is equivalent to the C^*-algebra being an MF algebra, so that the C^*-algebraic quantum Gromov-Hausdorff topology on $C^*\mathrm{M}$ is neither weaker nor stronger than the operator Gromov-Hausdorff topology. Section 6 establishes the nonseparability of OM_n for $n \geq 7$. What we in fact demonstrate is that for each $n \geq 7$ there is a nonseparable set of n-dimensional Lip-normed operator systems which are all isometric to each other as compact quantum metric spaces. Finally, in Section 7 we describe the generic complete order structure of a Lip-normed 1-exact operator system under operator Gromov-Hausdorff distance.

Terminology and notation related to quantum and operator metric spaces is established at the beginning of Section 3. For general references on operator spaces and systems see [8, 16, 18]. The notation d_{cb} as appears in Sections 5 and 6 refers to completely bounded Banach-Mazur distance. In this paper we will not assume operator spaces and systems to be complete.

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2. Amalgamated sums of operator spaces and systems

To show that the operator Gromov-Hausdorff distance defined in Section 3 yields a metric on the set of isometry classes of Lip-normed operator systems, we will need the notion of an amalgamated sum of operator systems. Since the methods for dealing with amalgamated sums of operator spaces and operator systems are the same, we will first discuss the former.

In analogy with the full amalgamated free product of C^*-algebras [1], we define the amalgamated sum of operator spaces via a universal property in the category of operator spaces with complete contractions as morphisms:

Definition 2.1. Given operator spaces X, Y, and V and completely isometric linear maps $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the *amalgamated sum of X and Y over V* is an operator space E with complete contractions $\psi_X : X \to E$ and $\psi_Y : Y \to E$ satisfying the following:

(i) $\psi_X \circ \varphi_X = \psi_Y \circ \varphi_Y$,

(ii) whenever F is an operator space and $\pi_X : X \to F$ and $\pi_Y : Y \to F$ are complete contractions satisfying $\pi_X \circ \varphi_X = \pi_Y \circ \varphi_Y$, there is a unique complete contraction $\pi : E \to F$ such that $\pi_X = \pi \circ \psi_X$ and $\pi_Y = \pi \circ \psi_Y$.

We denote E by $X +_V Y$.

Clearly $X +_V Y$ is unique up to complete isometry, if it exists.

The case $V = 0$ is discussed in Section 2.6 of [18].

For any operator space X we denote by $C^*(X)$ the universal C^*-algebra associated to X, meaning that there is a fixed completely isometric embedding $\psi_X : X \hookrightarrow C^*(X)$ such that $\psi_X(X)$ generates $C^*(X)$ as a C^*-algebra and for any C^*-algebra A and complete contraction $\varphi : X \to A$ there exists a (unique) *-homomorphism $\Phi : C^*(X) \to A$ with $\varphi = \Phi \circ \psi_X$ [17, Thm. 3.2] [18, Thm. 8.14]. Identifying X with $\psi(X)$ we may regard X as an operator subspace of $C^*(X)$. If Y is a subspace of X, then the associated *-homomorphism $C^*(Y) \to C^*(X)$ is injective by Wittstock's extension theorem. We have the following result, which is easy to see.

Proposition 2.2. For any completely isometric linear maps $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the sum $X + Y$ in the full amalgamated free product $C^*(X) *_{C^*(V)} C^*(Y)$ is an amalgamated sum $X +_V Y$. Furthermore, $C^*(X +_V Y) = C^*(X) *_{C^*(V)} C^*(Y)$.

Proposition 2.2 generalizes Theorem 8.15 of [18].

Since the canonical *-homomorphisms $C^*(X) \to C^*(X) *_{C^*(V)} C^*(Y)$ and $C^*(Y) \to C^*(X) *_{C^*(V)} C^*(Y)$ are both faithful [1], we get:

Corollary 2.3. For any completely isometric linear maps $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the canonical maps $X \to X +_V Y$ and $Y \to X +_V Y$ are both complete isometries.

The following was pointed out to the second author by Gilles Pisier.

Proposition 2.4 (Pisier)**.** Denote by $X +^{\mathrm{a}}_V Y$ the algebraic amalgamated sum $(X \oplus Y)/\{(\varphi_X(v), -\varphi_Y(v)) : v \in V\}$. If V is closed in both X and Y, then the natural map $\varphi : X +^{\mathrm{a}}_V Y \to X +_V Y$ is a linear isomorphism. If both X and Y are complete, then so is $X +_V Y$.

Proof. Obviously φ is surjective. When $V = 0$, φ is injective and the norm on $X +_0 Y$ is the ℓ_1-norm [18, Sect. 2.6]. Supposing that V is closed in both X and Y, we have that $U = \{\varphi_X(v) - \varphi_Y(v) \in X +_0 Y : v \in V\}$ is a closed subspace of $X +_0 Y$. Clearly $X +_V Y$ is the quotient space $(X +_0 Y)/U$. Thus φ is always injective.

When both X and Y are complete the amalgamated sum $X +_V Y$ does not change if we replace V by its completion. Thus we may assume that V is also complete. Then $X +_V Y = (X +_0 Y)/U$ is complete. □

We now pass to the operator system setting. In this case the morphisms are u.c.p. (unital completely positive) maps. Recall that a *complete order embedding* is a completely positive isometry φ from an operator system X to an operator system Y such that $\varphi^{-1} : \varphi(X) \to X$ is completely positive. A completely positive map from X to Y is a complete order embedding if and only if it is completely isometric. A *complete order isomorphism* is a surjective complete order embedding.

Definition 2.5. Given operator systems X, Y, and V with unital complete order embeddings $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the *amalgamated sum of* X *and* Y *over* V is an operator system E with u.c.p. maps $\psi_X : X \to E$ and $\psi_Y : Y \to E$ satisfying the following:

(i) $\psi_X \circ \varphi_X = \psi_Y \circ \varphi_Y$,
(ii) whenever F is an operator system and $\pi_X : X \to F$ and $\pi_Y : Y \to F$ are u.c.p. maps satisfying $\pi_X \circ \varphi_X = \pi_Y \circ \varphi_Y$, there is a unique u.c.p. map $\pi : E \to F$ such that $\pi_X = \pi \circ \psi_X$ and $\pi_Y = \pi \circ \psi_Y$.

We denote E by $X +_V Y$.

Clearly $X +_V Y$ is unique up to unital complete order isomorphism, if it exists.

For any operator system X we denote by $C^*_{\mathrm{h}}(X)$ the universal unital C^*-algebra associated to X, meaning that there is a fixed unital complete order embedding $\psi_X : X \hookrightarrow C^*_{\mathrm{h}}(X)$ such that $\psi_X(X)$ generates $C^*_{\mathrm{h}}(X)$ as a C^*-algebra and for any unital C^*-algebra A and u.c.p. map $\varphi : X \to A$ there exists a (unique) *-homomorphism $\Phi : C^*_{\mathrm{h}}(X) \to A$ with $\varphi = \Phi \circ \psi_X$ [12, Prop. 8]. Identifying X with $\psi(X)$ we may regard X as an operator subsystem of $C^*_{\mathrm{h}}(X)$. If Y is a subsystem of X, then the associated *-homomorphism

$C^*_{\mathrm{h}}(Y) \to C^*_{\mathrm{h}}(X)$ is injective [12, Prop. 9]. In parallel to Proposition 2.2 we have the following easily verified facts.

Proposition 2.6. For any unital complete order embeddings $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the sum $X + Y$ in the full amalgamated free product $C^*_{\mathrm{h}}(X) *_{C^*_{\mathrm{h}}(V)} C^*_{\mathrm{h}}(Y)$ is an amalgamated sum $X +_V Y$. Furthermore, $C^*_{\mathrm{h}}(X +_V Y) = C^*_{\mathrm{h}}(X) *_{C^*_{\mathrm{h}}(V)} C^*_{\mathrm{h}}(Y)$.

Corollary 2.7. For any unital complete order embeddings $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$, the canonical maps $X \to X +_V Y$ and $Y \to X +_V Y$ are both unital complete order embeddings.

Consider now the algebraic amalgamated sum $X +^{\mathrm{a}}_V Y$ defined as $(X \oplus Y)/\{(\varphi_X(v), -\varphi_Y(v)) : v \in V\}$. Suppose that V is closed in both X and Y. The referee has indicated to us that the linear isomorphism in Proposition 2.4 also holds in the operator system case by the following argument. A matrix ordered space structure may be put on $X +^{\mathrm{a}}_V Y$ by declaring that $(x + y)^* = x^* + y^*$ and that $z \in M_n(X +^{\mathrm{a}}_V Y)$ is positive if for every $\varepsilon > 0$ there exist $x \in M_n(X)_+$ and $y \in M_n(Y)_+$ such that $z + \varepsilon I_n = x + y$. The Choi-Effros axioms for an abstract operator system [6] are readily verified, and we thereby obtain an operator system satisfying the universal property in Definition 2.5. We thus have the analogue of Proposition 2.4:

Proposition 2.8. If V is closed in both X and Y, then the natural map $\varphi : X +^{\mathrm{a}}_V Y \to X +_V Y$ is a linear isomorphism. If both X and Y are complete, then so is $X +_V Y$.

We note finally that, as the following example demonstrates, the operator system and operator space amalgamated sums need not be canonically isometric despite our use of the common notation $X +_V Y$.

Example 2.9. Let $X = Y = M_2(\mathbb{C})$. Let V be the space of diagonal 2×2 matrices and $\varphi_X : V \to X$ and $\varphi_Y : V \to Y$ the natural embeddings. Set

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in X, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in Y.$$

As indicated in the proof of Proposition 2.4, the norm of (x, y) in the operator space amalgamated sum is equal to

$$\inf_{v \in V} \big(\|x + v\| + \|y - v\| \big) = 2.$$

Set

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} p & x \\ x^* & q \end{bmatrix} \in M_2(X)_+, \quad \begin{bmatrix} q & y \\ y^* & p \end{bmatrix} \in M_2(Y)_+.$$

Thus

$$\begin{bmatrix} I & (x,y) \\ (x,y)^* & I \end{bmatrix} = \begin{bmatrix} p & x \\ x^* & q \end{bmatrix} + \begin{bmatrix} q & y \\ y^* & p \end{bmatrix} \in M_2(X +^{\mathrm{a}}_V Y)_+.$$

Since $\left[\begin{smallmatrix} I & w \\ w^* & I \end{smallmatrix}\right] \in M_2(W)_+$ if and only if $\|w\| \leq 1$ for an element w in an operator system W [8, Prop. 1.3.2], we see that the norm of (x, y) in the operator system amalgamated sum is at most 1. On the other hand, using the identity maps $X \to M_2(\mathbb{C})$ and $Y \to M_2(\mathbb{C})$ one sees that the norm of (x, y) in the operator system amalgamated sum is at least 1. Thus the norm of (x, y) in the operator system amalgamated sum is exactly 1.

3. Operator Gromov-Hausdorff distance

The proofs in this section and the next are modeled on ones from the order-unit and C^*-algebraic cases in [15, 14]. In the case that the argument directly translates and the aspects particular to the operator system setting have been dealt with, we will simply refer the reader to [15] or [14].

We begin by establishing some notation and terminology pertaining to quantum and operator metric structures.

We write $\mathrm{dist}_{\mathrm{H}}^{\rho}$ and $\mathrm{dist}_{\mathrm{GH}}$ to designate Hausdorff distance (with respect to a metric ρ, which will be omitted when it is given by a norm on a linear space) and Gromov-Hausdorff distance, respectively.

Given an operator system X we denote by $S_n(X)$ its nth matrix state space, i.e., the set of all unital completely positive maps from X into the matrix algebra M_n. We have a canonical identification $S_n(\overline{X}) = S_n(X)$ where $\overline{X}$ is the completion of X. We write X_{sa} for the set of self-adjoint elements of X.

A *Lip-norm* on an order-unit space A is a seminorm L (which we will allow to take the value $+\infty$) on A such that

(i) $L(a) = 0$ for all $a \in \mathbb{R}1$, and
(ii) the metric ρ_L defined on the state space $S(A)$ of A by

$$\rho_L(\sigma, \omega) = \sup\{|\sigma(a) - \omega(a)| : a \in A \text{ and } L(a) \leq 1\}$$

induces the weak* topology.

Notice that L must in fact vanish precisely on $\mathbb{R}1$. The *closure* of L is the Lip-norm L^{c} on the completion $\overline{A}$ given by

$$L^{\mathrm{c}}(a) = \inf\left\{ \liminf_{n\to\infty} L(a_n) : \{a_n\} \text{ is a sequence in } A \text{ with } \lim_{n\to\infty} a_n = a \right\}.$$

We say that L is *closed* if $L = L^{\mathrm{c}}$. A *compact quantum metric space* [24, Defn. 2.2] is a pair (A, L) consisting of an order-unit space A equipped with a Lip-norm L (by permitting infinite values we are not strictly observing the definition from [24], but this does not cause any problems since we can always restrict to the subspace on which L is finite if we wish).

Let (A, L_A) and (B, L_B) be compact quantum metric spaces. We say that (A, L_A) and (B, L_B) are *isometric* if there is an isometry from (A, L_A) to (B, L_B), i.e., a unital order isomorphism $\varphi : \overline{A} \to \overline{B}$ such that $L_X^{\mathrm{c}} = L_Y^{\mathrm{c}} \circ \varphi$. A Lip-norm L on the order-unit direct sum $A \oplus B$ is said to be *admissible* if it induces L_A and L_B under the natural quotient maps onto the respective summands. The *quantum Gromov-Hausdorff distance* between (A, L_A) and (B, L_B) is defined by

$$\mathrm{dist}_{\mathrm{q}}(A, B) = \inf \mathrm{dist}_{\mathrm{H}}^{\rho_L}(S(A), S(B))$$

where the infimum is taken over all admissible Lip-norms L on $A \oplus B$. This yields a metric on the set of isometry classes of compact quantum metric spaces [24, Thm. 7.8].

By a *Lip-normed operator system* we mean a pair (X, L) where X is an operator system and L is a Lip-norm on X_{sa} (which is the same as saying that (X_{sa}, L) forms a compact quantum metric space). We will also speak of Lip-normed unital C^*-algebras, Lip-normed exact operator systems, etc., when we need to qualify or specialize the class of operator systems under consideration. The *closure* of (X, L) is the Lip-normed operator system $(\overline{X}, L^{\mathrm{c}})$ where L^{c} is the closure of L on $\overline{X}_{\mathrm{sa}}$. We say that (X, L) is *closed* if it is equal to it closure.

Let (X, L) be a Lip-normed operator system. We denote by $\mathrm{rad}(X)$ its radius, i.e., the common value over $n \in \mathbb{N}$ of the radii of the metrics

$$\rho_{L,n}(\varphi, \psi) = \sup\{\|\varphi(x) - \psi(x)\| : x \in X \text{ and } L(x) \leq 1\}$$

defined on the respective matrix state spaces $S_n(X)$ (see [10, Prop. 2.9]). The closure $(\overline{X}, L^{\mathrm{c}})$ satisfies $\rho_{L^{\mathrm{c}},n} = \rho_{L,n}$ for every n (cf. [21, Sect. 4]). The Lip-norm unit ball $\{x \in X_{\mathrm{sa}} : L(x) \leq 1\}$ will be denoted by $\mathcal{E}(X)$, or $\mathcal{E}(X, L)$ in case of confusion. For $R \geq 0$ we set

$$\mathcal{D}_R(X) = \{x \in X_{\mathrm{sa}} : L(x) \leq 1 \text{ and } \|x\| \leq R\},$$

and in the case $R = \mathrm{rad}(X)$ we will simply write $\mathcal{D}(X)$. We will frequently use the fact that $\mathcal{E}(X) = \mathcal{D}(X) + \mathbb{R}1$ [15, Lemma 4.1].

Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. The *complete distance* between (X, L_X) and (Y, L_Y) is defined by

$$\mathrm{dist}_{\mathrm{s}}(X, Y) = \inf \sup_{n \in \mathbb{N}} \mathrm{dist}_{\mathrm{H}}^{\rho_{L,n}}(S_n(X), S_n(Y))$$

where the infimum is taken over all admissible Lip-norms L on $(X \oplus Y)_{\mathrm{sa}}$ [10, Defn. 3.2]. By an *isometry* from (X, L_X) to (Y, L_Y) we mean a unital complete order isomorphism $\varphi : \overline{X} \to \overline{Y}$ such that $L_X^{\mathrm{c}} = L_Y^{\mathrm{c}} \circ \varphi$ on $\overline{X}_{\mathrm{sa}}$. When there exists an isometry from (X, L_X) to (Y, L_Y), we say that (X, L_X) and (Y, L_Y) are *isometric*. We denote by OM the set of isometry classes of Lip-normed operator systems, and by OM^R the subset consisting of isometry classes of Lip-normed operator systems with radius no bigger than R. The set of isometry classes of Lip-normed unital C^*-algebras will be denoted $C^*\mathrm{M}$. Complete distance defines a metric on OM [10, Thm. 4.10].

In analogy to Definition 4.2 of [15] and Definition 3.3 of [14] we introduce the following notion of distance for Lip-normed operator systems.

Definition 3.1. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. We define the *operator Gromov-Hausdorff distance*

$$\mathrm{dist}_{\mathrm{op}}(X, Y) = \inf \mathrm{dist}_{\mathrm{H}}(h_X(\mathcal{E}(X)), h_Y(\mathcal{E}(Y)))$$

where the infimum is taken over all triples (V, h_X, h_Y) consisting of an operator system V and unital complete order embeddings $h_X : X \to V$ and $h_Y : Y \to V$. We also define

$$\mathrm{dist}'_{\mathrm{op}}(X, Y) = \inf \mathrm{dist}_{\mathrm{H}}(h_X(\mathcal{D}(X)), h_Y(\mathcal{D}(Y)))$$

and, for $R \geq 0$,

$$\mathrm{dist}^R_{\mathrm{op}}(X, Y) = \inf \mathrm{dist}_{\mathrm{H}}(h_X(\mathcal{D}_R(X)), h_Y(\mathcal{D}_R(Y)))$$

with the infima being taken over the same set of triples.

The distance $\mathrm{dist}'_{\mathrm{op}}$ is the more immediate analogue of order-unit and C^*-algebraic quantum Gromov-Hausdorff distance [15, 14]. However, since we are considering unital embeddings in our present context, we can remove the norm restriction to obtain the simpler definition $\mathrm{dist}_{\mathrm{op}}$. Indeed we will show that $\mathrm{dist}_{\mathrm{op}}$ and $\mathrm{dist}'_{\mathrm{op}}$ define Lipschitz equivalent metrics on OM. The reason for the R version is to facilitate the proof of completeness (see Section 4) as well as some arguments involving continuity in continuous fields (see [15, Sect. 7] and the beginning of Section 5 below).

Although $\mathcal{E}(X)$ is not itself totally bounded, it is, as pointed out above, equal to the set of scalar translations of the totally bounded set $\mathcal{D}(X)$, so that for any triple (V, h_X, h_Y) as in Definition 3.1 we have

$$\mathrm{dist_H}(h_X(\mathcal{E}(X)), h_Y(\mathcal{E}(Y))) \leq \mathrm{dist_H}(h_X(\mathcal{D}(X)), h_Y(\mathcal{D}(Y)))$$

and hence $\mathrm{dist}_{\mathrm{op}} \leq \mathrm{dist}'_{\mathrm{op}}$. It is also evident from the definitions that for $R \geq \max(\mathrm{rad}(X), \mathrm{rad}(Y))$ we have $\mathrm{dist}_{\mathrm{op}}(X, Y) \leq \mathrm{dist}^R_{\mathrm{op}}(X, Y)$.

To show that $\mathrm{dist}_{\mathrm{op}}$, $\mathrm{dist}'_{\mathrm{op}}$, and $\mathrm{dist}^R_{\mathrm{op}}$ define metrics on OM, we can proceed in the same manner as in Section 3 of [14], granted that we have the appropriate operator systems facts at hand. We will thus only explicitly indicate what we require at the operator system level and refer the reader to Section 3 of [14] for the main line of argument.

First we have the triangle inequality, for which we make use of amalgamated sums of operator systems and in particular the fact that the summands unitally complete order embed into the sum, as asserted by Corollary 2.7. More precisely, given any operator system V_X (resp. V_Z) containing X and Y (resp. Z and Y) as operator subsystems, we embed everything into the amalgamated sum $V_X +_Y V_Z$ to obtain the desired estimate.

Lemma 3.2. For any Lip-normed operator systems (X, L_X), (Y, L_Y), and (Z, L_Z) we have

$$\mathrm{dist}_{\mathrm{op}}(X, Z) \leq \mathrm{dist}_{\mathrm{op}}(X, Y) + \mathrm{dist}_{\mathrm{op}}(Y, Z),$$
$$\mathrm{dist}'_{\mathrm{op}}(X, Z) \leq \mathrm{dist}'_{\mathrm{op}}(X, Y) + \mathrm{dist}'_{\mathrm{op}}(Y, Z),$$

and, for $R \geq 0$,

$$\mathrm{dist}^R_{\mathrm{op}}(X, Z) \leq \mathrm{dist}^R_{\mathrm{op}}(X, Y) + \mathrm{dist}^R_{\mathrm{op}}(Y, Z).$$

Secondly we must show that two Lip-normed operator systems are distance zero apart if and only if they are isometric. For this we need to consider direct limits of operator systems. Direct limits of operator spaces are discussed on page 39 of [8]. The following lemma is a direct consequence of the abstract characterization of operator systems as matrix order unit spaces by Choi and Effros [6, Thm. 4.4].

Lemma 3.3. Let $\{X_j\}_{j\in J}$ be an inductive system of operator systems where the maps are unital complete order embeddings. The algebraic inductive limit $\varinjlim X_j$ equipped with the natural *-vector space structure, order unit, matricial order structure, and matricial norms is an operator system.

With Corollary 2.7 and Lemma 3.3 at our disposal we now can argue as in the proof of Theorem 3.15 in [14] to deduce that distance zero is equivalent to being isometric. In view of Lemma 3.2 we thereby conclude the following.

Theorem 3.4. The distances $\mathrm{dist}_{\mathrm{op}}$, $\mathrm{dist}'_{\mathrm{op}}$, and $\mathrm{dist}^R_{\mathrm{op}}$ define metrics on OM and OM^R, respectively.

Corollary 3.5. The restrictions of $\mathrm{dist}_{\mathrm{op}}$, $\mathrm{dist}'_{\mathrm{op}}$, and $\mathrm{dist}^R_{\mathrm{op}}$ define metrics on $C^*\mathrm{M}$ and $C^*\mathrm{M}^R$, respectively.

Next we establish several inequalities involving $\mathrm{dist}_{\mathrm{op}}$, ${\mathrm{dist}_{\mathrm{op}}}'$, and $\mathrm{dist}^R_{\mathrm{op}}$, including comparisons with complete distance $\mathrm{dist}_{\mathrm{s}}$ as introduced in [10]. Notice that for any operator system X the pairing between X and $S_n(X)$ gives us a natural u.c.p. map $X \to C(S_n(X), M_n)$. We need the following well-known fact.

Lemma 3.6. Let X be an operator system, and consider the C^*-algebraic direct product $\prod_{n=1}^\infty C(S_n(X), M_n)$, defined as

$$\big\{(a_n)_{n\in\mathbb{N}} : a_n \in C(S_n(X), M_n) \text{ for all } n \text{ and } \sup_{n\in\mathbb{N}} \|a_n\| < \infty\big\}.$$

Then the natural linear map $\varphi : X \to \prod_{n=1}^\infty C(S_n(X), M_n)$ is a unital complete order embedding.

Proof. Clearly φ is a u.c.p. map. Say $X \subseteq \mathcal{B}(\mathcal{H})$. Then $M_m(X) \subseteq \mathcal{B}\big(\bigoplus_{j=1}^m \mathcal{H}\big)$. Let $w \in M_m(X)$ and $\varepsilon > 0$ be given. Then we can find a finite-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ such that $\|qwq\| > \|w\| - \varepsilon$, where $q = \mathrm{diag}(p, \dots, p)$ and p is the orthogonal projection onto $\mathcal{K}$. Say $\dim(\mathcal{K}) = n$. Notice that the map $\psi : X \to \mathcal{B}(\mathcal{K}) = M_n$ sending x to pxp is in $S_n(X)$, and we have

$$\begin{aligned}\|(\mathrm{id}_{M_m} \otimes \varphi)(w)\| &\geq \|((\mathrm{id}_{M_m} \otimes \varphi)(w))(\psi)\| = \|(\mathrm{id}_{M_m} \otimes \psi)(w)\| \\ &= \|qwq\| > \|w\| - \varepsilon.\end{aligned}$$

Thus $\|(\mathrm{id}_{M_m} \otimes \varphi)(w)\| = \|w\|$, and hence φ is a complete isometry. □

Theorem 3.7. For any Lip-normed operator systems (X, L_X) and (Y, L_Y) we have $\mathrm{dist}_{\mathrm{op}}(X, Y) = \mathrm{dist}_{\mathrm{s}}(X, Y)$.

Proof. Apply the same argument as in the proof of Proposition 4.7 in [15], only this time using Lemma 3.6 and substituting Arveson's extension theorem for the Hahn-Banach theorem.

□

Using Theorem 3.7 we get a new proof of Theorem 4.10(ii) in [10].

The following is the analogue of Proposition 4.8 in [15] and Proposition 3.9 in [14].

Proposition 3.8. For any Lip-normed operator systems (X, L_X) and (Y, L_Y) we have

$$|\mathrm{rad}(X) - \mathrm{rad}(Y)| \leq \mathrm{dist}_{\mathrm{GH}}(\mathcal{D}(X), \mathcal{D}(Y)) \leq \mathrm{dist}'_{\mathrm{op}}(X, Y) \leq \mathrm{rad}(X) + \mathrm{rad}(Y), \tag{1}$$

$$|\mathrm{dist}'_{\mathrm{op}}(X, Y) - \mathrm{dist}^{\mathrm{rad}(X)}_{\mathrm{op}}(X, Y)| \leq |\mathrm{rad}(X) - \mathrm{rad}(Y)|, \tag{2}$$

$$\mathrm{dist}'_{\mathrm{op}}(X, Y) \leq 3\,\mathrm{dist}_{\mathrm{op}}(X, Y). \tag{3}$$

For $R \geq 0$ we also have

$$\mathrm{dist}^R_{\mathrm{op}}(X, Y) \leq 2\,\mathrm{dist}_{\mathrm{op}}(X, Y). \tag{4}$$

Proof. The proofs of (1) and (2) are similar to those of (5) and (6) of Proposition 4.8 in [15]. By the definition of $\mathrm{dist}_{\mathrm{s}}$ one has $\mathrm{dist}_{\mathrm{q}}(X,Y) \leq \mathrm{dist}_{\mathrm{s}}(X,Y)$. The inequality (3) then follows from (2), (4), and the fact that $|\mathrm{rad}(X) - \mathrm{rad}(Y)| \leq \mathrm{dist}_{\mathrm{q}}(X,Y)$. The proof of (4) parallels those of (8) of Proposition 4.8 in [15] and (6) of Proposition 3.9 in [14]. □

Putting together Theorem 3.7, Proposition 3.8, and the observations in the second paragraph after Definition 3.1, we have proved:

Theorem 3.9. The metrics $\mathrm{dist}_{\mathrm{s}}$, $\mathrm{dist}_{\mathrm{op}}$, and $\mathrm{dist}'_{\mathrm{op}}$ are Lipschitz equivalent on OM, and on OM^R they are Lipschitz equivalent to $\mathrm{dist}^R_{\mathrm{op}}$.

As a consequence of Theorem 3.9 we can speak of *operator Gromov-Hausdorff convergence* and the *operator Gromov-Hausdorff topology* without any ambiguity. Throughout the rest of the paper we will typically use the operator Gromov-Hausdorff distance $\mathrm{dist}_{\mathrm{op}}$ in the formulation of convergence and completeness results with the tacit understanding that these apply equally well to the complete distance $\mathrm{dist}_{\mathrm{s}}$, as well as to $\mathrm{dist}'_{\mathrm{op}}$.

We also have the following two facts, which can be established along the lines of the proofs of Theorem 3.16 and Proposition 3.17, respectively, in [14]. We denote by CM the set of isometry classes of compact metric spaces, and for a compact metric space (X,ρ) we write L_ρ for the associated Lipschitz seminorm on $C(X)$.

Theorem 3.10. The map $(X,\rho) \mapsto (C(X), L_\rho)$ is a homeomorphism from $(\mathrm{CM}, \mathrm{dist}_{\mathrm{GH}})$ onto a closed subspace of $(\mathrm{OM}, \mathrm{dist}_{\mathrm{op}})$.

Proposition 3.11. Let (X,ρ_X) and (Y,ρ_Y) be compact metric spaces. For any $R \geq 0$ we have

$$\mathrm{dist}^R_{\mathrm{op}}(C(X), C(Y)) \leq \mathrm{dist}_{\mathrm{GH}}(X,Y).$$

4. Completeness

We establish in this section a completeness theorem for operator Gromov-Hausdorff distance.

One way to obtain a Lip-normed unital C^*-algebra is to restrict the Lip-norm of a C^*-algebraic compact quantum metric space [14, Defn. 2.2]. The latter type of Lip-norm is a complex scalar version of an order-unit Lip-norm which is defined on the whole C^*-algebra (but possibly taking the value $+\infty$) and required to be adjoint invariant, vanish precisely on $\mathbb{C}1$, and induce the weak* topology on the state space via the associated metric (defined the same way as in the order-unit case). Such Lip-norms appear naturally in many examples (e.g., quantum metrics arising from ergodic actions of compact groups [20]), and in the C^*-algebraic case one may wish to study convergence questions in the presence of the Leibniz rule or generalizations thereof such as the F-Leibniz property, which we now recall (see [10, Sect. 5] [14, Sect. 4]).

Let $F : \mathbb{R}^4_+ \to \mathbb{R}_+$ be a continuous function which is nondescreasing with respect to the partial order on $\mathbb{R}^4_+$ under which $(x_1,x_2,x_3,x_4) \leq (y_1,y_2,y_3,y_4)$ if and only $x_j \leq y_j$ for each j. A C^*-algebraic compact quantum metric space (A,L) is said to satisfy the *F-Leibniz property* if

$$L(ab) \leq F(L(a), L(b), \|a\|, \|b\|)$$

for all $a,b \in A$. Note that taking $F(x_1,x_2,x_3,x_4) = x_1x_4 + x_2x_3$ yields the Leibniz rule.

We write $C^*\mathrm{M}_F$ for the subset of $C^*\mathrm{M}$ whose elements come from C^*-algebraic compact quantum metric spaces satisfying the F-Leibniz property.

Theorem 4.1. The metric space $(\mathrm{OM}, \mathrm{dist}_{\mathrm{op}})$ is complete. Let $F : \mathbb{R}_+^4 \to \mathbb{R}_+$ be a continuous nondecreasing function. Then $(C^*\mathrm{M}_F, \mathrm{dist}_{\mathrm{op}})$ is also complete.

Proof. Let $\{(X_n, L_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathrm{OM}, \mathrm{dist}_{\mathrm{op}})$. We may assume each L_n to be closed. By (1) we have $R := 1 + \sup_{n\in\mathbb{N}} \mathrm{rad}_{X_n} < +\infty$. Thus $\{(X_n, L_n)\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $(\mathrm{OM}^R, \mathrm{dist}^R_{\mathrm{op}})$ by Theorem 3.9. To show that $\{(X_n, L_n)\}_{n\in\mathbb{N}}$ converges, it suffices to show that a subsequence converges under $\mathrm{dist}^R_{\mathrm{op}}$. Thus, passing to a subsequence, we may assume that $\mathrm{dist}^R_{\mathrm{op}}(X_n, X_{n+1}) < 2^{-n}$ for all n. By Corollary 2.7 and Lemma 3.3 we can find a complete operator system V containing all of the X_n as operator subsystems such that $\mathrm{dist}_{\mathrm{H}}(\mathcal{D}_R(X_n), \mathcal{D}_R(X_{n+1})) < 2^{-n}$ for all n. Since V is a complete metric space, the set of non-empty closed compact subsets of V is complete with respect to Hausdorff distance. Denote by W the limit of the sequence $\{\mathcal{D}_R(X_n)\}_{n\in\mathbb{N}}$ with respect to Hausdorff distance.

Since each $\mathcal{D}_R(X_n)$ is $\mathbb{R}$-balanced (i.e., $\lambda x \in \mathcal{D}_R(X_n)$ for all $x \in \mathcal{D}_R(X_n)$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$) and convex, has radius R, and contains 0_V and $R \cdot 1_V$, we can clearly say the same about W. Thus the set $\mathbb{R}_+ \cdot W = \{\lambda w : \lambda \in \mathbb{R}_+, w \in W\}$ is a real linear subspace of V_{sa} containing 1_V. Denote it by B, and denote by X the closure of $B + iB$. Then X is an operator subsystem of V. Notice that Lemmas 4.8 and 4.9 of [14] hold in our current context, that is, (X, L) is a Lip-normed operator system with $\mathrm{rad}(X) \leq R$ and $W = \mathcal{D}_R(B)$, where L is defined by

$$L(x) := \inf\left\{ \limsup_{n\to\infty} L_n(x_n) : x_n \in X_n \text{ for all } n \text{ and } \lim_{n\to\infty} x_n = x \right\} \tag{5}$$

for all $x \in X$. Now we have $\mathrm{dist}^R_{\mathrm{op}}(X_n, X) \leq \mathrm{dist}_{\mathrm{H}}(\mathcal{D}_R(X_n), \mathcal{D}_R(X)) \to 0$ as $n \to \infty$. This proves the first assertion.

Now assume further that (X_n, L_n) lies in $C^*\mathrm{M}_F$ for all n. Let Z be a countable dense subset of $W + iW$. Passing to a subsequence, we may assume that for any $x, y \in Z$ there exist $x_n, y_n \in X_n$ for each n such that $x_n \to x$, $y_n \to y$, and $\{x_n y_n\}_{n\in\mathbb{N}}$ converges to an element in $B + iB$. This is sufficient for Lemma 4.7 of [14] to hold, so that if $x, y \in X$ and we have $x_n, y_n \in X_n$ for each n such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $\{x_n y_n\}_{n\in\mathbb{N}}$ converges to an element in X and the limit depends only on x and y. Then we can define a product on X by setting $x \cdot y$ to be the limit of $\{x_n y_n\}_{n\in\mathbb{N}}$. An argument similar to that after Lemma 4.7 in [14] shows that X becomes a unital C^*-algebra. Since each (X_n, L_n) satisfies the F-Leibniz property, so does (X, L) in view of (5). This proves the second assertion. $\square$

Note that the second assertion of Theorem 4.1 is equivalent to Theorem 5.3 of [10] in view of Theorem 3.7.

5. Continuity, $\mathrm{dist}_{\mathrm{nu}}$, $\mathrm{dist}_{\mathrm{cq}}$, and matrix approximability

In Remark 5.5 of [14] the second author introduced a distance $\mathrm{dist}_{\mathrm{nu}}$ for C^*-algebraic compact quantum metric spaces. We will apply the unital version of the definition to Lip-normed unital C^*-algebras, keeping the same notation. Thus for Lip-normed unital

C^*-algebras (A, L_A) and (B, L_B) we set

$$\mathrm{dist}_{\mathrm{nu}}(A,B) = \inf \mathrm{dist}_{\mathrm{H}}(h_A(\mathcal{E}(A)), h_B(\mathcal{E}(B))),$$
$$\mathrm{dist}'_{\mathrm{nu}}(A,B) = \inf \mathrm{dist}_{\mathrm{H}}(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B))),$$

where the infima are taken over all triples (D, h_A, h_B) consisting of a unital C^*-algebra D and unital faithful *-homomorphisms $h_A : A \to D$ and $h_B : B \to D$. The notation reflects the fact that these distances behave well for unital nuclear C^*-algebras in the context of continuity problems. Applying the same arguments as in the operator system case (see Section 3), it can be shown that $\mathrm{dist}_{\mathrm{nu}}(A,B)$ and $\mathrm{dist}'_{\mathrm{nu}}(A,B)$ define Lipschitz equivalent metrics on the set $C^*\mathrm{M}$ of isometry classes of Lip-normed unital C^*-algebras (more precisely, $\mathrm{dist}_{\mathrm{nu}} \leq \mathrm{dist}'_{\mathrm{nu}} \leq 3\,\mathrm{dist}_{\mathrm{nu}}$), and so it suffices for our purposes to work with the simpler definition $\mathrm{dist}_{\mathrm{nu}}$. It is easily seen that everything in [14, Sect. 5] pertaining to $\mathrm{dist}_{\mathrm{nu}}$ works in the unital situation as well.

It follows from the definitions that for any Lip-normed unital C^*-algebras (A, L_A) and (B, L_B) we have $\mathrm{dist}_{\mathrm{nu}}(A,B) \geq \mathrm{dist}_{\mathrm{op}}(A,B)$. Thus Theorems 5.2 and 5.3 in [14] hold with $\mathrm{dist}_{\mathrm{cq}}$ and $\mathrm{dist}^R_{\mathrm{cq}}$ replaced by $\mathrm{dist}_{\mathrm{op}}$ and $\mathrm{dist}^R_{\mathrm{op}}$, respectively, when T is a compact metric space and each fibre A_t is nuclear. Since C^*-algebras admitting ergodic actions of compact groups are automatically nuclear, we see that Theorem 5.11 and Corollary 5.12 in [14] hold with $\mathrm{dist}_{\mathrm{cq}}$ replaced by $\mathrm{dist}_{\mathrm{op}}$ when T is a compact metric space.

In particular, this gives us continuity with respect to $\mathrm{dist}_{\mathrm{nu}}$ and $\mathrm{dist}_{\mathrm{op}}$ in quantum tori, Berezin-Toeplitz quantizations, and θ-deformations (see [14, Sect. 5]). Corollary 2.2.13 and Proposition 3.1.4 in [13] also enable us to conclude approximation of quantum tori by finite quantum tori under $\mathrm{dist}_{\mathrm{nu}}$ and $\mathrm{dist}_{\mathrm{op}}$ (see Theorem 1.0.1 in [13]).

We turn now to the problem of matrix approximability. This will in particular enable us to distinguish the $\mathrm{dist}_{\mathrm{op}}$ and $\mathrm{dist}_{\mathrm{nu}}$ topologies on the set of isometry classes of Lip-normed unital C^*-algebras.

Recall that an operator space X is said to be 1-*exact* if for every finite-dimensional subspace $E \subseteq X$ and $\lambda > 1$ there is an isomorphism α from E onto a subspace of a matrix algebra such that $\|\alpha\|_{\mathrm{cb}} \|\alpha^{-1}\|_{\mathrm{cb}} \leq \lambda$ (i.e., if X is exact with exactness constant 1). This is equivalent to requiring that for every C^*-algebra A and closed two-sided ideal $I \subseteq A$ the natural complete contraction

$$(A \otimes_{\min} X)/(I \otimes_{\min} X) \to (A/I) \otimes_{\min} X$$

is isometric (see [8, Sect. 14.4] or [18, Sect. 17]). An operator system is said to be 1-exact if it is 1-exact as an operator space.

Lemma 5.1. Let X be a 1-exact operator system. Let $\mathcal{H}$ be a Hilbert space and $\iota : X \to \mathcal{B}(\mathcal{H})$ a unital complete order embedding. Then there is a net

$$X \xrightarrow{\varphi_\lambda} M_{n_\lambda} \xrightarrow{\psi_\lambda} \mathcal{B}(\mathcal{H})$$

of unital completely positive maps through matrix algebras such that $\psi_\lambda \circ \varphi_\lambda$ converges pointwise to ι.

Proof. Since X is 1-exact a standard application of Wittstock's extension theorem produces a net

$$X \xrightarrow{\varphi_\lambda} M_{n_\lambda} \xrightarrow{\psi_\lambda} \mathcal{B}(\mathcal{H})$$

of completely contractive maps through matrix algebras such that $\psi_\lambda \circ \varphi_\lambda$ converges pointwise to ι. Applying the construction in the proof of Proposition 3.6 in [19] we can then produce a net

$$X \xrightarrow{\varphi'_\lambda} M_{m_\lambda} \xrightarrow{\psi'_\lambda} \mathcal{B}(\mathcal{H})$$

of completely positive contractive maps through matrix algebras such that $\psi'_\lambda \circ \varphi'_\lambda$ converges pointwise to ι and $\lim_\lambda \|\varphi'_\lambda(1) - 1\| = 0$. We may assume that for all λ we have $\|\varphi'_\lambda(1) - 1\| < 1/2$ and $\|\psi'_\lambda \circ \varphi'_\lambda(1) - 1\| < 1/2$ so that both $\varphi'_\lambda(1)$ and $\psi'_\lambda(1)$ are invertible, which permits us to define the maps

$$\varphi''_\lambda(\cdot) = \varphi'_\lambda(1)^{-1/2}\varphi'_\lambda(\cdot)\varphi'_\lambda(1)^{-1/2},$$
$$\psi''_\lambda(\cdot) = \psi'_\lambda(1)^{-1/2}\psi'_\lambda(\cdot)\psi'_\lambda(1)^{-1/2}.$$

We thereby obtain a net

$$X \xrightarrow{\varphi''_\lambda} M_{m_\lambda} \xrightarrow{\psi''_\lambda} \mathcal{B}(\mathcal{H})$$

of unital completely positive maps such that $\psi''_\lambda \circ \varphi''_\lambda$ converges pointwise to ι, as desired. □

Lemma 5.2. Let X be a separable operator system and (Y, L) a Lip-normed finite-dimensional operator subsystem of X. Let $\varepsilon > 0$. Then there is a Lip-norm L' on X with respect to which we have $\mathrm{dist}_{\mathrm{H}}(\mathcal{E}(X), \mathcal{E}(Y)) \le \varepsilon$.

Proof. Take any Lip-norm L'' on X_{sa}. Since Y_{sa} is finite-dimensional we may assume by scaling L'' if necessary that $L''|_{Y_{\mathrm{sa}}} \le L$. The finite-dimensionality of Y_{sa} also guarantees the existence of a bounded projection $P : X_{\mathrm{sa}} \to Y_{\mathrm{sa}}$ (see for example [2, Lemma 3.2.3]). Define the seminorm L' on X by

$$L'(x) = \max\left(L(P(x)), L''(x), \varepsilon^{-1}\|x - P(x)\|\right).$$

Since L' vanishes precisely on $\mathbb{R}1$, dominates L'', and is finite on a dense subspace of X_{sa}, we deduce that L' is a Lip-norm, so that (X, L') forms a Lip-normed operator system.

Now if $x \in \mathcal{E}(X)$ then $P(x) \in \mathcal{E}(Y)$ and $\|x - P(x)\| \le \varepsilon$. Since $\mathcal{E}(Y) \subseteq \mathcal{E}(X)$ (because $L'|_{Y_{\mathrm{sa}}} = L$), we thus conclude that the Hausdorff distance between $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ is at most ε, as desired. □

Theorem 5.3. For a Lip-normed operator system (X, L) the following are equivalent:

(i) X is 1-exact,

(ii) for every $\varepsilon > 0$ there is a Lip-normed operator system (Y, L') such that Y is an operator subsystem of a matrix algebra and $\mathrm{dist}_{\mathrm{op}}(X, Y) \le \varepsilon$,

(iii) for every $\varepsilon > 0$ there is a Lip-normed matrix algebra (M_n, L') such that $\mathrm{dist}_{\mathrm{op}}(X, M_n) \le \varepsilon$.

Proof. (i)⇒(ii). By Lemma 5.1 there is a unital complete order embedding $\iota : X \to \mathcal{B}(\mathcal{H})$ and a net

$$X \xrightarrow{\varphi_\lambda} M_{n_\lambda} \xrightarrow{\psi_\lambda} \mathcal{B}(\mathcal{H})$$

of unital completely positive maps through matrix algebras such that $\psi_\lambda \circ \varphi_\lambda$ converges pointwise to ι. In view of the equality of $\mathrm{dist}_{\mathrm{op}}$ and $\mathrm{dist}_{\mathrm{s}}$ (Theorem 3.7) we can now proceed as in the proof of Proposition 3.10 in [10] to obtain (ii).

(ii)⇒(iii). Apply Lemma 5.2.

(iii)⇒(i). Let $E \subseteq X$ be a finite-dimensional operator subsystem, set $d = \dim E$, and let $\varepsilon > 0$. Let $\{(x_k, x_k^*)\}_{k=1}^d$ be an Auerbach system for $E_{\rm sa}$ (see for example [8, page 335]). By complexifying we may view this as a biorthogonal system for E with $\|x_k^*\| \leq 2$ for each $k = 1, \dots, d$. For every $k = 1, \dots, d$ choose a $x_k' \in X_{\rm sa}$ with $L(x_k') < \infty$ and $\|x_k - x_k'\| < \varepsilon/d$. Take a $\gamma > 0$ such that for each $k = 1, \dots, d$ we have $\gamma L(x_k') \leq 1$, that is, $\gamma x_k' \in \mathcal{E}(X)$. By (iii) there is a Lip-normed matrix algebra (M_n, L') such that $\mathrm{dist}_{\rm op}(X, M_n) < \gamma\varepsilon/d$. By the definition of $\mathrm{dist}_{\rm op}$ we may view X and M_n as operator subsystems of an operator system V such that for each $k = 1, \dots, d$ there exists a $y_k \in \mathcal{E}(M_n)$ with $\|\gamma x_k' - y_k\| < \gamma\varepsilon/d$. Then

$$\sum_{k=1}^d \|x_k - \gamma^{-1} y_k\| \leq \sum_{k=1}^d \|x_k - x_k'\| + \sum_{k=1}^d \gamma^{-1} \|\gamma x_k' - y_k\| < 2\varepsilon,$$

and thus setting $Y = \mathrm{span}\{y_1, \dots, y_d\} \subseteq M_n$ we conclude by Lemma 2.13.2 of [18] that $d_{\rm cb}(E, Y) \leq (1 + 4\varepsilon)(1 - 4\varepsilon)^{-1}$. Since ε was arbitrary this yields (i). □

Remark 5.4. A perturbation argument as in the proof of (iii)⇒(i) in Theorem 5.3 shows that if a sequence $\{(X_n, L_n)\}_{n\in\mathbb{N}}$ of Lip-normed operator systems converges in the operator Gromov-Hausdorff topology to a Lip-normed operator system (X, L), then for every finite-dimensional subspace $E \subseteq X$ there exist, for all sufficiently large n, subspaces $E_n \subseteq X_n$ with $\dim E_n = \dim E$ such that $\lim_{n\to\infty} d_{\rm cb}(E_n, E) = 1$. In the next section we will require some quantitative information concerning the relationship between $\mathrm{dist}_{\rm op}$ and $d_{\rm cb}$ in the finite-dimensional case (see Lemma 6.1).

The above remark shows in particular that the operator space exactness constant (see [8, 18]) is lower semicontinuous on OM. In other words, if for $\lambda \geq 1$ we denote by $\mathrm{OM}_{\lambda\text{-ex}}$ the set of isometry classes of Lip-normed λ-exact operator systems in OM, then we have:

Proposition 5.5. For each $\lambda \geq 1$, the set $\mathrm{OM}_{\lambda\text{-ex}}$ is closed in OM.

Following the notation of Section 6 of [10], for a Lip-normed operator system (X, L) and $\varepsilon > 0$ we denote by $\mathrm{Afn}_L(\varepsilon)$ the smallest positive integer k such that there is a Lip-normed operator system (Y, L_Y) with Y an operator subsystem of the matrix algebra M_k and $\mathrm{dist}_{\rm s}(X, Y) \leq \varepsilon$, and put $\mathrm{Afn}_L(\varepsilon) = \infty$ if no such k exists. By Theorems 3.9 and 5.3, for a Lip-normed operator system (X, L) we have that X is 1-exact if and only if $\mathrm{Afn}_L(\varepsilon)$ is finite for all $\varepsilon > 0$. In other words, the set $\mathrm{OM}_{\rm ex}$ of isometry classes of Lip-normed 1-exact operator systems coincides with $\mathcal{R}_{\rm fa}$, in the notation of Section 6 in [10]. We can thus restate the compactness theorem of [10] as follows.

Theorem 5.6. [10, Thm. 6.3] Let $\mathcal{C}$ be a subset of $\mathrm{OM}_{\rm ex}$. Then $\mathcal{C}$ is totally bounded if and only if

(i) there is a $D > 0$ such that the diameter of every element of $\mathcal{C}$ is bounded by D, and
(ii) there is a function $F : (0, \infty) \to (0, \infty)$ such that $\mathrm{Afn}_L(\varepsilon) \leq F(\varepsilon)$ for all $(X, L) \in \mathcal{C}$.

To establish the analogue of Theorem 5.3 for $\mathrm{dist}_{\rm nu}$ we will need the following characterization of being quasidiagonal and exact for separable unital C^*-algebras. This combines results of Voiculescu [25] and Dadarlat [7] and the equivalence of exactness and nuclear embeddability due to Kirchberg [11].

Theorem 5.7. A separable unital C^*-algebra A is quasidiagonal and exact if and only if for every finite set $\{x_1, \dots, x_n\} \subseteq A$ and $\varepsilon > 0$ there is a unital C^*-algebra D, a finite-dimensional unital C^*-subalgebra B of D, elements $y_1, \dots, y_n \in B$, and an injective unital *-homomorphism $\Phi : A \to D$ such that $\|\Phi(x_k) - y_k\| < \varepsilon$ for every $k = 1, \dots, n$.

Theorem 5.8. For a Lip-normed unital C^*-algebra (A, L) the following are equivalent:

(i) A is quasidiagonal and exact,

(ii) for every $\varepsilon > 0$ there is a Lip-normed finite-dimensional C^*-algebra (B, L') such that $\mathrm{dist}_{\mathrm{nu}}(A, B) \leq \varepsilon$,

(iii) for every $\varepsilon > 0$ there is a Lip-normed matrix algebra (M_n, L') such that $\mathrm{dist}_{\mathrm{nu}}(A, M_n) \leq \varepsilon$.

Proof. (i)⇒(ii). Let $\varepsilon > 0$. Since $\mathcal{D}(A)$ is totally bounded and $\mathcal{E}(A) = \mathcal{D}(A) + \mathbb{R}1$, we can find a finite-dimensional subspace X of A_{sa} containing 1 such that $\mathrm{dist}_{\mathrm{H}}(\mathcal{E}(A), X \cap \mathcal{E}(A)) \leq \varepsilon/3$. Take a linear basis $\{1, x_1, \dots, x_n\}$ for X.

By hypothesis A is quasidiagonal and exact, and since it admits a Lip-norm it must also be separable. Thus, by Theorem 5.7, given a $\delta > 0$ we may view A as a unital C^*-subalgebra of a unital C^*-algebra D such that there exists a unital finite-dimensional C^*-subalgebra B of D and a $y_x \in B$ with $\|x - y_x\| \leq \delta$ for each $x \in X \cap \mathcal{E}(A)$. Choose a $\gamma > 0$ such that $\gamma \mathrm{rad}(X) \leq \varepsilon/3$. By taking δ small enough we may assume by a standard perturbation argument that the unital linear map $\varphi : X \to B_{\mathrm{sa}}$ defined by $\varphi(1) = 1$ and $\varphi(x_k) = \mathrm{re}\, y_{x_k}$ for $k = 1, \dots, n$, is injective and satisfies $\|x - \varphi(x)\| \leq \gamma \min(\|x\|, \|\varphi(x)\|)$ for all $x \in X$. Define a Lip-norm L'' on $Y = \varphi(X)$ by setting $L''(y) = L(\varphi^{-1}(y))$ for all $y \in Y$.

Now if $x \in X \cap \mathcal{D}(A)$ then setting $y = \varphi(x)$ we have $y \in \mathcal{E}(Y)$ and $\|x - y\| \leq \gamma\|x\| \leq \gamma \mathrm{rad}(X) \leq \varepsilon/3$. Since $\mathcal{E}(A) = \mathcal{D}(A) + \mathbb{R}1$ and $\mathcal{E}(Y) = \varphi(X \cap \mathcal{E}(A))$, it follows that $\mathrm{dist}_{\mathrm{H}}(X \cap \mathcal{E}(A), \mathcal{E}(Y)) \leq \varepsilon/3$.

By Lemma 5.2 we can define a Lip-norm L' on B such that $\mathrm{dist}_{\mathrm{H}}(\mathcal{E}(Y), \mathcal{E}(B)) \leq \varepsilon/3$. The triangle inequality then yields $\mathrm{dist}_{\mathrm{H}}(\mathcal{E}(A), \mathcal{E}(B)) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ so that $\mathrm{dist}_{\mathrm{nu}}(A, B) \leq \varepsilon$, as desired.

(ii)⇒(iii). Every finite-dimensional C^*-algebra embeds as a unital C^*-subalgebra of a matrix algebra and so we can apply Lemma 5.2.

(iii)⇒(i). We can apply an approximation argument similar to the one in the proof of (iii)⇒(i) in Theorem 5.3 and appeal to Theorem 5.7. ☐

It follows from Theorems 5.3 and 5.8 that on the set $C^*\mathrm{M}$ of isometry classes of Lip-normed unital C^*-algebras the operator Gromov-Hausdorff topology is strictly weaker than the $\mathrm{dist}_{\mathrm{nu}}$ topology.

We round out this section by giving a characterization of matrix approximability for the C^*-algebraic quantum Gromov-Hausdorff distance $\mathrm{dist}_{\mathrm{cq}}$ introduced in [14]. This will allow us to compare the C^*-algebraic quantum Gromov-Hausdorff and operator Gromov-Hausdorff topologies on $C^*\mathrm{M}$.

The distance $\mathrm{dist}_{\mathrm{cq}}$ was defined in [14] for C^*-algebraic compact quantum metric spaces (see the beginning of Section 4), but it only requires the Lip-norm on self-adjoint elements, and so the definition makes sense for general Lip-normed unital C^*-algebras. Thus given Lip-normed unital C^*-algebras (A, L_A) and (B, L_B) we define

$$\mathrm{dist}_{\mathrm{cq}}(A, B) = \inf \mathrm{dist}_{\mathrm{H}}\big(h_A^{(3)}(\mathcal{D}(A)^{\mathrm{m}}), h_B^{(3)}(\mathcal{D}(B)^{\mathrm{m}})\big)$$

where the infimum is taken over all triples (V, h_A, h_B) consisting of a normed linear space V and isometric linear maps $h_A : A \to V$ and $h_B : B \to V$. As in [14], for a subset X of a C^*-algebra A we are using X^{m} to denote $\{(x, y, xy) \in A \oplus A \oplus A : x, y \in X\}$ and for a linear map $h : V \to W$ between normed linear spaces we are writing $h^{(3)}$ for the induced map $V \oplus V \oplus V \to W \oplus W \oplus W$ between the threefold ℓ_∞-direct sums.

In fact given a Lip-normed unital C^*-algebra (A, L) we can produce a C^*-algebraic compact quantum metric space (A, L') by setting

$$L(a) = \sup\left\{ \frac{|\omega(a) - \sigma(a)|}{\rho_L(\omega, \sigma)} : \omega, \sigma \in S(A) \text{ and } \omega \neq \sigma \right\}$$

for all $a \in A$, in which case the set $\mathcal{D}(A)$ associated to L' is the closure of the set $\mathcal{D}(A)$ associated to L. We can thus equivalently view $C^*\mathrm{M}$ with the metric $\mathrm{dist}_{\mathrm{cq}}$ as the set of isometry classes of C^*-algebraic compact quantum metric spaces ("isometry" has the same meaning in this case [14, Def. 3.14]) with the metric $\mathrm{dist}_{\mathrm{cq}}$ as originally defined in [14, Def. 3.3].

Recall that a separable C^*-algebra A is said to be an *MF algebra* if it can be expressed as the inductive limit of a generalized inductive system of finite-dimensional C^*-algebras [2, Def. 3.2.1]. By Theorem 3.2.2 of [2] this is equivalent to each of the following conditions:

(i) there exists an injective *-homomorphism $\Phi : A \to (\prod_{k=1}^\infty M_{n_k})/(\bigoplus_{k=1}^\infty M_{n_k})$ for some sequence $\{n_k\}$ of positive integers,
(ii) A admits an essential quasidiagonal extension by the compact operators $\mathcal{K}$,
(iii) there exists a continuous field (A_t) of C^*-algebras over $\mathbb{N} \cup \{\infty\}$ with A_t finite-dimensional for every $t \in \mathbb{N}$ and $A_\infty = A$.

If A is unital then the *-homomorphism Φ in (i) may be taken to be unital, since the image of the unit under Φ can be lifted to a projection (p_k) in $\prod_{k=1}^\infty M_{n_k}$, yielding an injective unital *-homomorphism from A to $(\prod_{k=1}^\infty p_k M_{n_k} p_k)/(\bigoplus_{k=1}^\infty p_k M_{n_k} p_k)$. Thus, in view of the proof of [2, Prop. 2.2.3], we may also in this case take the fibre C^*-algebras to be unital and the unit section to be continuous in the continuous field in (iii). In the proof of Theorem 5.10 below we will implicitly use this unital version of (iii) as a characterization of being an MF algebra for separable unital C^*-algebras.

Note that a Lip-normed unital C^*-algebra is automatically separable as a C^*-algebra.

Lemma 5.9. Let (A_t) be a continuous field of C^*-algebras over $\mathbb{N} \cup \{\infty\}$ with separable unital fibres such that the unit section is continous. Let L be a Lip-norm on A_∞. Then for each $t \in \mathbb{N}$ there is a Lip-norm L_t on A_t so that (A_t, L_t) forms a continuous field of Lip-normed unital C^*-algebras over $\mathbb{N} \cup \{\infty\}$ (with the same meaning as that given in [14, Def. 5.1]) and $\lim_{t\to\infty} \mathrm{dist}_{\mathrm{GH}}(\mathcal{D}(A_t), \mathcal{D}(A_\infty)) = 0$.

Proof. By [4, Cor. 2.8] we can find a normed linear space V containing each A_t isometrically such that for every continuous section f the map $t \mapsto f_t$ is continuous at $t = \infty$. Let $\varepsilon > 0$. Take a finite-dimensional subspace X of A_{sa} such that $\mathrm{dist}_{\mathrm{H}}(\mathcal{D}(A_\infty), X \cap \mathcal{D}(A_\infty)) \leq \varepsilon/3$. As in the proof of (i)⇒(ii) in Theorem 5.6, by a standard perturbation argument we can construct, for any sufficiently large $t \in \mathbb{N}$, a unital linear map $\varphi : X \to (A_t)_{\rm sa}$ such that by taking the Lip-norm L'' on $Y = \varphi(X)$ defined by $L''(y) = L(\varphi^{-1}(y))$ we get $\mathrm{dist}_{\rm H}(X \cap \mathcal{D}(A), \mathcal{D}(Y)) \leq \varepsilon/3$. By Lemma 5.2 there is a Lip-norm L' on A_t such that $\mathrm{dist}_{\rm H}(\mathcal{D}(Y), \mathcal{D}(B)) \leq \varepsilon/3$. The triangle inequality then yields

$$\mathrm{dist}_{\rm GH}(\mathcal{D}(A_\infty), \mathcal{D}(A_t)) \leq \mathrm{dist}_{\rm H}(\mathcal{D}(A_\infty), \mathcal{D}(A_t)) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

We can thus apply this procedure to produce a Lip-norm L_t on A_t for each $t \in \mathbb{N}$ so that $\lim_{t\to\infty} \mathrm{dist}_{\rm GH}(\mathcal{D}(A_t), \mathcal{D}(A_\infty)) = 0$, and it is readily checked that (A_t, L_t) forms a continuous field of Lip-normed unital C^*-algebras. □

Theorem 5.10. Let (A, L) be a Lip-normed unital C^*-algebra. Then the following are equivalent:

(i) A is an MF algebra,
(ii) for every $\varepsilon > 0$ there is a finite-dimensional Lip-normed unital C^*-algebra (B, L') such that $\mathrm{dist}_{\rm cq}(A, B) \leq \varepsilon$,
(iii) for every $\varepsilon > 0$ there is a Lip-normed matrix algebra (M_n, L') such that $\mathrm{dist}_{\rm cq}(A, M_n) \leq \varepsilon$.

Proof. The implication (i)⇒(ii) is a consequence of Lemma 5.9 above and Theorem 5.2 of [14], while (ii)⇒(iii) follows from Lemma 5.2 and (iii)⇒(i) from Proposition 5.4 of [14]. □

Theorem 5.10 shows that on the set $C^*\mathrm{M}$ of isometry classes of Lip-normed unital C^*-algebras the $\mathrm{dist}_{\rm cq}$ topology is strictly weaker than the $\mathrm{dist}_{\rm nu}$ topology and is neither weaker nor stronger than the operator Gromov-Hausdorff topology.

Finally, we remark that the compactness theorem recorded above as Theorems 5.6 also holds, with the appropriate substitutions, for $\mathrm{dist}_{\rm nu}$ and $\mathrm{dist}_{\rm cq}$. In these cases we take $\mathrm{Afn}_L(\varepsilon)$ for a Lip-normed unital C^*-algebra (A, L) to be the smallest positive integer k such that there is a Lip-normed matrix algebra (M_k, L') with $\mathrm{dist}_{\rm nu}(A, M_k) \leq \varepsilon$ (resp. $\mathrm{dist}_{\rm cq}(A, M_k) \leq \varepsilon$) and setting $\mathrm{Afn}_L(\varepsilon) = \infty$ if no such k exists. Then for $\mathrm{dist}_{\rm nu}$ we should replace $\mathrm{OM}_{\rm ex}$ by the set of isometry classes of Lip-normed unital quasidiagonal exact C^*-algebras, as Theorem 5.8 shows, while for $\mathrm{dist}_{\rm cq}$ we should instead substitute the set of isometry classes of Lip-normed unital MF C^*-algebras, by Theorem 5.10. To establish the compactness theorems, we use the observation that the set of isometry classes of Lip-normed matrix algebras (M_k, L) for a fixed k and with a fixed upper bound on the radius is totally bounded under both $\mathrm{dist}_{\rm nu}$ and $\mathrm{dist}_{\rm cq}$. For $\mathrm{dist}_{\rm nu}$ this follows from the fact that, given Lip-norms L_1 and L_2 on a matrix algebra M_k, the quantity $\mathrm{dist}_{\rm nu}((M_k, L_1), (M_k, L_2))$ is bounded above by $\mathrm{dist}_{\rm H}(\mathcal{E}(M_k, L_1), \mathcal{E}(M_k, L_2))$, which coincides with the Hausdorff distance in $M_k/\mathbb{C}1$ between the images of $\mathcal{E}(M_k, L_1)$ and $\mathcal{E}(M_k, L_2)$ under the quotient. Compare the proof of Theorem 6.3 in [10].

6. Nonseparability

We write OS_n for the set of n-dimensional operator spaces, with two such operator spaces being considered the same if they are completely isometric. The subset of n-dimensional Hilbertian operator spaces (i.e., operator spaces which are isometric to ℓ_2^n as normed spaces) will be denoted HOS_n. We write OM_n for the set of isometry classes of Lip-normed n-dimensional operator systems equipped with the operator Gromov-Hausdorff topology, i.e., the $\mathrm{dist_{op}}$ (or equivalently $\mathrm{dist_s}$) topology. Note that for $n \geq 2$ the set OM_n is not closed in OM since it is possible to have dimension collapse (as happens already for finite metric spaces). However, $\bigcup_{1\leq n\leq m} \mathrm{OM}_n$ is closed in OM for each $m \in \mathbb{N}$, as can be gathered from Remark 5.4.

Let (X, L) be a Lip-normed operator system. We denote by $\tilde{L}$ the norm on the Banach space quotient $X_{\mathrm{sa}}/\mathbb{R}1$ induced from L. Suppose that X is of some finite dimension strictly greater than one. Then the formal identity map $I_X : (X_{\mathrm{sa}}/\mathbb{R}1, \tilde{L}) \to (X_{\mathrm{sa}}/\mathbb{R}1, \|\cdot\|)$ is an isomorphism, and we can define $\mu_X = \|I_X^{-1}\|$. In the case that $X = C(F)$ for a finite metric space F, μ_X is equal to the inverse of half of the smallest distance between any two points of F.

Lemma 6.1. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems of some finite dimension $n \geq 2$. Set $\kappa_{X,Y} = 2n \min(\mu_X, \mu_Y)\mathrm{dist_{op}}(X, Y)$. If $0 \leq \kappa_{X,Y} < 1$ then we have

$$d_{\mathrm{cb}}(X, Y) \leq \frac{1 + \kappa_{X,Y}}{1 - \kappa_{X,Y}}.$$

Proof. Without loss of generality we may assume that $\mu_X \leq \mu_Y$. Let $\{(x_k, x_k^*)\}_{k=1}^n$ be an Auerbach system for X_{sa} (see for example [8, page 335]). By complexifying we regard this as a biorthogonal system for X with $\|x_k^*\| \leq 2$ for each $k = 1, \dots, n$. For each $k = 1, \dots, n$ we have $L(x_k) \leq \mu_X\|x_k\| = \mu_X$, that is, $\mu_X^{-1}x_k \in \mathcal{E}(X)$. By the definition of $\mathrm{dist_{op}}$ we may view X and Y as operator subsystems of an operator system Z such that for each $k = 1, \dots, n$ there is a $y_k \in \mathcal{E}(Y)$ with $\|\mu_X^{-1}x_k - y_k\| \leq \mathrm{dist_{op}}(X, Y)$. We then have

$$\sum_{k=1}^n \|x_k^*\|\|x_k - \mu_X y_k\| \leq 2\sum_{k=1}^n \mu_X\|\mu_X^{-1}x_k - y_k\| \leq 2n\mu_X \mathrm{dist_{op}}(X, Y),$$

and since X and Y are of the same dimension we conclude by Lemma 2.13.2 of [18] that $d_{\mathrm{cb}}(X, Y) \leq (1 + \kappa_{X,Y})(1 - \kappa_{X,Y})^{-1}$, as desired. ☐

Theorem 6.2. For every $n \geq 7$ there is a nonseparable (and in particular non-totally-bounded) subset of OM_n whose elements are all isometric to each other as compact quantum metric spaces.

Proof. For each $V \in \mathrm{HOS}_3$ we take a representation of V as an operator subspace of a unital C^*-algebra A and define the 7-dimensional operator system

$$X_V = \left\{ \begin{bmatrix} \lambda 1 & a \\ b^* & \lambda 1 \end{bmatrix} \in M_2(A) : \lambda \in \mathbb{C} \text{ and } a, b \in V \right\}.$$

Note that we have a completely isometric embedding $V \hookrightarrow X_V$ given by $a \mapsto \left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right]$. The self-adjoint elements of X_V are those of the form $\left[\begin{smallmatrix} \lambda 1 & a \\ a^* & \lambda 1 \end{smallmatrix}\right]$ for $\lambda \in \mathbb{R}$ and $a \in V$, and by Lemma 3.1 of [16] such an element is positive if and only if $\lambda \geq \|a\|$. Thus if we define Y_V as the operator system direct sum of X_V and the commutative C^*-algebra $\mathbb{C}^{n-7}$ we see that the operator systems Y_V for $V \in \mathrm{HOS}_3$ are all order-isomorphic to each other. Moreover if we define a Lip-norm L_V on $(Y_V)_{\mathrm{sa}}$ by taking $L_V(y)$ to be the norm of the image of y under the quotient map $(Y_V)_{\mathrm{sa}} \to (Y_V)_{\mathrm{sa}}/\mathbb{R}1$ then the Lip-normed operator systems (Y_V, L_V) for $V \in \mathrm{HOS}_3$ are all isometric to each other as compact quantum metric spaces. Denote by Θ the subset of OM_n consisting of the (Y_V, L_V) and by Γ the subset of OS_n consisting of the Y_V.

We claim that Γ is nonseparable. Suppose that this is not the case. Then since for any integers $s \geq r \geq 1$ the set of r-dimensional subspaces of a given s-dimensional operator space is compact in OS_r (this can be shown using Lemma 2.13.2 of [18]) we infer the separability of the subset of OS_3 consisting of all 3-dimensional operator spaces which appear as a subspace of some operator system in Γ. But this subset of OS_3 contains HOS_3, which is nonseparable [9, Remark 2.4], producing a contradiction. Thus Γ is nonseparable, and so by Lemma 6.1 we conclude that Θ is nonseparable, as desired. □

Theorem 6.2 shows that, in contrast to the order-unit case [24, Thm. 13.5][15, Thm. 5.5], there can be no compactness theorem for $\mathrm{dist}_{\mathrm{op}}$ or $\mathrm{dist}_{\mathrm{s}}$ which at the "topological" level makes reference only to the state space or norm structure (cf. Theorem 5.6). It also follows that for every $n \geq 7$ and $D > 0$ the set of isometry classes of Lip-normed n-dimensional operator systems of diameter at most D is not separable and, in particular, not totally bounded, thus answering Question 6.5 of [10] for $n \geq 7$.

7. Generic complete order structure

In Theorem 7.5 we describe the type of complete order structure possessed by a generic element in $\mathrm{OM}_{\mathrm{ex}}$ under operator Gromov-Hausdorff distance, where $\mathrm{OM}_{\mathrm{ex}}$ is viewed as consisting of Lip-normed 1-exact operator systems (X, L) with X complete and L closed. Note that $\mathrm{OM}_{\mathrm{ex}}$ is a separable closed subset of the complete metric space OM by Theorem 5.3, and so $\mathrm{OM}_{\mathrm{ex}}$ is a Baire space.

Lemma 7.1. Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists a $\delta > 0$ such that whenever $n \in \mathbb{N}$ and $\varphi : M_d \to M_n$ is an injective unital linear map with $\max(\|\varphi\|_{\mathrm{cb}}, \|\varphi^{-1}\|_{\mathrm{cb}}) < 1 + \delta$, there exists a unital complete order embedding $\psi : M_d \to M_n$ with $\|\psi - \varphi\| < \varepsilon$.

Proof. Suppose that the lemma is not true. Then there is an $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exists an injective unital linear map φ_k from M_d to a matrix algebra M_{n_k} such that $\max(\|\varphi_k\|_{\mathrm{cb}}, \|\varphi_k^{-1}\|_{\mathrm{cb}}) < 1 + 1/k$ and $\|\psi - \varphi\| \geq \varepsilon$ for every unital complete order embedding $\psi : M_d \to M_{n_k}$. Define the map $\varphi : M_d \to (\prod_{k=1}^{\infty} M_{n_k})/(\bigoplus_{k=1}^{\infty} M_{n_k})$ by $\varphi(x) = \pi((\varphi_k(x))_k)$ where $\pi : \prod_k M_{n_k} \to (\prod_{k=1}^{\infty} M_{n_k})/(\bigoplus_{k=1}^{\infty} M_{n_k})$ is the quotient map. Then φ is a unital complete isometry and hence a complete order embedding [8, Cor. 5.1.2].

Let $\{e_{ij}\}_{1 \leq i,j \leq d}$ be the set of standard matrix units for M_d. By Proposition 4.2.8 of [2] (or rather the unital version which follows from the same proof) there is a $\delta > 0$ such that whenever γ is a u.c.p. map from M_d to a finite-dimensional C^*-algebra B with $\|\gamma(e_{12})\gamma(e_{23}) \cdots \gamma(e_{d-1,d})\| \geq 1 - \delta$, there is a unital complete order embedding $\gamma' : M_d \to B$ with $\|\gamma' - \gamma\| < \varepsilon/2$.

By the Choi-Effros lifting theorem [5] there is a u.c.p. map $\theta : M_d \to \prod_{k=1}^{\infty} M_{n_k}$ such that $\pi \circ \theta = \varphi$. By the compactness of the unit ball of M_d we can find a $j \in \mathbb{N}$ such that $\|\pi_j \circ \theta - \varphi_j\| < \varepsilon/2$, where $\pi_j : \prod_{k=1}^{\infty} M_{n_k} \to M_{n_j}$ is the projection map onto the jth coordinate. Since $\theta(e_{12})\theta(e_{23}) \cdots \theta(e_{d-1,d})$ is a lift of $\varphi(e_{12})\varphi(e_{23}) \cdots \varphi(e_{d-1,d})$ under π and

$$\|\varphi(e_{12})\varphi(e_{23}) \cdots \varphi(e_{d-1,d})\| \geq \|\varphi(e_{12}e_{23} \cdots e_{d-1,d})\| = \|\varphi(e_{1d})\| = 1$$

by the supermultiplicativity of φ [2, Prop. 4.2.5], we may assume that j is large enough so that we additionally have

$$\|\pi_j(\theta(e_{12}))\pi_j(\theta(e_{23}))\cdots\pi_j(\theta(e_{d-1,d}))\| = \|\pi_j(\theta(e_{12})\theta(e_{23})\cdots\theta(e_{d-1,d}))\| \geq 1-\delta.$$

Because $\pi_j\circ\theta$ is a u.c.p. map, it follows by our choice of δ that there is a unital complete order embedding $\psi : M_d \to M_{n_j}$ with $\|\psi - \pi_j\circ\theta\| < \varepsilon/2$. But then

$$\|\psi - \varphi_j\| \leq \|\psi - \pi_j\circ\theta\| + \|\pi_j\circ\theta - \varphi_j\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

producing a contradiction. □

Lemma 7.2. Let (M_d, L) be a Lip-normed matrix algebra and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that whenever (X, L_X) is a Lip-normed operator system, A is a unital C^*-algebra, and $\beta : M_d \to A$ and $\gamma : X \to A$ are unital complete order embeddings for which $\mathrm{dist_H}(\beta(\mathcal{E}(M_d)), \gamma(\mathcal{E}(X))) < \delta$, there exists an injective unital linear map $\varphi : M_d \to X$ such that $\max(\|\varphi\|_{\mathrm{cb}}, \|\varphi^{-1}\|_{\mathrm{cb}}) \leq 1+\varepsilon$ and $\|\beta - \gamma\circ\varphi\| < \varepsilon$.

Proof. We may assume that $d > 1$. Letting $\{e_{ij}\}_{1\leq i,j\leq d}$ be the set of standard matrix units for M_d, for $i,j = 1,\dots,d$ we set $a_{ij} = e_{ij} + e_{ji}$ if $i > j$, $a_{ij} = i(e_{ij} - e_{ji})$ if $i < j$, and $a_{ij} = e_{ii}$ if $i = j$. Then $\{a_{ij}\}_{1\leq i,j\leq d}$ is an Auerbach basis for M_d. Since M_d is finite-dimensional there exists a $K > 0$ such that $L(a) \leq K\|a\|$ for all self-adjoint $a \in M_d$. Let $0 < \eta < \min(\varepsilon, 1)$, and suppose that (X, L_X) is a Lip-normed finite-dimensional C^*-algebra, A is a unital C^*-algebra, and $\beta : M_d \to A$ and $\gamma : X \to A$ are unital complete order embeddings for which

$$\mathrm{dist_H}(\beta(\mathcal{E}(M_d)), \gamma(\mathcal{E}(X)) < \frac{\eta}{Kd^3}.$$

Then the image under β of the norm unit ball of $(M_d)_{\mathrm{sa}}$ is contained in $K\beta(\mathcal{E}(M_d))$, and so for all $i,j = 1,\dots,d$ we can find an $x_{ij} \in X_{\mathrm{sa}}$ such that $\|\beta(a_{ij}) - \gamma(x_{ij})\| < d^{-3}\eta$. Redefining x_{11} as $1 - \sum_{i=2}^d x_{ii}$, we then have $\|\beta(a_{ij}) - \gamma(x_{ij})\| < d^{-2}\eta$ for all $1 \leq i,j \leq d$, and the unital linear map $\varphi : M_d \to B$ determined by $\psi(a_{ij}) = x_{ij}$ satisfies $\|\beta - \gamma\circ\psi\| < \eta < \varepsilon$. Moreover by Lemma 2.13.2 of [18] ψ is injective and $\|\psi\|_{\mathrm{cb}} \leq 1+\eta$ and $\|\psi^{-1}\|_{\mathrm{cb}} \leq (1-\eta)^{-1}$. We can thus find an η small enough as a function of ε to ensure that $\max(\|\varphi\|_{\mathrm{cb}}, \|\varphi^{-1}\|_{\mathrm{cb}}) \leq 1+\varepsilon$ and then take $\delta = \eta(Kd^3)^{-1}$ to obtain the lemma. □

Lemma 7.3. Let (M_d, L) be a Lip-normed matrix algebra and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that whenever (M_n, L') is a Lip-normed matrix algebra, A is a unital C^*-algebra, and $\beta : M_d \to A$ and $\gamma : M_n \to A$ are unital complete order embeddings for which $\mathrm{dist_H}(\beta(\mathcal{E}(M_d)), \gamma(\mathcal{E}(M_n))) < \delta$, there exists a unital complete order embedding $\varphi : M_d \to M_n$ satisfying $\|\beta - \gamma\circ\varphi\| < \varepsilon$.

Proof. By Lemmas 7.1 and 7.2, if δ is sufficiently small as a function of d and ε then whenever (M_n, L') is a Lip-normed matrix algebra, A is a unital C^*-algebra, and $\beta : M_d \to A$ and $\gamma : M_n \to A$ are unital complete order embeddings for which $\mathrm{dist_H}(\beta(\mathcal{E}(M_d)), \gamma(\mathcal{E}(M_n))) < \delta$, there exist an injective unital linear map $\psi : M_d \to M_n$ with $\|\beta - \gamma\circ\psi\| < \varepsilon/2$ and a unital complete order embedding $\varphi : M_d \to M_n$ with $\|\psi - \varphi\| < \varepsilon/2$, in which case

$$\|\beta - \gamma\circ\varphi\| \leq \|\beta - \gamma\circ\psi\| + \|\gamma\|\|\psi - \varphi\| < \varepsilon.$$

□

Lemma 7.4. Let I be an infinite subset of $\mathbb{N}$, and let Λ_I be the subset of $\mathrm{OM}_{\mathrm{ex}}$ consisting of all Lip-normed matrix algebras (M_d, L) such that $d \in I$. Then Λ_I is dense in $\mathrm{OM}_{\mathrm{ex}}$.

Proof. For positive integers $d \leq n$ there always exists a unital complete order embedding $\varphi : M_d \to M_n$. For example, take a state σ on M_d, a rank d projection $p \in M_n$, and a *-isomorphism $\Phi : M_d \to pM_np$, and define $\psi(x) = \Phi(x) + \sigma(x)(1 - p)$ for all $x \in M_d$. Using this fact in conjunction with Lemma 5.2 and Theorem 5.3 yields the lemma. □

Theorem 7.5. Let I be an infinite subset of $\mathbb{N}$. Then, with respect to the operator Gromov-Hausdorff topology, a generic element of $\mathrm{OM}_{\mathrm{ex}}$ is, as an operator system, unitally completely order isomorphic to an operator system inductive limit $\varinjlim(M_{n_k}, \varphi_k)$ over $\mathbb{N}$ where $\{n_k\}_k$ is a strictly increasing sequence in I and each φ_k is a unital complete order embedding.

Proof. For every Lip-normed matrix algebra (M_d, L) and every $\varepsilon > 0$ we take a $\delta(d, L, \varepsilon) \in (0, \varepsilon)$ that works for both Lemma 7.2 and Lemma 7.3 and define $\Gamma_\varepsilon(M_d, L)$ to be the set of all (X, L_X) in $\mathrm{OM}_{\mathrm{ex}}$ such that

$$\mathrm{dist}_{\mathrm{op}}((M_d, L), (X, L_X)) < \delta(d, L, \varepsilon)/2.$$

For every subset $I \subseteq \mathbb{N}$ and $\varepsilon > 0$ we define the open subset $\Theta(I, \varepsilon)$ of $\mathrm{OM}_{\mathrm{ex}}$ as the union of the sets $\Gamma_\varepsilon(M_d, L)$ over all Lip-normed matrix algebras (M_d, L) with $d \in I$. By Lemma 7.4, for all infinite subsets $I \subseteq \mathbb{N}$ and $\varepsilon > 0$ the set $\Theta(I, \varepsilon)$ is dense in $\mathrm{OM}_{\mathrm{ex}}$, and so the countable intersection

$$R := \bigcap_{\substack{J \subseteq I \\ I \setminus J \text{ finite}}} \bigcap_{k \in \mathbb{N}} \Theta(J, 1/k)$$

is a dense G_δ subset of $\mathrm{OM}_{\mathrm{ex}}$. Letting (X, L) be an element of R, it thus suffices to show that X can be expressed as an inductive limit of the type described in the theorem statement.

Let $\{\varepsilon_k\}_k$ be a summable sequence of positive real numbers. By the definition of R there is a sequence $\{(M_{n_k}, L_k)\}_k$ of Lip-normed matrix algebras such that $\{n_k\}_k$ is a strictly increasing sequence in I and

$$\mathrm{dist}_{\mathrm{op}}((M_{n_k}, L_k), (X, L)) < \delta(n_k, L, \varepsilon_k)/2.$$

By Corollary 2.7 and Lemma 3.3 there is a unital C^*-algebra A containing X as an operator subsystem and unital complete order embeddings $\beta_k : M_{n_k} \to A$ such that $\mathrm{dist}_{\mathrm{H}}(\beta_k(\mathcal{E}(M_{n_k})), \mathcal{E}(X)) < \delta(n_k, L, \varepsilon_k)/2$ for all $k \in \mathbb{N}$. By passing to a subsequence and relabeling if necessary we may assume that $\delta(n_k, L, \varepsilon_k)$ decreases with k, and so by the triangle inequality for Hausdorff distance and the definition of $\delta(n_k, L, \varepsilon_k)$ there exist for each $k \in \mathbb{N}$ a unital complete order embedding $\varphi_k : M_{n_k} \to M_{n_{k+1}}$ and an injective unital linear map $\theta_k : M_{n_k} \to X$ such that $\|\beta_k - \beta_{k+1} \circ \varphi_k\| < \varepsilon_k$, $\max(\|\theta_k\|_{\mathrm{cb}}, \|\theta_k^{-1}\|_{\mathrm{cb}}) < 1 + \varepsilon_k$, and $\|\beta_k - \theta_k\| < \varepsilon_k$. Using the summability of $\{\varepsilon_k\}_k$, a simple estimate shows that for every $k \in \mathbb{N}$ and $x \in M_{n_k}$ the sequence $\{(\theta_{k+j} \circ \varphi_{k+j-1} \circ \cdots \circ \varphi_{k+1} \circ \varphi_k)(x)\}_j$ in X is Cauchy; denote by $\psi_k(x)$ its limit. Since $\max(\|\theta_k\|_{\mathrm{cb}}, \|\theta_k^{-1}\|_{\mathrm{cb}}) \to 1$ as $k \to \infty$, each of the resulting unital linear maps $\psi_k : M_{n_k} \to X$ is completely isometric (cf. the proof of Lemma 2.13.2 in [18]). These maps are compatible with the inductive system $\{(M_{n_k}, \varphi_k)\}_k$ and thus give rise to a unital map $\psi : \varinjlim(M_{n_k}, \varphi_k) \to X$ which is surjective and completely isometric and hence a unital complete order isomorphism [8, Cor. 5.1.2], completing the proof. $\square$

Remark 7.6. An operator system which can be expressed as an inductive limit as in Theorem 7.5 for a given I is far from being unique. Indeed it can be seen from the results and arguments of [3] (see Proposition 5.12 and Theorem 5.13 therein) that the unital C^*-algebras that as operator systems can be so expressed for a given I are precisely the infinite-dimensional unital prime strong NF algebras, and C^*-algebras are determined up to *-isomorphism by their complete order structure.

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