COMBINATORIAL INDEPENDENCE IN MEASURABLE DYNAMICS

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Abstract. We develop a fine-scale local analysis of measure entropy and measure sequence entropy based on combinatorial independence. The concepts of measure IE-tuples and measure IN-tuples are introduced and studied in analogy with their counterparts in topological dynamics. Local characterizations of the Pinsker von Neumann algebra and its sequence entropy analogue are given in terms of combinatorial independence, ℓ_1 geometry, and Voiculescu's completely positive approximation entropy. Among the novel features of our local study is the treatment of general discrete acting groups, with the structural assumption of amenability in the case of entropy.

1. Introduction

Many of the fundamental concepts in measurable dynamics revolve around the notion of probabilistic independence as an indicator of randomness or unpredictability. Ergodicity, weak mixing, and mixing are all expressions of asymptotic independence, whether in a mean or strict sense. At a stronger level, completely positive entropy can be characterized by a type of uniform asymptotic independence (see [12]).

In topological dynamics the appropriate notion of independence is the combinatorial (or set-theoretic) one, according to which a family of tuples of subsets of a set is independent if when picking any one subset from each of finitely many tuples one always ends up with a collection having nonempty intersection. Combinatorial independence manifests itself dynamically in many ways and has long played an important role in the topological theory, although it has not received the same kind of systematic attention as probabilistic independence has in measurable dynamics. In fact it has only been recently that precise relationships have been established between independence and the properties of nullness, tameness, and positive entropy [22, 30]. For example, a topological $\mathbb{Z}$-system has uniformly positive entropy if and only if the orbit of each pair of nonempty open subsets of the space is independent along a positive density subset of $\mathbb{Z}$ [22] (see [30] for a combinatorial proof that applies more generally to actions of discrete amenable groups).

The aim of this paper is to develop a theory of combinatorial independence in measurable dynamics. Among other things, this will provide the missing link for a geometric understanding of local entropy production in connection with Voiculescu's operator-algebraic notion of approximation entropy [46]. One of our main motivations is to establish local combinatorial and linear-geometric characterizations of positive entropy and positive sequence entropy. For automorphisms of a Lebesgue space, the extreme situation of complete positive entropy was characterized in terms of combinatorial independence by Glasner and

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Weiss in Section 3 of [16] using Karpovsky and Milman's generalization of the Sauer-Perles-Shelah lemma. What we see in this case however is an essentially topological phenomenon whereby independence over positive density subsets of iterates occurs for every finite partition of the space into sets of positive measure (cf. Theorem 3.9 in this paper). This does not help us much in the analysis of entropy production for other kinds of systems, as it can easily happen that combinatorial independence is present but not in a robust enough way to be measure-theoretically meaningful (indeed every free ergodic $\mathbb{Z}$-system has a minimal topological model with uniformly positive entropy [15]). We seek moreover a fine-scale localization predicated not on partitions but rather on tuples of subsets that together compose only a very small fraction of the space, which the Glasner-Weiss result provides for $\mathbb{Z}$-systems with completely positive entropy but in the purely topological sense of [30].

It turns out that we should ask whether combinatorial independence can be observed to the appropriate degree in orbits of tuples of subsets whenever we hide from view a small portion of the ambient space at each stage of the dynamics. Thus the recognition of positive entropy or positive sequence entropy becomes a purely combinatorial issue, with the measure being relegated to the role of observational control device. This way of counting sets appears in the global entropy formulas of Katok for metrizable topological $\mathbb{Z}$-systems with an ergodic invariant measure [26], which rely on the Shannon-McMillan-Breiman theorem for the uniformization of entropy measurement. Here we avoid the Shannon-McMillan-Breiman theorem in our focus on local entropy production and its relation to independence for arbitrary systems. What is particularly important at the technical level is that we be able to vary the obscured part of the space across group elements when making an observation (see Subsection 2.1), as this will permit us to work with L^2 perturbations and thereby establish the link with Voiculescu's approximation entropy. We will thus be developing probabilistic arguments that will render the theory rather different from the topological one, despite the obvious analogies in the statements of the main results, although we will make use of the key combinatorial lemma from [30].

Our basic framework will be that of a discrete group acting on a compact Hausdorff space with an invariant Borel probability measure, with the structural assumption of amenability on the group in the context of entropy. With a couple of exceptions, our results do not require any restrictions of metrizability on the space or countability on the group. In analogy with topological IE-tuples and IN-tuples [30], we introduce the notions of measure IE-tuple (in the entropy context) and measure IN-tuple (in the sequence entropy context) as tuples of points in the space such that the orbit of every tuple of neigbourhoods of the respective points exhibits independence with fixed density on certain finite subsets. For IE-tuples these finite subsets will be required to be approximately invariant in the sense of the Følner characterization of amenability, while for IN-tuples we will demand that they can be taken to be arbitrarily large.

Our main application of measure IE-tuples will be the derivation of a series of local descriptions of the Pinsker σ-algebra (or maximal zero entropy factor) in terms of combinatorial independence, ℓ_1 geometry, and Voiculescu's c.p. (completely positive) approximation entropy (Theorem 3.7). These local descriptions are formulated as conditions on an L^∞ function f which are equivalent to the containment of f in the Pinsker von Neumann algebra, i.e., the von Neumann subalgebra corresponding to the Pinsker σ-algebra. These conditions include:

(1) there exist $\lambda \geq 1$ and $d > 0$ such that every L^2 perturbation of the orbit of f exhibits λ-equivalence to the standard basis of ℓ_1 over subsets of Følner sets with density at least d,
(2) the local c.p. approximation entropy with respect to f is positive.

If the action is ergodic we can add:

(3) every L^2 perturbation of the orbit of f contains a subset of positive asymptotic density which is equivalent to the standard basis of ℓ_1.

In the case that f is continuous we can add:

(4) f separates a measure IE-pair.

This provides new geometric insight into the phenomenon of positive c.p. approximation entropy, in parallel to what was done in the topological setting for Voiculescu-Brown approximation entropy in [28, 29]. In fact the only way to establish positive c.p. approximation entropy until now has been by means of a comparison with Connes-Narnhofer-Thirring entropy, whose definition is based on Abelian models (see Proposition 3.6 in [46]). We also do not require the Shannon-McMillan-Breiman theorem, which factors critically into Voiculescu's proof for *-automorphisms in the separable commutative ergodic case that c.p. approximation entropy coincides with the underlying measure entropy [46, Cor. 3.8]. One consequence of the characterization of elements in the Pinsker von Neumann algebra given by condition (1) is a linear-geometric explanation for the well-known disjointness between zero entropy systems and systems with completely positive entropy, as discussed at the end of Section 3.

The notion of measure entropy tuple was introduced in [4] in the pair case and in [22] in general and has been a key tool in the local study of both measure entropy and topological entropy for $\mathbb{Z}$-systems (see Section 19 of [12]). We show in Theorem 2.27 that nondiagonal measure IE-tuples are the same as measure entropy tuples. The argument depends in part on a theorem of Huang and Ye for $\mathbb{Z}$-systems from [22], whose proof involves taking powers of the generating automorphism and thus does not extend as is to actions of amenable groups. For more general systems we reduce to Huang and Ye's result by applying the orbit equivalence technique of Rudolph and Weiss [42]. We point this out in particular because, with the exception of the product formula of Theorem 2.30 and the characterizations of completely positive entropy in Theorem 3.9, our study of measure IE-tuples and their relation to the topological theory does not otherwise rely on orbit equivalence or any special treatment of the integer action case, in contrast to what the measure entropy tuple approach in its present $\mathbb{Z}$-system form seems to demand (see [11, 22]). It is worth emphasizing however that we do need the relation with measure entropy tuples to establish the product formula for measure IE-tuples (Theorem 2.30), while the corresponding product formula for topological IE-tuples as established in Theorem 3.15 of [30] completely avoids the entropy tuple perspective, which would only serve to complicate matters (compare the proof of the entropy pair product formula for topological $\mathbb{Z}$-systems in [11]). We also show (without the use of orbit equivalence) that the set of topological IE-tuples is the closure of the union of the sets of measure IE-tuples over all invariant Borel probability measures (Theorem 2.21), and furthermore that when the space is metrizable there exists an invariant Borel probability measure such that the sets of measure IE-tuples and topological IE-tuples coincide (Theorem 2.23). In the $\mathbb{Z}$-system setting, the latter result for entropy pairs was established in [3] and more generally for entropy tuples in [22].

One of the major advantages of the combinatorial viewpoint is the universal nature of its application to entropy and independence density problems, as was demonstrated in the topological-dynamical domain in [30]. This means that many of the methods we develop for the study of measure IE-tuples apply equally well to the sequence entropy context of measure IN-tuples. Accordingly, using measure IN-tuples we are able to establish various local descriptions of the maximal null von Neumann algebra, i.e., the sequence entropy analogue of the Pinsker von Neumann algebra (Theorem 5.5). We thus have the following types of conditions on a L^∞ function f characterizing its containment in the maximal null von Neumann algebra:

(1) there exist $\lambda \geq 1$ and $d > 0$ such that every L^2 perturbation of the orbit of f contains arbitrarily large finite subsets possessing subsets of density at least d which are λ-equivalent to the standard basis of ℓ_1 in the corresponding dimension,
(2) the local sequence c.p. approximation entropy with respect to f is positive for some sequence,

and, in the case that f is continuous,

(3) f separates a measure IN-pair.

Here, however, additional equivalent conditions arise that have no counterpart on the entropy side, such as:

(4) every L^2 perturbation of the orbit of f contains an infinite subset which is equivalent to the standard basis of ℓ_1, and
(5) every L^2 perturbation of the orbit of f contains arbitrarily large finite subsets which are λ-equivalent to the standard basis of ℓ_1 for some $\lambda > 0$.

The presence of such conditions reflects the fact that there is a strong dichotomy between nullness and nonnullness, which registers as compactness vs. noncompactness for orbit closures in L^2 and is thus tied to weak mixing and the issue of finite-dimensionality for group subrepresentations. Notice that the appearance of condition (4) indicates that the distinction between tameness and nullness in topological dynamics collapses in the measurable setting. In parallel with measure IE-tuples, it turns out (Theorem 4.9) that nondiagonal measure IN-tuples are the same as measure sequence entropy tuples as introduced in [21], which leads in particular to a simple product formula (Theorem 4.12).

The main body of the paper is divided into four sections. Section 2 consists of four subsections. The first discusses measure independence density for tuples of subsets, while in the second we define measure IE-tuples and establish several basic properties. In the third subsection we address the problem of realizing IE-tuples as measure IE-tuples. The fourth subsection contains the proof that nondiagonal measure IE-tuples are the same as measure entropy tuples and includes the product formula for measure IE-tuples. Section 3 furnishes the local characterizations of the Pinsker von Neumann algebra. In Section 4 we define measure IN-tuples, record their basic properties, show that nondiagonal measure IN-tuples are the same as sequence measure entropy tuples, and derive the measure IN-tuple product formula. Finally, in Section 5 we establish the local characterizations of the maximal null von Neumann algebra.

We now describe some of the basic concepts and notation used in the paper. A collection $\{(A_{i,1}, \dots, A_{i,k}) : i \in I\}$ of k-tuples of subsets of a given set is said to be *independent* if $\bigcap_{i \in J} A_{i,\sigma(i)} = \emptyset$ for every finite set $J \subseteq I$ and $\sigma \in \{1, \dots, k\}^J$. The following definition captures a relativized version of this idea of combinatorial independence in a group action context and forms the basis for our analysis of measure-preserving dynamics. The relativized form is not necessary for topological dynamics (cf. Definition 2.1 of [30]) but becomes crucial in the measure-preserving case, where we will need to consider independence relative to subsets of nearly full measure.

Definition 1.1. Let G be a group acting on a set X. Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X. Let D be a map from G to the power set 2^X of X, with the image of $s \in G$ written as D_s. We say that a set $J \subseteq G$ is an *independence set for* $\boldsymbol{A}$ *relative to* D if for every nonempty finite subset $F \subseteq J$ and map $\sigma : F \to \{1, \dots, k\}$ we have $\bigcap_{s \in F}(D_s \cap s^{-1}A_{\sigma(s)}) \neq \emptyset$. For a subset D of X, we say that J is an *independence set for* $\boldsymbol{A}$ *relative to* D if for every nonempty finite subset $F \subseteq J$ and map $\sigma : F \to \{1, \dots, k\}$ we have $D \cap \bigcap_{s \in F} s^{-1}A_{\sigma(s)} \neq \emptyset$, i.e., if J is an independence set for $\boldsymbol{A}$ relative to the map $G \to 2^X$ with constant value D.

By a *topological dynamical system* we mean a pair (X, G) where X is a compact Hausdorff space and G is a discrete group acting on X by homeomorphisms. We will also speak of a *topological G-system*. In this context we will always use $\mathscr{B}$ to denote the Borel σ-algebra of X. Given a G-invariant Borel probability measure μ on X, we will invariably write α for the induced action of G on $L^\infty(X, \mu)$ given by $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in L^\infty(X, \mu)$, and $x \in X$. Given another topological G-system (Y, G), a continuous surjective G-equivariant map $X \to Y$ will be called a *topological G-factor map*. In this situation we will regard $C(Y)$ as a unital C^*-subalgebra of $C(X)$.

By a *measure-preserving dynamical system* we mean a quadruple $(X, \mathscr{X}, \mu, G)$ where $(X, \mathscr{X}, \mu)$ is a probability space and G is a discrete group acting on $(X, \mathscr{X}, \mu)$ by μ-preserving bimeasurable transformations. The expression *measure-preserving G-system* will also be used. The action of G is said to be *free* if for every $s \in G \setminus \{e\}$ the fixed-point set $\{x \in X : sx = x\}$ has measure zero. A *topological model* for $(X, \mathscr{X}, \mu, G)$ is a measure-preserving G-system $(Y, \mathscr{Y}, \nu, G)$ isomorphic to $(X, \mathscr{X}, \mu, G)$ such that (Y, G) is a topological dynamical system.

We will actually work for the most part with an invariant Borel probability measure for a topological dynamical system instead of an abstract measure-preserving dynamical system, since the local study of independence properties requires the specification of a topological model and such a specification entails no essential loss of generality from the measure-theoretic viewpoint. So our basic setting will consist of (X, G) along with a G-invariant Borel probability measure μ. In Sections 2 and 3 we will also suppose G to be amenable, as the entropy context naturally requires.

For a finite $K \subseteq G$ and $\varepsilon > 0$ we write $M(K, \varepsilon)$ for the set of all nonempty finite subsets F of G which are (K, ε)-invariant in the sense that

$$|\{s \in F : Ks \subseteq F\}| \geq (1 - \varepsilon)|F|.$$

The Følner characterization of amenability asserts that $M(K, \varepsilon)$ is nonempty for every finite set $K \subseteq G$ and $\varepsilon > 0$. Given a real-valued function φ on the finite subsets of G we define the *limit supremum* and *limit infimum of* $\varphi(F)/|F|$ *as* F *becomes more and more invariant* by

$$\lim_{(K,\varepsilon)} \sup_{F\in M(K,\varepsilon)} \frac{\varphi(F)}{|F|} \qquad \text{and} \qquad \lim_{(K,\varepsilon)} \inf_{F\in M(K,\varepsilon)} \frac{\varphi(F)}{|F|}$$

respectively, where the net is constructed by stipulating that $(K,\varepsilon) \succ (K',\varepsilon')$ if $K \supseteq K'$ and $\varepsilon \leq \varepsilon'$. These limits coincide under the following conditions:

(1) $0 \leq \varphi(A) < +\infty$ and $\varphi(\emptyset) = 0$,
(2) $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$,
(3) $\varphi(As) = \varphi(A)$ for all finite $A \subseteq G$ and $s \in G$,
(4) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ if $A \cap B = \emptyset$.

See Section 6 of [32] and the last part of Section 3 in [30]. These conditions hold in the definition of measure entropy, which we recall next.

The entropy of a finite measurable partition $\mathcal{P}$ of a probability space $(X, \mathscr{X}, \mu)$ is defined by $H(\mathcal{P}) = \sum_{p\in\mathcal{P}} -\mu(P)\ln\mu(P)$ (sometimes we write $H_\mu(\mathcal{P})$ for precision). Let $(X, \mathscr{X}, \mu, G)$ be a measure-preserving dynamical system. For a finite set $F \subseteq G$, we abbreviate the join $\bigvee_{s\in F} s^{-1}\mathcal{P}$ to $\mathcal{P}^F$. When G is amenable, we write $h_\mu(\mathcal{P})$ (or sometimes $h_\mu(X,\mathcal{P})$) for the limit of $\frac{1}{|F|}H(\mathcal{P}^F)$ as F becomes more and more invariant, and we define the measure entropy $h_\mu(X)$ to be the supremum of $h_\mu(\mathcal{P})$ over all finite Borel partitions $\mathcal{P}$ of X. For general G, given a sequence $\mathfrak{s} = \{s_i\}_{i\in\mathbb{N}}$ in G we set $h_\mu(\mathcal{P};\mathfrak{s}) = \limsup_{n\to\infty} \frac{1}{n}H(\bigvee_{i=1}^n s_i^{-1}\mathcal{P})$ and define the measure sequence entropy $h_\mu(X;\mathfrak{s})$ to be the supremum of $h_\mu(\mathcal{P};\mathfrak{s})$ over all finite measurable partitions $\mathcal{P}$. The system is said to be *null* if $h_\mu(X;\mathfrak{s}) = 0$ for all sequences $\mathfrak{s}$ in G.

The conditional entropy of a finite measurable partition $\mathcal{P} = \{P_1,\dots,P_n\}$ with respect to a σ-subalgebra $\mathscr{A} \subseteq \mathscr{X}$ is defined by

$$H(\mathcal{P}|\mathscr{A}) = \int I^{\mathscr{A}}(\mathcal{P})(x)\, d\mu(x)$$

where $I^{\mathscr{A}}(\mathcal{P})(x) = -\sum_{i=1}^n \mathbf{1}_{P_i}(x)\ln\mu(P_i|\mathscr{A})(x)$ is the conditional information function. For references on entropy see [12, 47, 35].

A unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ of a discrete group G is said to be *weakly mixing* if for all $\xi,\zeta \in \mathcal{H}$ the function $f_{\xi,\zeta}(s) = \langle\pi(s)\xi,\zeta\rangle$ on G satisfies $\mathfrak{m}(|f_{\xi,\zeta}|) = 0$, where $\mathfrak{m}$ is the unique invariant mean on the space of weakly almost periodic bounded functions on G. A subset J of G is *syndetic* if there is a finite set $F \subseteq G$ such that $FJ = G$ and *thickly syndetic* if for every finite set $F \subseteq G$ the set $\bigcap_{s\in F} sJ$ is syndetic. Weak mixing is equivalent to each of the following conditions:

(1) π has no nonzero finite-dimensional subrepresentations,
(2) for every finite set $F \subseteq \mathcal{H}$ and $\varepsilon > 0$ there exists an $s \in G$ such that $|\langle\pi(s)\xi,\zeta\rangle| < \varepsilon$ for all $\xi,\zeta \in F$,
(3) for all $\xi,\zeta \in \mathcal{H}$ and $\varepsilon > 0$ the set of all $s \in G$ such that $|\langle\pi(s)\xi,\zeta\rangle| < \varepsilon$ is thickly syndetic.

We say that a measure-preserving dynamical system $(X, \mathscr{X}, \mu, G)$ is *weakly mixing* if the associated unitary representation of G on $L^2(X,\mu) \ominus \mathbb{C}\mathbf{1}$ is weakly mixing. For references on weak mixing see [2, 12].

For a probability space $(X, \mathscr{X}, \mu)$ we write $\|\cdot\|_\mu$ for the corresponding Hilbert space norm on elements of $L^\infty(X, \mu)$, i.e., $\|f\|_\mu = \mu(|f|^2)^{1/2}$.

After this paper was completed we received a preprint by Huang, Ye, and Zhang [24] which uses orbit equivalence to establish a local variational principle for measure-preserving actions of countable discrete amenable groups on compact metrizable spaces. For such systems they provide an entropy tuple variational relation (cf. Subsection 2.3 herein) and a positive answer to our Question 2.10. They also obtained what appears here as Lemma 2.24 [24, Thm. 5.11].

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2. Measure IE-tuples

Throughout this section (X, G) is a topological dynamical system with G amenable and μ is a G-invariant Borel probability measure on X.

2.1. **Measure independence density for tuples of subsets.** Our concept of measure IE-tuple is based on a notion of independence density for tuples of subsets, which in turn is formulated in terms of the concept of independence set from Definition 1.1. In the purely topological framework, we defined in [30] the independence density of a finite tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of X by taking the limit of $\frac{1}{|F|}\varphi_{\boldsymbol{A}}(F)$ as F becomes more and more invariant, where $\varphi_{\boldsymbol{A}}(F)$ denotes the maximum of $|J|$ over all $J \subseteq F$ such that $\bigcap_{s\in F} s^{-1}A_{\sigma(s)} \neq \emptyset$ for all $\sigma : F \to \{1, \dots, k\}$, i.e., J is an independence set relative to X in the terminology of Definition 1.1. In the measure setting, we only want to consider independent behaviour that is robust enough to be observable when a small portion of the space is obscured. This will translate at the function level into stability under L^2 perturbations, as illustrated by Theorem 3.7.

So for $\delta > 0$ denote by $\mathscr{B}(\mu, \delta)$ the collection of all Borel subsets D of X such that $\mu(D) \geq 1 - \delta$, and by $\mathscr{B}'(\mu, \delta)$ the collection of all maps $D : G \to \mathscr{B}(X)$ such that $\inf_{s\in G} \mu(D_s) \geq 1 - \delta$. Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X and let $\delta > 0$. For every finite subset F of G we define

$$\varphi_{\boldsymbol{A},\delta}(F) = \min_{D\in\mathscr{B}(\mu,\delta)} \max\big\{|F \cap J| : J \text{ is an independence set for } \boldsymbol{A} \text{ relative to } D\big\},$$

$$\varphi'_{\boldsymbol{A},\delta}(F) = \min_{D\in\mathscr{B}'(\mu,\delta)} \max\big\{|F \cap J| : J \text{ is an independence set for } \boldsymbol{A} \text{ relative to } D\big\}.$$

Since the action of G on X is μ-preserving, we have $\varphi_{\boldsymbol{A},\delta}(Fs) = \varphi_{\boldsymbol{A},\delta}(F)$ and $\varphi'_{\boldsymbol{A},\delta}(Fs) = \varphi'_{\boldsymbol{A},\delta}(F)$ for all finite sets $F \subseteq G$ and $s \in G$. However, neither $\varphi_{\boldsymbol{A},\delta}$ nor $\varphi'_{\boldsymbol{A},\delta}$ satisfy the subadditivity condition in Proposition 3.22 of [30], so that the limit of $\frac{1}{|F|}\varphi_{\boldsymbol{A},\delta}(F)$ or $\frac{1}{|F|}\varphi'_{\boldsymbol{A},\delta}(F)$ as F becomes more and more invariant might not exist. We define $\overline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta)$ to be the limit supremum of $\frac{1}{|F|}\varphi_{\boldsymbol{A},\delta}(F)$ as F becomes more and more invariant, and $\underline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta)$ to be the corresponding limit infimum. Similarly, we define $\overline{\mathrm{I}}'_\mu(\boldsymbol{A}, \delta)$ to be the limit supremum of $\frac{1}{|F|}\varphi'_{\boldsymbol{A},\delta}(F)$ as F becomes more and more invariant, and $\underline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta)$ to be the corresponding limit infimum. Note that $\overline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta) \leq \overline{\mathrm{I}}_\mu(\boldsymbol{A},\delta)$ and $\underline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta) \leq \underline{\mathrm{I}}_\mu(\boldsymbol{A},\delta)$.

Definition 2.1. We set

$$\overline{\mathrm{I}}_\mu(\boldsymbol{A}) = \sup_{\delta>0} \overline{\mathrm{I}}_\mu(\boldsymbol{A},\delta) \quad \text{and} \quad \underline{\mathrm{I}}_\mu(\boldsymbol{A}) = \sup_{\delta>0} \underline{\mathrm{I}}_\mu(\boldsymbol{A},\delta)$$

and refer to these quantities respectively as the *upper μ-independence density* and *lower μ-independence density* of $\boldsymbol{A}$.

In the case $G = \mathbb{Z}$, we could alternatively take the limit infimum and supremum of averages over larger and larger intervals instead of general Følner sets. The suprema of these respective quantities over $\delta > 0$ would then lie between $\underline{\mathrm{I}}_\mu(\boldsymbol{A})$ and $\overline{\mathrm{I}}_\mu(\boldsymbol{A})$ and thus lead to the same definition of measure IE-tuples in the next subsection in view of Lemma 2.15.

In order to relate independence and c.p. approximation entropy in the local description of the Pinsker von Neumann algebra (Theorem 3.7), we will need to be able to estimate $\overline{\mathrm{I}}_\mu(\boldsymbol{A})$ and $\underline{\mathrm{I}}_\mu(\boldsymbol{A})$ from above in terms of the primed quantities $\overline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta)$ and $\underline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta)$. More precisely, if the subsets of X of measure at least $1-\delta$ relative to which independence is gauged are not required to be uniform over G, then the resulting versions of upper and lower independence density are no more than four times smaller than the original ones. This is the content of Proposition 2.4, which we now aim to establish.

By Karpovsky and Milman's generalization of the Sauer-Perles-Shelah lemma [43, 44, 25], for $k \geq 2$, positive integers $n \geq m$, and a subset S of $\{1,\dots,k\}^{\{1,\dots,n\}}$ of cardinality greater than $\sum_{j=0}^{m-1}\binom{n}{j}(k-1)^{n-j}$, there exists an $I \subseteq \{1,\dots,n\}$ with $|I| \geq m$ such that $S|_I = \{1,\dots,k\}^I$. For a fixed k, we thus see by Stirling's formula that for every $\lambda \in (\log_k(k-1),1)$ there is an $a \in (0,1)$ such that for every $n \in \mathbb{N}$ and $S \subseteq \{1,\dots,k\}^{\{1,\dots,n\}}$ with $|S| \geq k^{\lambda n}$ there exists an $I \subseteq \{1,\dots,n\}$ with $|I| \geq an$ such that $S|_I = \{1,\dots,k\}^I$. Write $a(k)$ for the supremum of all a which witness this statement for some $\lambda \in (\log_k(k-1),1)$ (this depends on k, and tends to zero as $k \to \infty$).

For the remainder of this subsection the length k of $\boldsymbol{A}$ is assumed to be at least 2.

Lemma 2.2. *For every λ in the interval $(\log_k(k-1),1)$ there are $a,b>0$ such that for every $n \in \mathbb{N}$ and $S \subseteq \{0,1,\dots,k\}^{\{1,\dots,n\}}$ with $|S| \geq k^{\lambda n}$ and $\max_{\sigma\in S}|\sigma^{-1}(0)| \leq bn$ there exists an $I \subseteq \{1,\dots,n\}$ with $|I| \geq an$ and $S|_I \supseteq \{1,\dots,k\}^{\{1,\dots,n\}}$. Moreover as $\lambda \nearrow 1$ we may choose $a \nearrow a(k)$.*

Proof. Let $\lambda \in (\log_k(k-1),1)$. Set $f(\lambda) = (1-\lambda)(\lambda - \log_k(k-1))$. Then the quantity $\lambda - f(\lambda)$ lies in the interval $(\log_k(k-1),1)$ and tends to one as $\lambda \nearrow 1$. By the result of Karpovsky and Milman as discusssed above, there is an $a \in (0,1)$ such that for every $n \in \mathbb{N}$ and $S \subseteq \{1,\dots,k\}^{\{1,\dots,n\}}$ with $|S| \geq k^{(\lambda - f(\lambda))n}$ there exists an $I \subseteq \{1,\dots,n\}$ with $|I| \geq an$ and $S|_I = \{1,\dots,k\}^I$, and we may choose $a \nearrow a(k)$ as $\lambda \nearrow 1$. By Stirling's formula there is a $b \in (0,1/2)$ such that $bn\binom{n}{bn} \leq k^{f(\lambda)n}$ for all $n \in \mathbb{N}$. Now suppose we are given an $n \in \mathbb{N}$ and $S \subseteq \{0,1,\dots,k\}^{\{1,\dots,n\}}$ with $|S| \geq k^{\lambda n}$ and $\max_{\sigma\in S}|\sigma^{-1}(0)| \leq bn$. Then we can find a $J \subseteq \{1,\dots,n\}$ with $|J| \geq (1-b)n$ such that the cardinality of the set $\{\sigma \in S : \sigma^{-1}\{1,\dots,k\} = J\}$ is at least $\frac{|S|}{bn\binom{n}{bn}} \geq k^{(\lambda - f(\lambda))n}$. Consequently there exists an

$I \subseteq J$ with $|I| \geq a|J| \geq (1-b)an$ and $S|_I \supseteq \{1,\dots,k\}^I$. Since b may be chosen to be arbitrarily small this yields the result. $\square$

Lemma 2.3. *For every $\delta > 0$ there is a $\delta' > 0$ such that $\frac{1}{|F|}\varphi'_{\boldsymbol{A},\delta'}(F) \geq a(k)\frac{1}{|F|}\varphi_{\boldsymbol{A},\delta}(F) - \delta$ for all finite sets $F \subseteq G$.*

Proof. Let $\delta > 0$, and d be a positive number to be further specified below as a function of δ. Set $\delta' = \delta d$. Let F be a finite subset of G. To establish the inequality in the proposition statement we may assume that $a(k)\varphi_{\boldsymbol{A},\delta}(F) \geq \delta|F|$. Let D be an element of $\mathscr{B}'(\mu,\delta')$ such that $\varphi'_{\boldsymbol{A},\delta'}(F)$ is equal to the maximum of $|F \cap J|$ over all independence sets J for $\boldsymbol{A}$ relative to D. Put

$$E = \{x \in X : |\{s \in F : x \notin D_s\}| \leq d|F|\}.$$

Since $\mu(D_s) \geq 1 - \delta'$ for each $s \in F$ we have

$$\mu(E^{\mathrm{c}})d|F| \leq \sum_{s\in F} \mu(D_s^{\mathrm{c}}) \leq |F|\delta'$$

and so $\mu(E) \geq 1 - \frac{\delta'}{d} = 1 - \delta$, that is, $E \in \mathscr{B}(\mu,\delta)$. Hence there exists an $I \subseteq F$ with $|I| = \varphi_{\boldsymbol{A},\delta}(F)$ which is an independence set for $\boldsymbol{A}$ relative to E. For each $\sigma \in \{1,\dots,k\}^I$ we can find by the definition of E a set $I_\sigma \subseteq I$ with $|I \setminus I_\sigma| \leq d|F|$ such that $\bigcap_{s\in I_\sigma}(D_s \cap s^{-1}A_{\sigma(s)}) \neq \emptyset$, and we define $\rho_\sigma \in \{0,1,\dots,k\}^I$ by

$$\rho_\sigma(s) = \begin{cases} \sigma(s) & \text{if } s \in I_\sigma, \\ 0 & \text{if } s \notin I_\sigma. \end{cases}$$

Since for every $\rho \in \{0,1,\dots,k\}^I$ the number of $\sigma \in \{1,\dots,k\}^I$ for which $\rho_\sigma = \rho$ is at most $k^{d|F|}$, the set $\mathcal{S} = \{\rho_\sigma : \sigma \in \{1,\dots,k\}^I\}$ has cardinality at least $k^{|I|}/k^{d|F|} \geq k^{(1-a(k)d/\delta)|I|}$. It follows by Lemma 2.2 that if d is small enough as a function of δ then there exists a $J \subseteq I$ with $|J| \geq (1-\delta)a(k)|I|$ such that $\mathcal{S}|_J \supseteq \{1,\dots,k\}^J$. Such a J is an independence set for $\boldsymbol{A}$ relative to D, and so we conclude that $\frac{1}{|F|}\varphi'_{\boldsymbol{A},\delta'}(F) \geq \frac{1}{|F|}(1-\delta)a(k)\varphi_{\boldsymbol{A},\delta}(F) \geq a(k)\frac{1}{|F|}\varphi_{\boldsymbol{A},\delta}(F) - \delta$. Since our choice of δ' does not depend on F this completes the proof. $\square$

It follows from Lemma 2.3 that for every $\delta > 0$ there is a $\delta' > 0$ such that $\overline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta') \geq a(k)\overline{\mathrm{I}}_\mu(\boldsymbol{A},\delta) - \delta$ and $\underline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta') \geq a(k)\underline{\mathrm{I}}_\mu(\boldsymbol{A},\delta) - \delta$. We thus obtain the following alternative means of estimating upper and lower μ-independence density.

Proposition 2.4. *We have*

$$\overline{\mathrm{I}}_\mu(\boldsymbol{A}) \leq a(k)^{-1} \sup_{\delta>0} \overline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta),$$
$$\underline{\mathrm{I}}_\mu(\boldsymbol{A}) \leq a(k)^{-1} \sup_{\delta>0} \underline{\mathrm{I}}'_\mu(\boldsymbol{A},\delta).$$

2.2. **Definition and basic properties of measure IE-tuples.** In [30] we defined a tuple $\boldsymbol{x} = (x_1,\dots,x_k) \in X^k$ to be an IE-tuple (or an IE-pair in the case $k = 2$) if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the G-orbit of the tuple $(U_1,\dots,U_k)$ has an independent subcollection of positive density. The following is the measure-theoretic analogue.

Definition 2.5. We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ a *μ-IE-tuple* (or *μ-IE-pair* in the case $k = 2$) if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the tuple $(U_1, \dots, U_k)$ has positive upper μ-independence density. We denote the set of μ-IE-tuples of length k by $\mathrm{IE}^\mu_k(X)$.

Evidently every μ-IE-tuple is an IE-tuple. The problem of realizing IE-tuples as μ-IE-tuples for some μ will be addressed in Subsection 2.3.

We proceed now with a series of lemmas which will enable us to establish some properties of μ-IE-tuples as recorded in Proposition 2.16.

Lemma 2.6. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X which has positive upper μ-independence density. Suppose that $A_1 = A_{1,1} \cup A_{1,2}$. Then at least one of the tuples $\boldsymbol{A}_1 = (A_{1,1}, A_2, \dots, A_k)$ and $\boldsymbol{A}_2 = (A_{1,2}, A_2, \dots, A_k)$ has positive upper μ-independence density.*

Proof. By Lemma 3.6 of [30] there is a constant $c > 0$ depending only on k such that, for all $n \in \mathbb{N}$, if S is a subset of $(\{(1,0),(1,1)\} \cup \{2, \dots, k\})^{\{1,\dots,n\}}$ for which the restriction $\Gamma_n|_S$ is bijective, where $\Gamma_n : (\{(1,0),(1,1)\} \cup \{2,\dots,k\})^{\{1,\dots,n\}} \to \{1,\dots,k\}^{\{1,\dots,n\}}$ converts the coordinate values $(1,0)$ and $(1,1)$ to 1, then there is an $I \subseteq \{1,\dots,n\}$ with $|I| \geq cn$ and either $S|_I \supseteq (\{(1,0)\} \cup \{2,\dots,k\})^I$ or $S|_I \supseteq (\{(1,1)\} \cup \{2,\dots,k\})^I$. Thus, given sets $D_1, D_2 \subseteq X$, any finite set $I \subseteq G$ which is an independence set for $\boldsymbol{A}$ relative to $D_1 \cap D_2$ has a subset J of cardinality at least $c|I|$ which is either an independence set for $\boldsymbol{A}_1$ relative to $D_1 \cap D_2$ (and hence relative to D_1) or an independence set for $\boldsymbol{A}_2$ relative to $D_1 \cap D_2$ (and hence relative to D_2). Given a $\delta > 0$, we have $D_1 \cap D_2 \in \mathscr{B}(\mu, \delta)$ whenever $D_1, D_2 \in \mathscr{B}(\mu, \delta/2)$ and so we deduce that $\max\{\bar{\mathrm{I}}_\mu(\boldsymbol{A}_1, \delta/2), \bar{\mathrm{I}}_\mu(\boldsymbol{A}_2, \delta/2)\} \geq c \cdot \bar{\mathrm{I}}_\mu(\boldsymbol{A}, \delta)$. By hypothesis there is a $\delta > 0$ such that $\bar{\mathrm{I}}_\mu(\boldsymbol{A}, \delta) > 0$, from which we conclude that $\bar{\mathrm{I}}_\mu(\boldsymbol{A}_j, \delta/2) > 0$ for at least one $j \in \{0, 1\}$, yielding the proposition. □

Lemma 2.7. *For every $d > 0$ there exist $\delta > 0$, $c > 0$, and $M > 0$ such that if F is a finite subset of G with $|F| \geq M$, D is in $\mathscr{B}'(\mu, \delta)$, $\mathcal{P} = \{P_1, P_2\}$ is a Borel partition of X with $\frac{H(\mathcal{P}^F)}{|F|} \geq d$, and $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ are Borel sets with $\mu(P_1 \setminus A_1), \mu(P_2 \setminus A_2) < \delta$, then (A_1, A_2) has a μ-independence set $I \subseteq F$ relative to D with $|I| \geq c|F|$.*

Proof. Let $d > 0$. Given a finite set $F \subseteq G$, denote by $\mathcal{Y}$ the set of all $Y \in \mathcal{P}^F$ such that $\mu(Y) < e^{-\frac{d}{3}|F|}$ and by $\mathcal{Z}$ the set of all $Z \in \mathcal{P}^F$ such that $\mu(Z) \geq e^{-\frac{d}{3}|F|}$. Put $B = \bigcup \mathcal{Y}$. Since the function $f(x) = -x \ln x$ for $x \in [0,1]$ is concave downward and has maximal value e^{-1}, we have

$$\begin{aligned}\sum_{Y \in \mathcal{Y}} -\mu(Y) \ln \mu(Y) &\leq -\mu(B) \ln \frac{\mu(B)}{|\mathcal{Y}|} \\ &= \mu(B) \ln |\mathcal{Y}| - \mu(B) \ln \mu(B) \\ &\leq (\mu(B) \cdot \ln 2)|F| + e^{-1}.\end{aligned}$$

We also have

$$\sum_{Z \in \mathcal{Z}} -\mu(Z) \ln \mu(Z) \leq \sum_{Z \in \mathcal{Z}} \mu(Z) \ln e^{\frac{d}{3}|F|} \leq \frac{d}{3}|F|.$$

Thus

$$\begin{aligned} d \leq \frac{H(\mathcal{P}^F)}{|F|} &= \frac{\sum_{Y\in\mathcal{Y}} -\mu(Y)\ln\mu(Y) + \sum_{Z\in\mathcal{Z}} -\mu(Z)\ln\mu(Z)}{|F|} \\ &\leq \frac{(\mu(B)\cdot\ln 2)|F| + e^{-1} + \frac{d}{3}|F|}{|F|} \\ &= \mu(B)\cdot\ln 2 + \frac{d}{3} + \frac{e^{-1}}{|F|}. \end{aligned}$$

Choose an $M \geq \frac{3e^{-1}}{d(2-2\ln 2)}$ such that $\frac{d}{3}e^{\frac{d}{6}M} \geq 1$. We will suppose henceforth that $|F| \geq M$, in which case $\mu(B) \geq \frac{2}{3}d$.

By Lemma 2.2, there are $b, c > 0$ (depending on d) such that for every nonempty finite set K and $S \subseteq \{0,1,2\}^K$ with $|S| \geq e^{\frac{d}{12}|K|}$ and $\max_{\sigma\in S}|\sigma^{-1}(0)| \leq b|K|$ there exists an $I \subseteq K$ with $|I| \geq c|K|$ and $S|_I \supseteq \{1,2\}^I$. We may assume that $2^b \leq e^{\frac{d}{12}}$.

Set $\delta = \frac{db}{9}$. Then $\mu(X \setminus (D_s \cap s^{-1}(A_1 \cup A_2))) \leq 3\delta = \frac{db}{3}$ for every $s \in G$. Set

$$W = \{x \in X : |\{s \in F : x \in D_s \cap s^{-1}(A_1 \cup A_2)\}| \geq (1-b)|F|\},$$

which has measure at least

$$1 - \frac{1}{b|F|}\sum_{s\in F}\mu\big(X \setminus (D_s \cap s^{-1}(A_1 \cup A_2))\big) \geq 1 - \frac{1}{b|F|}\cdot|F|\frac{db}{3} = 1 - \frac{d}{3}.$$

Then $\mu(W \cap B) \geq \frac{d}{3}$. Thus the set $\mathcal{Y}'$ of all $Y \in \mathcal{Y}$ for which $\mu(W \cap Y) > 0$ has cardinality at least $\frac{d}{3}e^{\frac{d}{3}|F|} \geq e^{\frac{d}{6}|F|}$. For each $Y \in \mathcal{Y}'$ pick an $x_Y \in W \cap Y$. Define a map $\varphi : \mathcal{Y}' \to \{0,1,2\}^F$ by

$$\varphi(Y)(s) = \begin{cases} 0 & \text{if } x_Y \notin D_s \cap s^{-1}(A_1 \cup A_2), \\ 1 & \text{if } x_Y \in D_s \cap s^{-1}A_1, \\ 2 & \text{if } x_Y \in D_s \cap s^{-1}A_2, \end{cases}$$

for $Y \in \mathcal{Y}'$ and $s \in F$. If $\varphi(Y_1) = \varphi(Y_2)$, then Y_1 and Y_2 coincide on a subset of F with cardinality at least $(1-b)|F|$. Hence $|\varphi(\mathcal{Y}')| \geq |\mathcal{Y}'|/2^{b|F|} \geq e^{\frac{d}{12}|F|}$. Therefore there exists an $I \subseteq F$ such that $|I| \geq c|F|$ and $\varphi(\mathcal{Y}')|_I \supseteq \{1,2\}^I$. Then I is a μ-independence set for (A_1, A_2) relative to D. ☐

We remark that the constants δ, c, and M specified in the proof of Lemma 2.7 do not depend on (X, G) or μ.

Lemma 2.8. *Let $\mathcal{P} = \{P_1, P_2\}$ be a two-element Borel partition of X such that $h_\mu(\mathcal{P}) > 0$. Then there exists an $\varepsilon > 0$ such that $\underline{\mathrm{I}}_\mu(\boldsymbol{A}) > 0$ whenever $\boldsymbol{A} = (A_1, A_2)$ for Borel subsets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ with $\mu(P_1 \setminus A_1), \mu(P_2 \setminus A_2) < \varepsilon$.*

Proof. Apply Lemma 2.7. ☐

Lemma 2.9. *Let A be a Borel subset of X with $\mu(A) > 0$. Then there are $d > 0$ and $\delta > 0$ such that for every finite subset $F \subseteq G$ and $D \in \mathscr{B}(\mu, \delta)$ there is an $H \subseteq F$ with $|H| \geq d|F|$ and $D \cap \big(\bigcap_{s\in H} s^{-1}A\big) \neq \emptyset$.*

Proof. Choose a $d > 0$ less than $\mu(A)$ and set $E = \{x \in X : |\{g \in F : gx \in A\}| \geq d|F|\}$. Then $(1-d)|F|1_{X\setminus E} \leq \sum_{g\in F} 1_{g^{-1}(X\setminus A)}$ so that

$$(1-d)|F|(1-\mu(E)) = \int (1-d)|F|1_{X\setminus E}\, d\mu \leq \int \sum_{g\in F} 1_{g^{-1}(X\setminus A)}\, d\mu = |F|(1-\mu(A))$$

and hence $\mu(E) \geq 1 - \frac{1-\mu(A)}{1-d} > 0$. We can thus take δ to be any strictly positive number less than $1 - \frac{1-\mu(A)}{1-d}$. $\square$

In order to determine the behaviour of measure IE-tuples under taking factors and to establish the main results of the next two subsections, we need to consider several auxiliary entropy quantities. Let $\mathcal{U}$ be a finite Borel cover of X. For a subset D of X denote by $N_D(\mathcal{U})$ the minimal number of members of $\mathcal{U}$ needed to cover D. For $\delta > 0$ we set $N_\delta(\mathcal{U}) = \min_{D\in\mathscr{B}(\mu,\delta)} N_D(\mathcal{U})$ and write $\underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)$ for the limit infimum of $\frac{1}{|F|}\ln N_\delta(\mathcal{U}^F)$ as F becomes more and more invariant and $\overline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)$ for the limit supremum of $\frac{1}{|F|}\ln N_\delta(\mathcal{U}^F)$ as F becomes more and more invariant. We then define

$$\underline{h}_{\mathrm{c},\mu}(\mathcal{U}) = \sup_{\delta>0} \underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta),$$
$$\overline{h}_{\mathrm{c},\mu}(\mathcal{U}) = \sup_{\delta>0} \overline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta).$$

The metric versions of $\underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)$ and $\overline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)$ in the ergodic $\mathbb{Z}$-system case appear in the entropy formulas of Katok from [26]. Writing $H(\mathcal{U})$ for the infimum of $H(\mathcal{P})$ over all Borel parititions $\mathcal{P}$ of X refining $\mathcal{U}$, we define $h^-_\mu(\mathcal{U})$ to be the limit of $\frac{1}{|F|}H(\mathcal{U}^F)$ as F becomes more and more invariant. Finally, we define $h^+_\mu(\mathcal{U})$ to be the infimum of $h_\mu(\mathcal{P})$ over all Borel parititions $\mathcal{P}$ of X refining $\mathcal{U}$. The quantities $h^-_\mu(\mathcal{U})$ and $h^+_\mu(\mathcal{U})$ were introduced by Romagnoli in the case $G = \mathbb{Z}$ [39]. We have the trivial inequalities $\underline{h}_{\mathrm{c},\mu}(\mathcal{U}) \leq \overline{h}_{\mathrm{c},\mu}(\mathcal{U})$ and $h^-_\mu(\mathcal{U}) \leq h^+_\mu(\mathcal{U})$. Huang, Ye, and Zhang observed in [23] that results in [17, 20, 39] can be combined to deduce that $h^-_\mu(\mathcal{U}) = h^+_\mu(\mathcal{U})$ for all open covers $\mathcal{U}$ when X is metrizable and $G = \mathbb{Z}$.

Question 2.10. Is it always the case that $h^-_\mu(\mathcal{U}) = h^+_\mu(\mathcal{U})$ for an open cover $\mathcal{U}$?

The following fact was established by Romagnoli [39, Eqn. (8)].

Lemma 2.11. *Let $\pi : X \to Y$ be a factor of X. Then*

$$H_\mu(\pi^{-1}\mathcal{U}) = H_{\pi_*(\mu)}(\mathcal{U})$$

for every finite Borel cover $\mathcal{U}$ of Y.

One direct consequence of Lemma 2.11 is the following, which in the case $G = \mathbb{Z}$ is recorded as Proposition 6 in [39].

Lemma 2.12. *Let $\pi : X \to Y$ be a factor of X. Then*

$$h^-_\mu(\pi^{-1}\mathcal{U}) = h^-_{\pi_*(\mu)}(\mathcal{U})$$

for every finite Borel cover $\mathcal{U}$ of Y.

Lemma 2.13. *For a finite Borel cover $\mathcal{U}$ of X and $\delta > 0$ we have*

$$\delta \cdot \overline{h}_{c,\mu}(\mathcal{U}, \delta) \leq h_\mu^-(\mathcal{U}) \leq \underline{h}_{c,\mu}(\mathcal{U}).$$

Proof. Let $\varepsilon > 0$ and $\delta > 0$. When a finite subset F of G is sufficiently invariant, we have $\frac{1}{|F|} H_\mu(\mathcal{U}^F) \leq h_\mu^-(\mathcal{U}) + \varepsilon$. Then we can find a finite Borel partition $\mathcal{P} \succeq \mathcal{U}^F$ with $\frac{1}{|F|} H_\mu(\mathcal{P}) \leq h_\mu^-(\mathcal{U}) + 2\varepsilon$. Consider the set $\mathcal{Y}$ consisting of members of $\mathcal{P}$ with μ-measure at least $e^{-|F|(h_\mu^-(\mathcal{U})+2\varepsilon)/\delta}$ and set $D = \bigcup \mathcal{Y}$. Then $\mu(D^c) \leq \delta$. Thus $D \in \mathscr{B}(\mu, \delta)$ and hence $N_\delta(\mathcal{U}^F) \leq |\mathcal{Y}| \leq e^{|F|(h_\mu^-(\mathcal{U})+2\varepsilon)/\delta}$. Consequently, $\overline{h}_{c,\mu}(\mathcal{U}, \delta) \leq (h_\mu^-(\mathcal{U}) + 2\varepsilon)/\delta$. Letting $\varepsilon \to 0$ we obtain $\delta \cdot \overline{h}_{c,\mu}(\mathcal{U}, \delta) \leq h_\mu^-(\mathcal{U})$.

For the second inequality, let $\varepsilon > 0$ and $\delta \in (0, e^{-1})$. Take a finite subset F of G sufficiently invariant so that $\frac{1}{|F|} \ln N_\delta(\mathcal{U}^F) < \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon$. Then we can find a $D \in \mathscr{B}(\mu, \delta)$ with $\frac{1}{|F|} \ln N_D(\mathcal{U}^F) < \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon$. Take a Borel partition $\mathcal{Y}$ of D finer than the restriction of $\mathcal{U}^F$ to D with cardinality $N_D(\mathcal{U}^F)$ and a Borel partition $\mathcal{Z}$ of D^c finer than the restriction of $\mathcal{U}^F$ to D^c with cardinality $N_{D^c}(\mathcal{U}^F)$. Since the function $x \mapsto -x \ln x$ is concave on $[0, 1]$ and increasing on $[0, e^{-1}]$ and decreasing on $[e^{-1}, 1]$, we have

$$\begin{aligned} -\sum_{P \in \mathcal{Y}} \mu(P) \ln \mu(P) &\leq -\mu(D) \ln \frac{\mu(D)}{|\mathcal{Y}|} \\ &\leq -(1 - \delta) \ln(1 - \delta) + \ln N_D(\mathcal{U}^F) \\ &\leq -(1 - \delta) \ln(1 - \delta) + |F|(\underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon), \end{aligned}$$

and

$$\begin{aligned} -\sum_{P \in \mathcal{Z}} \mu(P) \ln \mu(P) &\leq -\mu(D^c) \ln \frac{\mu(D^c)}{|\mathcal{Z}|} \\ &\leq -\delta \ln \delta + \delta \ln N_{D^c}(\mathcal{U}^F) \\ &\leq -\delta \ln \delta + \delta |F| \ln |\mathcal{U}|. \end{aligned}$$

Thus $\frac{1}{|F|} H_\mu(\mathcal{U}^F) \leq -(1 - \delta) \ln(1 - \delta) - \delta \ln \delta + \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon + \delta \ln |\mathcal{U}|$ and hence

$$h_\mu^-(\mathcal{U}) \leq -(1 - \delta) \ln(1 - \delta) - \delta \ln \delta + \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon + \delta \ln |\mathcal{U}|.$$

Letting $\varepsilon \to 0$ and $\delta \to 0$ we get $h_\mu^-(\mathcal{U}) \leq \underline{h}_{c,\mu}(\mathcal{U})$. □

Let $k \geq 2$ and let Z be a nonempty finite set. We write $\mathcal{W}$ for the cover of $\{0, 1, \ldots, k\}^Z = \prod_{z \in Z} \{0, 1, \ldots, k\}$ consisting of subsets of the form $\prod_{z \in Z} \{i_z\}^c$, where $1 \leq i_z \leq k$ for each $z \in Z$. For a set $S \subseteq \{0, 1, \ldots, k\}^Z$ we denote by F_S the minimal number of sets in $\mathcal{W}$ one needs to cover S. The following lemma provides a converse to [30, Lemma 3.3].

Lemma 2.14. *Let $k \geq 2$. For every finite set Z and $S \subseteq \{0, 1, \ldots, k\}^Z$, if $S|_W \supseteq \{1, \ldots, k\}^W$ for some nonempty set $W \subseteq Z$, then $F_S \geq \left(\frac{k}{k-1}\right)^{|W|}$.*

Proof. Replacing S by $S|_W$ we may assume that $W = Z$. We prove the assertion by induction on $|Z|$. The case $|Z| = 1$ is trivial. Suppose that the assertion holds for $|Z| = n$. Consider the case $|Z| = n + 1$. Take $z \in Z$ and set $Y = Z \setminus \{z\}$. For each $1 \leq j \leq k$ write S_j for the set of all elements of S taking value j at z. Then $S_j|_Y \supseteq \{1, \ldots, k\}^Y$, and so $F_{S_j} \geq \left(\frac{k}{k-1}\right)^{|Y|}$. Now suppose that some $\mathcal{V} \subseteq \mathcal{W}$ covers S. Write $\mathcal{V}_j$ for the set of all elements of $\mathcal{V}$ that have nonempty intersection with S_j. Then $|\mathcal{V}_j| \geq F_{S_j} \geq \left(\frac{k}{k-1}\right)^{|Y|}$. Note that each element of $\mathcal{V}$ is contained in at most $k-1$ many of the sets $\mathcal{V}_1, \dots, \mathcal{V}_k$. Thus $(k-1)|\mathcal{V}| \geq \sum_{j=1}^{k} |\mathcal{V}_j| \geq k\left(\frac{k}{k-1}\right)^{|Y|}$, and hence $|\mathcal{V}| \geq \left(\frac{k}{k-1}\right)^{|Z|}$, completing the induction. ☐

Lemma 2.15. *For a finite Borel cover $\mathcal{U}$ of X, the three quantities $h^-_\mu(\mathcal{U})$, $\underline{h}_{c,\mu}(\mathcal{U})$, and $\overline{h}_{c,\mu}(\mathcal{U})$ are either all zero or all nonzero. If the complements in X of the members of $\mathcal{U}$ are pairwise disjoint and $\boldsymbol{A}$ is a tuple consisting of these complements, then we may also add $\underline{\mathrm{I}}_\mu(\boldsymbol{A})$ and $\overline{\mathrm{I}}_\mu(\boldsymbol{A})$ to the list.*

Proof. The first assertion follows from Lemma 2.13. If $\boldsymbol{A}$ is a tuple as in the lemma statement, then Lemma 3.3 of [30] and Lemma 2.14 yield the equivalence of $\underline{h}_{c,\mu}(\mathcal{U}) > 0$ and $\underline{\mathrm{I}}_\mu(\boldsymbol{A}) > 0$ as well as the equivalence of $\overline{h}_{c,\mu}(\mathcal{U}) > 0$ and $\overline{\mathrm{I}}_\mu(\boldsymbol{A}) > 0$. ☐

Proposition 2.16. *The following hold:*

1. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of closed subsets of X which has positive upper μ-independence density. Then there exists a μ-IE-tuple $(x_1, \dots, x_k)$ with $x_j \in A_j$ for $j = 1, \dots, k$.*
2. *$\mathrm{IE}^\mu_2(X) \setminus \Delta_2(X)$ is nonempty if and only if $h_\mu(X) > 0$.*
3. *$\mathrm{IE}^\mu_1(X) = \mathrm{supp}(\mu)$.*
4. *$\mathrm{IE}^\mu_k(X)$ is a closed G-invariant subset of X^k.*
5. *Let $\pi : X \to Y$ be a topological G-factor map. Then $\pi^k(\mathrm{IE}^\mu_k(X)) = \mathrm{IE}^{\pi_*(\mu)}_k(Y)$.*

Proof. (1) Apply Lemma 2.6 and a compactness argument.

(2) As is well known and easy to show, $h_\mu(X) > 0$ if and only if there is a two-element Borel partition of X with positive entropy. We can thus apply (1) and Lemma 2.8 to obtain the "if" part. The "only if" part follows from Lemma 2.15.

(3) This follows from Lemma 2.9.

(4) Trivial.

(5) This follows from (1), (3), (4), and Lemmas 2.12 and 2.15. ☐

2.3. **IE-tuples and measure IE-tuples.** Here we will show that the set of IE-tuples of length k is equal to the closure of the union of the sets $\mathrm{IE}^k_\mu(X)$ over all G-invariant Borel probability measures μ on X, and furthermore that when X is metrizable there exists a G-invariant Borel probability measure μ on X such that the sets of μ-IE-tuples and IE-tuples coincide.

We will need a version of the Rokhlin tower lemma. Following [42], for a finite set $F \subseteq G$ and a Borel subset V of X we say that $F \times V$ maps to an *ε-quasi-tower* if there exists a measurable subset $A \subseteq F \times V$ such that the map $A \to X$ sending (s, x) to sx is one-to-one and for each $x \in V$ the cardinality of $\{s \in F : (s, x) \in A\}$ is at least $(1 - \varepsilon)|F|$. The case $\delta = 0$ of the following theorem is a direct consequence of Theorem 5 on page 59 of [35]. The general case $\delta > 0$ follows from the proof given there. Note that although the acting groups are generally assumed to be countable in [35], this assumption is not necessary here.

Theorem 2.17. *Let $1 > \varepsilon > 0$ and $\frac{\varepsilon^2}{4} > \delta > 0$. Then whenever the action of G is free with respect to μ, $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ are nonempty finite subsets of G such that F_{j+1} is $(F_j F_j^{-1}, \eta_j)$-invariant and $\eta_j |F_j| < \frac{\varepsilon^2}{4}$ for all $1 \leq j < k$, $(1 - \frac{\varepsilon}{2})^k < \varepsilon$, and $D_1, \dots, D_k$ are Borel subsets of X with μ-measure at least $1 - \delta$, one can find Borel subsets $V_1, \dots, V_k$ such that*

1. *each $F_j \times V_j$ maps to an ε-quasi-tower,*
2. *$F_i V_i \cap F_j V_j = \emptyset$ for $i \neq j$,*
3. *$\mu\big(\bigcup_{j=1}^k F_j V_j\big) > 1 - \varepsilon$,*
4. *$V_j \subseteq D_j$ for each j.*

For the definitions of the quantities $h_\mu^+(\mathcal{U})$ and $\underline{h}_{c,\mu}(\mathcal{U})$ see the discussion after Lemma 2.9.

Lemma 2.18. *Suppose that G is infinite and the action of G is free with respect to μ. Let $\mathcal{U}$ be a finite Borel cover of X. Then $h_\mu^+(\mathcal{U}) \leq \underline{h}_{c,\mu}(\mathcal{U})$.*

Proof. Let $1 > \varepsilon > 0$ and $\frac{\varepsilon^2}{4} > \delta > 0$. Then we can find nonempty finite subsets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ of G satisfying the conditions of Theorem 2.17 and $\frac{1}{|F_j|} \ln N_\delta(\mathcal{U}^{F_j}) < \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon$ for $j = 1, \dots, k$. For each $j = 1, \dots, k$ take a $D_j \in \mathscr{B}(\mu, \delta)$ such that $\frac{1}{|F_j|} \ln N_{D_j}(\mathcal{U}^{F_j}) < \underline{h}_{c,\mu}(\mathcal{U}, \delta) + \varepsilon$. Then we can find Borel sets $V_1, \dots, V_k \subseteq X$ satisfying the conclusion of Theorem 2.17.

For $j = 1, \dots, k$ pick a Borel partition $\mathcal{P}_j$ of D_j which is finer than the restriction of $\mathcal{U}^{F_j}$ to D_j and has cardinality $N_{D_j}(\mathcal{U}^{F_j})$. For each $P \in \mathcal{P}_j$ fix a $U_{P,s} \in \mathcal{U}$ for each $s \in F_j$ such that $P \subseteq \bigcap_{s \in F_j} s^{-1} U_{P,s}$. Since $F_j \times V_j$ maps to an ε-quasi-tower, we can find a measurable subset A_j of $F_j \times V_j$ such that $T|_{A_j} : A_j \to X$ is one-to-one, where $T : G \times X \to X$ is the map $(s, x) \mapsto sx$, and $|\{s \in F_j : (s, x) \in A_j\}| \geq (1 - \varepsilon)|F_j|$ for each $x \in V_j$. Define a Borel partition $\mathcal{Y} = \{Y_U : U \in \mathcal{U}\}$ of $\bigcup_j T(A_j)$ finer than the restriction of $\mathcal{U}$ to $\bigcup_j T(A_j)$ by stipulating that, for each $(s, x) \in A_j$ with $x \in P \in \mathcal{P}_j$, $sx \in Y_U$ exactly when $U = U_{P,s}$. Take a Borel partition $\mathcal{Z} = \{Z_U : U \in \mathcal{U}\}$ of $(\bigcup_j T(A_j))^c$ with $Z_U \subseteq U$ for each $U \in \mathcal{U}$. Set $P_U = Y_U \cup Z_U$ for each $U \in \mathcal{U}$. Then $\mathcal{P} = \{P_U : U \in \mathcal{U}\}$ is a Borel partition of X finer than $\mathcal{U}$. Note that $\mu(T(A_j)) \geq (1 - \varepsilon)\mu(F_j V_j)$ for each j. Thus $\mu(\bigcup_j T(A_j)) > (1 - \varepsilon)^2$.

Next we estimate $h_\mu(\mathcal{P})$. Suppose that F is a finite subset of G which is $((\bigcup_j F_j)(\bigcup_j F_j)^{-1}, \sqrt{\varepsilon})$-invariant. Set $F_x = \big\{s \in F : sx \in \bigcup_j T(A_j)\big\}$ for each $x \in X$ and put $W = \{x \in X : |F_x| \geq (1 - \sqrt{\varepsilon})|F|\}$. It is easy to see that $\mu(W^c) \leq \mu((\bigcup_j T(A_j))^c)/\sqrt{\varepsilon} < 2\sqrt{\varepsilon}$. Replacing W by $W \setminus \bigcup_{s \in F^{-1}F \setminus \{e_G\}} \{x \in X : sx = x\}$ we may assume that $s_1 x \neq s_2 x$ for all $x \in W$ and all distinct $s_1, s_2 \in F$. Let us estimate the number M of atoms of $\mathcal{P}^F$ which have nonempty intersection with W. Write $\mathfrak{H}_j$ for the collection of all subsets of F_j with cardinality at least $(1 - \varepsilon)|F_j|$. For each $x \in W$, setting $F'_x = F_x \cap \big\{s \in F : (\bigcup_j F_j)(\bigcup_j F_j)^{-1} s \subseteq F\big\}$, we have $|F'_x| \geq (1 - 2\sqrt{\varepsilon})|F|$. Note that if $(s, y) \in A_j$ for some $1 \leq j \leq k$ and $sy = s'x$ for some $s' \in F'_x$, setting $c = s^{-1}s'$ and $H = \{h \in F_j : (h, y) \in A_j\}$, we have $y = cx$, $Hc \subseteq F_x$ and $H \in \mathfrak{H}_j$. Thus for each $w \in W$ we can find a finite set $C_{j,H} \subseteq G$ for every $H \in \mathfrak{H}_j$ such that the following hold:

1. $Hc \cap H'c' = \emptyset$ for all $c \in C_{j,H}, c' \in C_{j',H'}$ unless $H = H'$, $c = c'$, and $j = j'$,
2. $\bigcup_{j,H} HC_{j,H} \subseteq F$ and $\big|\bigcup_{j,H} HC_{j,H}\big| \geq (1 - 2\sqrt{\varepsilon})|F|$,

(3) $cx \in V_j$ and $H = \{h \in F_j : (h, cx) \in A_j\}$ for each $c \in C_{j,H}$.

Note that the atom of $\mathcal{P}$ to which hcx for $h \in H$ belongs is determined by h and the atom of $\mathcal{P}_j$ to which cx belongs. Thus, for each fixed choice of sets $C_{j,H}$ satisfying (1) and (2) above, the number of atoms of $\mathcal{P}^F$ containing some $x \in W$ with such a choice of $C_{j,H}$ is at most

$$\begin{aligned}
|\mathcal{U}|^{2\sqrt{\varepsilon}|F|} \cdot \prod_j |\mathcal{P}_j|^{\sum_{H\in\mathfrak{H}_j}|C_{j,H}|} &\leq |\mathcal{U}|^{2\sqrt{\varepsilon}|F|} \cdot \prod_j \exp\big[(\underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)+\varepsilon)|F_j|{\textstyle\sum_{H\in\mathfrak{H}_j}}|C_{j,H}|\big] \\
&= |\mathcal{U}|^{2\sqrt{\varepsilon}|F|} \cdot \exp\big[(\underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)+\varepsilon){\textstyle\sum_j}\big(|F_j|{\textstyle\sum_{H\in\mathfrak{H}_j}}|C_{j,H}|\big)\big] \\
&\leq |\mathcal{U}|^{2\sqrt{\varepsilon}|F|} \cdot \exp\bigg[\frac{(\underline{h}_{\mathrm{c},\mu}(\mathcal{U},\delta)+\varepsilon)|F|}{1-\varepsilon}\bigg].
\end{aligned}$$

By Stirling's formula, the number of subsets of an n-element set with cardinality at least $(1-\varepsilon)n$ is at most $e^{f(\varepsilon)n}$ for all $n \geq 0$ with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Fix an element $g_{j,H} \in H$ for each j and $H \in \mathfrak{H}_j$. Then $C_{j,H}$ is determined by the set $g_{j,H}C_{j,H}$ in F. Thus, for a fixed $Q \subseteq F$, writing $a = \min_j |F_j|$ and summing as appropriate over nonnegative integers $t_{j,H}$, t_j, or t subject to the indicated constraints, the number of choices of sets $C_{j,H}$ satisfying (1) and (2) and $\bigcup_{j,H} HC_{j,H} = Q$ is at most

$$\begin{aligned}
&\sum_{\sum_{j,H} t_{j,H}|H|=|Q|} \frac{|F|!}{(|F|-\sum_{j,H} t_{j,H})!\prod_{j,H} t_{j,H}!} \\
&\qquad\leq \sum_{(1-\varepsilon)\sum_j t_j|F_j|\leq|Q|} \frac{|F|!}{(|F|-\sum_j t_j)!\prod_j t_j!} \cdot \prod_j \sum_{\sum_{H\in\mathfrak{H}_j} t_{j,H}=t_j} \frac{|t_j|!}{\prod_{H\in\mathfrak{H}_j} t_{j,H}!} \\
&\qquad= \sum_{(1-\varepsilon)\sum_j t_j|F_j|\leq|Q|} \frac{|F|!}{(|F|-\sum_j t_j)!\prod_j t_j!} \cdot \prod_j |\mathfrak{H}_j|^{t_j} \\
&\qquad\leq \sum_{(1-\varepsilon)\sum_j t_j|F_j|\leq|Q|} \frac{|F|!}{(|F|-\sum_j t_j)!\prod_j t_j!} \cdot \prod_j e^{f(\varepsilon)t_j|F_j|} \\
&\qquad\leq \sum_{(1-\varepsilon)\sum_j t_j|F_j|\leq|Q|} \frac{|F|!}{(|F|-\sum_j t_j)!\prod_j t_j!} \cdot e^{f(\varepsilon)|F|/(1-\varepsilon)} \\
&\qquad\leq \sum_{(1-\varepsilon)at\leq|Q|} \frac{|F|!}{(|F|-t)!t!} \cdot \sum_{\sum_j t_j=t} \frac{t!}{\prod_j t_j!} \cdot e^{f(\varepsilon)|F|/(1-\varepsilon)} \\
&\qquad= \sum_{(1-\varepsilon)at\leq|Q|} \frac{|F|!}{(|F|-t)!t!} \cdot k^t \cdot e^{f(\varepsilon)|F|/(1-\varepsilon)} \\
&\qquad\leq \sum_{(1-\varepsilon)at\leq|Q|} \frac{|F|!}{(|F|-t)!t!} \cdot k^{|F|/((1-\varepsilon)a)} \cdot e^{f(\varepsilon)|F|/(1-\varepsilon)} \\
&\qquad\leq e^{f(1/((1-\varepsilon)a))|F|} \cdot k^{|F|/((1-\varepsilon)a)} \cdot e^{f(\varepsilon)|F|/(1-\varepsilon)}.
\end{aligned}$$

The number of choices of $Q \subseteq F$ with $|Q| \geq (1-2\sqrt{\varepsilon})|F|$ is at most $e^{f(2\sqrt{\varepsilon})|F|}$. Therefore, M is at most

$$|\mathcal{U}|^{2\sqrt{\varepsilon}|F|} \cdot \exp\left[\frac{(\underline{h}_{c,\mu}(\mathcal{U},\delta)+\varepsilon)|F|}{1-\varepsilon}\right] \cdot \exp\left[f\big(1/((1-\varepsilon)a)\big)|F|\right] \\ \cdot k^{|F|/((1-\varepsilon)a)} \cdot \exp\left[\frac{f(\varepsilon)|F|}{1-\varepsilon}\right] \cdot \exp\left[f(2\sqrt{\varepsilon})|F|\right].$$

Since the function $x \mapsto -x\ln x$ is concave on $[0,1]$, we have

$$\sum_{P\in\mathcal{P}^F} -\mu(P\cap W)\ln\mu(P\cap W) \leq -\mu(W)\ln\frac{\mu(W)}{M} \leq -\mu(W)\ln\mu(W) + \ln M$$

and

$$\sum_{P\in\mathcal{P}^F} -\mu(P\cap W^{c})\ln\mu(P\cap W^{c}) \leq -\mu(W^{c})\ln\frac{\mu(W^{c})}{|\mathcal{P}|^{|F|}-M} \\ \leq -\mu(W^{c})\ln\mu(W^{c}) + \mu(W^{c})|F|\ln|\mathcal{U}|.$$

Set $\mathcal{Q} = \{W, W^{c}\}$. Since the function $x \mapsto -x\ln x$ on $[0,1]$ has maximal value e^{-1}, we get

$$\begin{aligned} H(\mathcal{P}^F) \leq H(\mathcal{P}^F \vee \mathcal{Q}) &= \sum_{P\in\mathcal{P}^F} -\mu(P\cap W)\ln\mu(P\cap W) + \sum_{P\in\mathcal{P}^F} -\mu(P\cap W^{c})\ln\mu(P\cap W^{c}) \\ &\leq -\mu(W)\ln\mu(W) + \ln M - \mu(W^{c})\ln\mu(W^{c}) + \mu(W^{c})|F|\ln|\mathcal{U}| \\ &\leq 2e^{-1} + \ln M + 2\sqrt{\varepsilon}|F|\ln|\mathcal{U}|. \end{aligned}$$

Since G is infinite, $|F| \to \infty$ as F becomes more and more invariant. Therefore

$$h^{+}_{\mu}(\mathcal{U}) \leq h_{\mu}(\mathcal{P}) \leq 4\sqrt{\varepsilon}\ln|\mathcal{U}| + \frac{\underline{h}_{c,\mu}(\mathcal{U},\delta)+\varepsilon}{1-\varepsilon} + f\big(1/((1-\varepsilon)a)\big) \\ + \frac{\ln k}{(1-\varepsilon)a} + \frac{f(\varepsilon)}{1-\varepsilon} + f(2\sqrt{\varepsilon}).$$

Since we may choose $F_1,\dots,F_k$ to be as close as we wish to being invariant, we may let $a \to \infty$. Thus

$$\begin{aligned} h^{+}_{\mu}(\mathcal{U}) &\leq 4\sqrt{\varepsilon}\ln|\mathcal{U}| + \frac{\underline{h}_{c,\mu}(\mathcal{U},\delta)+\varepsilon}{1-\varepsilon} + \frac{f(\varepsilon)}{1-\varepsilon} + f(2\sqrt{\varepsilon}) \\ &\leq 4\sqrt{\varepsilon}\ln|\mathcal{U}| + \frac{\underline{h}_{c,\mu}(\mathcal{U})+\varepsilon}{1-\varepsilon} + \frac{f(\varepsilon)}{1-\varepsilon} + f(2\sqrt{\varepsilon}). \end{aligned}$$

Letting $\varepsilon \to 0$ we get $h^{+}_{\mu}(\mathcal{U}) \leq \underline{h}_{c,\mu}(\mathcal{U})$, as desired. □

Lemma 2.19. *Let μ be a Borel probability measure on X. Let $C_1,\dots,C_k$ be closed subsets of X. Then for every k-element Borel partition $\mathcal{P} = \{P_1,\dots,P_k\}$ with $P_i \cap C_i = \emptyset$ for $i = 1,\dots,k$ and every $\delta > 0$ there is a k-element Borel partition $\mathcal{Q} = \{Q_1,\dots,Q_k\}$ such that $Q_i \cap C_i = \emptyset$ and $\mu(\partial Q_i) = 0$ for $i = 1,\dots,k$ and $H_{\mu}(\mathcal{Q}|\mathcal{P}) < \delta$.*

Proof. Let $\mathcal{P} = \{P_1,\dots,P_k\}$ be a k-element Borel partition with $P_i \cap C_i = \emptyset$ for $i = 1,\dots,k$. Let $\delta > 0$. By the regularity of μ, for $i = 1,\dots,k-1$ we can find a compact set $K_i \subseteq P_i$ such that $\mu(P_i \setminus K_i) < \varepsilon$ and an open set $U_i \supseteq P_i$ such that $\mu(U_i \setminus P_i) < \varepsilon$ and $U_i \cap C_i = \emptyset$.

Then $U_1, \dots, U_{k-1}$ cover C_k. Thus we can find a closed cover $D_1, \dots, D_{k-1}$ of C_k such that $D_i \subseteq U_i$ for $i = 1, \dots, k-1$. For each $x \in K_i \cup D_i$ there exists an open neighbourhood V of x contained in U_i whose boundary has zero measure, for if we take a function $f \in C(X)$ with image in $[0,1]$ which is 0 at x and 1 on $U_i^{\rm c}$ then only countably many of the open sets $\{y \in X : f(y) < t\}$ for $t \in (0,1)$ can have boundary with positive measure. By compactness there is a finite union B_i of such V which covers $K_i \cup D_i$, and $\mu(\partial(B_i)) = 0$. Then $\mu(B_i \Delta P_i) < 2\varepsilon$ for $i = 1, \dots, k-1$. Now define the partition $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ by $Q_1 = B_1$, $Q_2 = B_2 \setminus B_1$, $Q_3 = B_3 \setminus (B_1 \cup B_2)$, $\dots$, $Q_k = X \setminus (B_1 \cup \dots \cup B_{k-1})$. Then $Q_i \cap C_i = \emptyset$ and $\mu(\partial Q_i) = 0$ for $i = 1, \dots, k$ and $H_\mu(\mathcal{Q}|\mathcal{P}) < \delta(\varepsilon)$ where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, yielding the lemma. □

Lemma 2.20. *Let $\boldsymbol{x} = (x_1, \dots, x_k)$ be an IE-tuple consisting of distinct points and let $U_1, \dots, U_k$ be pairwise disjoint open neighbourhoods of $x_1, \dots, x_k$, respectively. Then there exist a G-invariant Borel probability measure μ on X and a μ-IE-tuple $(x_1', \dots, x_k')$ such that $x_i' \in U_i$ for each $i = 1, \dots, k$.*

Proof. The case $k = 1$ follows from [30, Prop. 3.12] and Proposition 2.16(3). So we may assume $k \geq 2$.

Let $\{F_n\}_n$ be a Følner net in G. For each $i = 1, \dots, k$ choose a closed neighbourhood C_i of x_i contained in U_i. Since $\boldsymbol{x}$ is an IE-tuple there is a $d > 0$ such that for each n we can find an independence set $I_n \subseteq F_n$ for the tuple $\boldsymbol{C} = (C_1, \dots, C_k)$ such that $|I_n| \geq d|F_n|$. For each n pick an $x_\sigma \in \bigcap_{s \in I_n} s^{-1} C_{\sigma(s)}$ for every $\sigma \in \{1, \dots, k\}^{I_n}$ and define on X the following averages of point masses:

$$\nu_n = \frac{1}{k^{|I_n|}} \sum_{\sigma \in \{1,\dots,k\}^{I_n}} \delta_{x_\sigma}, \qquad \mu_n = \frac{1}{|F_n|} \sum_{s \in F_n} s\mu_n.$$

Take a weak* limit point μ of the net $\{\mu_n\}_n$. By passing to a cofinal subset of the net we may assume that μ_n converges to μ.

Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a Borel partition of X such that $P_i \cap U_i = \emptyset$ and $\mu(\partial P_i) = 0$ for each $i = 1, \dots, k$. Let E be a nonempty finite subset of G. We will use subadditivity and concavity as in the proof of the variational principle in Section 5.2 of [33]. The function $A \mapsto H_{\nu_n}(\mathcal{P}^A)$ on finite subsets of G is subadditive in the sense that if $\mathbf{1}_A = \sum \lambda_B \mathbf{1}_B$ is a finite decomposition of the indicator of a finite set $A \subseteq G$ over a collection of sets $B \subseteq A$ with each λ_B positive, then $H_{\nu_n}(\mathcal{P}^A) \leq \sum \lambda_B H_{\nu_n}(\mathcal{P}^B)$ (see Section 3.1 of [33]). Observe that $\varepsilon(n) := |E^{-1}F_n \setminus F_n|/|F_n|$ is bounded above by $|E^{-1}F_n \Delta F_n|/|F_n|$ and hence by the Følner property tends to zero along the net. Applying the subadditivity of $H_{\nu_n}(\cdot)$ to the decomposition $\mathbf{1}_{F_n} = \frac{1}{|E|} \sum_{s \in E^{-1}F_n} \mathbf{1}_{Es \cap F_n}$, we have

$$\begin{aligned} H_{\nu_n}(\mathcal{P}^{F_n}) &\leq \frac{1}{|E|} \sum_{s \in F_n} H_{\nu_n}(\mathcal{P}^{Es}) + \frac{1}{|E|} \sum_{s \in E^{-1}F_n \setminus F_n} H_{\nu_n}(\mathcal{P}^{Es}) \\ &\leq \frac{1}{|E|} \sum_{s \in F_n} H_{\nu_n}(\mathcal{P}^{Es}) + \varepsilon(n)|F_n| \ln k. \end{aligned}$$

Since $P_i \cap C_i = \emptyset$ for each i, every atom of $\mathcal{P}^{I_n}$ contains at most $(k-1)^{|I_n|}$ points from the set $\{x_\sigma : \sigma \in \{1, \dots, k\}^{I_n}\}$ and hence has ν_n-measure at most $(\frac{k-1}{k})^{|I_n|}$, so that

$$\begin{aligned} H_{\nu_n}(\mathcal{P}^{I_n}) &= \sum_{W \in \mathcal{P}^{I_n}} -\nu_n(W) \ln \nu_n(W) \\ &\geq \sum_{W \in \mathcal{P}^{I_n}} \nu_n(W) \ln \left(\frac{k}{k-1} \right)^{|I_n|} \\ &= |I_n| \ln \left(\frac{k}{k-1} \right) \end{aligned}$$

and thus

$$\frac{1}{|F_n|} H_{\nu_n}(\mathcal{P}^{F_n}) \geq \frac{1}{|F_n|} H_{\nu_n}(\mathcal{P}^{I_n}) \geq d \ln \left(\frac{k}{k-1} \right).$$

It follows using the concavity of the function $x \mapsto -x \ln x$ that

$$\frac{1}{|E|} H_{\mu_n}(\mathcal{P}^E) \geq \frac{1}{|F_n|} \sum_{s \in F_n} \frac{1}{|E|} H_{\nu_n}(\mathcal{P}^{Es}) \geq d \ln \left(\frac{k}{k-1} \right) - \varepsilon(n) \ln k.$$

Since the boundary of each P_i has zero μ-measure, the boundary of each atom of $\mathcal{P}^E$ has zero μ-measure, and so by [27, Thm. 17.20] the entropy of $\mathcal{P}^E$ is a continuous function of the measure with respect to the weak* topology, whence

$$\frac{1}{|E|} H_\mu(\mathcal{P}^E) = \lim_n \frac{1}{|E|} H_{\mu_n}(\mathcal{P}^E) \geq d \ln(k/(k-1)).$$

Since this holds for every nonempty finite set $E \subseteq G$, we obtain $h_\mu(\mathcal{P}) \geq d \ln(k/(k-1))$.

Now let $\mathcal{P} = \{P_1, \dots, P_k\}$ be any k-element Borel partition of X such that $P_i \cap U_i = \emptyset$ for each $i = 1, \dots, k$. By Lemma 2.19, for every $\delta > 0$ there is a k-element Borel partition $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ such that $Q_i \cap C_i = \emptyset$ and $\mu(\partial Q_i) = 0$ for $i = 1, \dots, k$ and $H_\mu(\mathcal{Q}|\mathcal{P}) < \delta$, so that $h_\mu(\mathcal{P}) \geq h_\mu(\mathcal{Q}) - \delta \geq d \ln(k/(k-1)) - \delta$ by the previous paragraph. Thus $h_\mu(\mathcal{P}) \geq d \ln(k/(k-1))$. This inequality holds moreover for any finite Borel partition $\mathcal{P}$ that refines $\mathcal{U} := \{U_1^{\mathrm{c}}, \dots, U_k^{\mathrm{c}}\}$ as a cover since we may assume that $\mathcal{P}$ is of the above form by coarsening it if necessary. Therefore $h_\mu^+(\mathcal{U}) > 0$.

Suppose that the action of G on X is (topologically) free, i.e., for all $x \in X$ and $s \in G$, $sx = x$ implies $s = e$. Then it is free with respect to μ, and hence $\underline{h}_{\mathrm{c},\mu}(\mathcal{U}) > 0$ by Lemma 2.18. Therefore by Lemma 2.15 and Proposition 2.16(1) there is a μ-IE-tuple $(x_1', \dots, x_k')$ contained in $U_1 \times \cdots \times U_k$.

Now suppose that the action of G on X is not free. Take a free action of G on a compact Hausdorff space (Y, G), e.g., the universal minimal G-system [10]. Then the product system $(X \times Y, G)$ is an extension of (X, G) which is free. By Proposition 3.9(4) of [30] we can find a lift $\tilde{\boldsymbol{x}}$ of the tuple $\boldsymbol{x}$ under this extension such that $\tilde{\boldsymbol{x}}$ is an IE-tuple. By the previous paragraph there are a G-invariant Borel probability measure μ on $X \times Y$ and a μ-IE-tuple $\tilde{\boldsymbol{x}}'$ contained in the inverse image of $U_1 \times \cdots \times U_k$. It then follows by Proposition 2.16(5) that the image $\boldsymbol{x}'$ of $\tilde{\boldsymbol{x}}'$ is a ν-IE-tuple contained in $U_1 \times \cdots \times U_k$ for the measure ν on X induced from μ, completing the proof. $\square$

From Lemma 2.20 we obtain:

Theorem 2.21. *For each $k \geq 1$ the set of IE-tuples of length k is equal to the closure of of the union of the sets $\mathrm{IE}^k_\mu(X)$ over all G-invariant Borel probability measures μ on X.*

Lemma 2.22. *Suppose that X is metrizable. Let $\boldsymbol{x} = (x_1, \dots, x_k)$ be an IE-tuple. Then there is a G-invariant Borel probability measure μ on X such that $\boldsymbol{x}$ is a μ-IE-tuple.*

Proof. We may assume that $\boldsymbol{x}$ consists of distinct points. Since X is metrizable, we can find for each $m \in \mathbb{N}$ pairwise disjoint open neighbourhoods $U_{m,1}, \dots, U_{m,k}$ of $x_1, \dots, x_k$, respectively, so that for each $i = 1, \dots, k$ the family $\{U_{m,i} : m \in \mathbb{N}\}$ forms a neighbourhood basis for x_i. For each m take a measure μ_m and a μ-IE-tuple $\boldsymbol{x}_m$ as given by Lemma 2.20 with respect to $U_{m,1}, \dots, U_{m,k}$ and define the G-invariant Borel probability measure $\mu = \sum_{m=1}^\infty 2^{-m}\mu_m$. Then $\boldsymbol{x}_m$ is a μ-IE-tuple for each m, and so $\boldsymbol{x}$ is a μ-IE-tuple by Proposition 2.16(4). □

Theorem 2.23. *Suppose that X is metrizable. Then there is a G-invariant Borel probability measure μ on X such that the sets of μ-IE-tuples and IE-tuples coincide.*

Proof. For each $k \geq 1$ take a countable dense subset $\{\boldsymbol{x}_{k,i}\}_{i\in I_k}$ of the set of IE-tuples of length k. By Lemma 2.22, for every $k \geq 1$ and $i \in I_k$ there is a G-invariant Borel probability measure $\mu_{k,i}$ on X such that $\boldsymbol{x}_{k,i}$ is a $\mu_{k,i}$-IE-tuple. Set $\mu = \sum_{k=1}^\infty \sum_{i\in I_k} \lambda_{k,i}\mu_{k,i}$ for some $\lambda_{k,i} > 0$ with $\sum_{k=1}^\infty \sum_{i\in I_k} \lambda_{k,i} = 1$. Then μ is a G-invariant Borel probability measure, and $\boldsymbol{x}_{k,i}$ is a μ-IE-tuple for every $k \geq 1$ and $i \in I_k$. Since the set of μ-IE-tuples of a given length is closed by Proposition 2.16(4) and μ-IE-tuples are always IE-tuples, we obtain the desired conclusion. □

In the case $G = \mathbb{Z}$, the conclusion of Theorem 2.23 for μ-entropy pairs and topological entropy pairs was established in [3] and then more generally for μ-entropy tuples and topological entropy tuples in [22].

2.4. **The relation between μ-IE-tuples and μ-entropy tuples.** For $G = \mathbb{Z}$ the notion of a μ-entropy pair was introduced in [4] and generalized to μ-entropy tuples in [22]. We will accordingly say for $k \geq 2$ that a nondiagonal tuple $(x_1, \dots, x_k) \in X^k$ is a *μ-entropy tuple* if whenever $U_1, \dots, U_l$ are pairwise disjoint Borel neighbourhoods of the distinct points in the list $x_1, \dots, x_k$, every Borel partition of X refining $\{U_1^{\mathrm{c}}, \dots, U_l^{\mathrm{c}}\}$ has positive measure entropy. In this subsection we aim to show that nondiagonal μ-IE-tuples are the same as μ-entropy tuples.

Our first task is to establish Lemma 2.24. For this we will use the orbit equivalence technique of Rudolph and Weiss [42], which will enable us to apply a result of Huang and Ye for $\mathbb{Z}$-actions [22]. In order to invoke Theorem 2.6 of [42], whose hypotheses include ergodicity, we will need the ergodic decomposition of entropy, which asserts that if $(Y, \mathscr{Y}, \nu)$ is a Lebesgue space equipped with an action of a countable discrete amenable group H and $\nu = \int_Z \nu_z \, d\omega(z)$ is the corresponding ergodic decomposition, then for every finite measurable partition $\mathcal{P}$ of Y we have $h_\nu(\mathcal{P}) = \int_Z h_{\nu_z}(\mathcal{P})\, d\omega(z)$. The standard proof of this for $G = \mathbb{Z}$ using symbolic representations (see for example Section 15.3 of [12]) also works in the general case. Given a tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of Borel subsets of X with $\bigcap_{i=1}^k A_i = \emptyset$, we say that a finite Borel partition $\mathcal{P}$ of X is *$\boldsymbol{A}$-admissible* if it refines $\{A_1^{\mathrm{c}}, \dots, A_k^{\mathrm{c}}\}$ as a cover of X. For the definitions of the quantities $h^+_\mu(\mathcal{U})$ and $\underline{h}_{\mathrm{c},\mu}(\mathcal{U})$ see the discussion after Lemma 2.9. As the proof below involves several different systems, we will explicitly indicate the action in our notation for the various entropy quantities.

Lemma 2.24. *Suppose that X is metrizable and G is countably infinite. Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of pairwise disjoint Borel subsets of X. Denote by $\mathcal{U}$ the Borel cover $\{A_1^{\mathrm{c}}, \dots, A_k^{\mathrm{c}}\}$ of X. Suppose that $h_\mu(\mathcal{P}) > 0$ for every $\boldsymbol{A}$-admissible finite Borel partition $\mathcal{P}$ of X. Then $\underline{h}_{\mathrm{c},\mu}(\mathcal{U}) > 0$.*

Proof. Denote by T the action of G on X. Take a free weakly mixing action S of G on a Lebesgue space $(Y, \mathscr{Y}, \nu)$ (for example a Bernoulli action). We will consider the product action $T \times S$ on $(X \times Y, \mathscr{B} \otimes \mathscr{Y}, \mu \times \nu)$ and view $\mathscr{B}$ and $\mathscr{Y}$ as sub-σ-algebras of $\mathscr{B} \otimes \mathscr{Y}$ when convenient. Since S is free and ergodic, by the Connes-Feldman-Weiss theorem [6] there is an integer action $\hat{R}$ on $(Y, \mathscr{Y}, \nu)$ with the same orbits as S and we may choose $\hat{R}$ to have zero measure entropy. Now we define an integer action R on $(X \times Y, \mathscr{B} \otimes \mathscr{Y}, \mu \times \nu)$ with the same orbits as $T \times S$ by setting $R(x, y) = (T \times S)_{s(y)}(x, y)$ where $s(y)$ is the element of G determined by $\hat{R}y = S_{s(y)}y$.

Let $\pi : (X, \mathscr{B}, \mu) \to (Z, \mathscr{Z}, \omega)$ be the dynamical factor defined by the σ-algebra $\mathscr{I}_T$ of T-invariant sets in $\mathscr{B}$. We write the disintegration of μ over ω as $\mu = \int_Z \mu_z \, d\omega(z)$ and for every $z \in Z$ put $X_z = \pi^{-1}(z)$ and $\mathscr{B}_z = \mathscr{B} \cap X_z$ and denote by T_z the restriction of T to $(X_z, \mathscr{B}_z, \mu_z)$. Since S is weakly mixing, the σ-algebra $\mathscr{I}_{T \times S}$ of $(T \times S)$-invariant sets in $\mathscr{B} \otimes \mathscr{Y}$ coincides with $\mathscr{I}_T$, viewing the latter as a sub-σ-algebra of $\mathscr{B} \otimes \mathscr{Y}$. The dynamical factor $(X \times Y, \mathscr{B} \otimes \mathscr{Y}, \mu \times \nu) \to (Z, \mathscr{Z}, \omega)$ defined by $\mathscr{I}_{T \times S}$ is the product of π and the trivial factor and gives the ergodic decomposition of $T \times S$ with ω-a.e. ergodic components $(X_z \times Y, \mathscr{B}_z \otimes \mathscr{Y}, \mu_z \times \nu)$ with action $T_z \times S$ for $z \in Z$. The orbit equivalence of R with $T \times S$ respects the ergodic decomposition and so for R we have ω-a.e. ergodic components $(X_z \times Y, \mathscr{B}_z \otimes \mathscr{Y}, \mu_z \times \nu)$ with action R_z for $z \in Z$. Note that for each $z \in Z$ the action R_z is free and the orbit change from $T_z \times S$ to R_z is $\mathscr{Y}$-measurable in the sense of Definition 2.5 in [42].

Write $\boldsymbol{B}$ for the tuple $(A_1 \times Y, \dots, A_k \times Y)$ of pairwise disjoint $\mathscr{B}$-measurable subsets of $X \times Y$. Let $\mathcal{Q} = \{Q_1, \dots Q_r\}$ be a $\boldsymbol{B}$-admissible finite measurable partition of $X \times Y$. We will show that there exists a set of $z \in Z$ of nonzero measure for which $h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z | \mathscr{Y}) > 0$, where $\mathcal{Q}_z = \{Q_j \cap (X_z \times Y) : j = 1, \dots, r\}$. Suppose to the contrary that $h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z | \mathscr{Y}) = 0$ for ω-a.e. $z \in Z$. Consider the conditional expectations $E^{\mathscr{B}} = \mathrm{id}_{L^1(X,\mu)} \otimes \nu : L^1(X \times Y, \mu \times \nu) = L^1(X, \mu) \widehat{\otimes} L^1(Y, \nu) \to L^1(X, \mu)$ and $E^{\mathscr{B}_z} = \mathrm{id}_{L^1(X_z,\mu_z)} \otimes \nu : L^1(X_z \times Y, \mu_z \times \nu) = L^1(X_z, \mu_z) \widehat{\otimes} L^1(Y, \nu) \to L^1(X_z, \mu_z)$ for $z \in Z$. As is easy to check using approximations in the algebraic tensor product, for every $f \in L^1(X \times Y, \mu \times \nu) = L^1(X, \mu) \widehat{\otimes} L^1(Y, \nu)$ there is a full-measure set of $z \in Z$ for which $E^{\mathscr{B}_z}(f|_{X_z})(x) = E^{\mathscr{B}}(f)(x)$ for μ_z-a.e. $x \in X_z$. For each $j = 1, \dots, r$ set $C_j = \{x \in X : E^{\mathscr{B}}(\mathbf{1}_{Q_j})(x) > 0\}$, which is defined up to a set of μ-measure zero and hence can be assumed to satisfy the condition that for every $i = 1, \dots, r$ it is disjoint from $A_i \times Y$ if and only if Q_j is. Then $\{C_j : j = 1, \dots, r\}$ is an $\boldsymbol{A}$-admissible Borel cover of X. Putting $\mathcal{P} = \{C_j \setminus \bigcup_{d=1}^{j-1} C_d : j = 1, \dots, r\}$ we obtain an $\boldsymbol{A}$-admissible measurable partition of X.

Now let $z \in Z$. Denote by $\mathscr{R}$ the relative Pinsker σ-algebra of $T_z \times S$ with respect to $\mathscr{Y}$, i.e., the σ-algebra generated by all measurable partitions $\mathcal{R}$ of $X_z \times Y$ such that $h_{\mu \times \nu}(T_z \times S, \mathcal{R} | \mathscr{Y}) = 0$. In the ω-a.e. situation that R_z is ergodic we have $\mathscr{R} = \mathscr{P}_{T_z} \otimes \mathscr{Y}$

by Theorem 4.10 of [42]. From the discussion in the previous paragraph we see that if z is assumed to belong to a certain set of full measure then for each $j = 1, \dots, r$ the sets $C_j \cap X_z$ and $\{x \in X_z : E^{\mathscr{B}_z}(\mathbf{1}_{Q_j \cap X_z})(x) > 0\}$ coincide up to a set of μ_z-measure zero. In this case, setting $\mathcal{P}_z = \{P \cap X_z : P \in \mathcal{P}\}$ we obtain a partition of X_z which is $\mathscr{P}_{T_z}$-measurable and hence satisfies $h_{\mu_z}(T_z, \mathcal{P}_z) = 0$. It follows using the ergodic decomposition of entropy that $h_\mu(T, \mathcal{P}) = \int_Z h_\mu(T_z, \mathcal{P}_z)\, d\omega(z) = 0$, contradicting our hypothesis. Therefore we must have $h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z | \mathscr{Y}) > 0$ for all z in a set $W \subseteq Z$ of nonzero measure.

For every z in a subset of W with the same measure as W the action R_z is ergodic and free, in which case we can apply Theorem 2.6 of [42] along with the fact that $\hat{R}$ has zero entropy to obtain

$$h_{\mu_z \times \nu}(R_z, \mathcal{Q}_z) = h_{\mu_z \times \nu}(R_z, \mathcal{Q}_z | \mathscr{Y}) = h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z | \mathscr{Y}) > 0.$$

The ergodic decomposition of entropy then yields

$$h_{\mu \times \nu}(R, \mathcal{Q}) = \int_Z h_{\mu_z \times \nu}(R_z, \mathcal{Q}_z)\, d\omega(z) > 0.$$

It follows by Theorem 4.6 of [22] that the infimum c of $h_{\mu \times \nu}(R, \mathcal{Q})$ over all $\boldsymbol{B}$-admissible finite measurable partitions $\mathcal{Q}$ of X is nonzero.

Denote by $\mathcal{V}$ the measurable cover $\{A_1^{\mathrm{c}} \times Y, \dots, A_k^{\mathrm{c}} \times Y\}$ of $X \times Y$. Suppose we are given a $\boldsymbol{B}$-admissible finite measurable partition $\mathcal{Q}$ of $X \times Y$. Applying the ergodic decomposition of entropy, Theorem 2.6 of [42], and the fact that $\hat{R}$ has zero entropy we get

$$\begin{aligned} h_{\mu \times \nu}(T \times S, \mathcal{Q}) &= \int_Z h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z)\, d\omega(z) \\ &\geq \int_Z h_{\mu_z \times \nu}(T_z \times S, \mathcal{Q}_z | \mathscr{Y})\, d\omega(z) \\ &= \int_Z h_{\mu_z \times \nu}(R_z, \mathcal{Q}_z | \mathscr{Y})\, d\omega(z) \\ &= \int_Z h_{\mu_z \times \nu}(R_z, \mathcal{Q}_z)\, d\omega(z) \\ &= h_{\mu \times \nu}(R, \mathcal{Q}) \\ &\geq c. \end{aligned}$$

Therefore $h^+_{\mu \times \nu}(T \times S, \mathcal{V}) \geq c > 0$, and since $T \times S$ is free it follows by Lemma 2.18 that $\underline{h}_{\mathrm{c}, \mu \times \nu}(T \times S, \mathcal{V}) > 0$. As we clearly have $\underline{h}_{\mathrm{c}, \mu}(T, \mathcal{U}) \geq \underline{h}_{\mathrm{c}, \mu \times \nu}(T \times S, \mathcal{V})$, this establishes the lemma. □

We remark that, in the last paragraph of the above proof, if $\mathcal{Q}$ is of the form $\{P \times Y : P \in \mathcal{P}\}$ for some finite $\boldsymbol{A}$-admissible Borel partition $\mathcal{P}$ of X, then $h_\mu(T, \mathcal{P}) = h_{\mu \times \nu}(T \times S, \mathcal{Q})$, in which case the display shows that $h^+_\mu(T, \mathcal{U}) \geq c > 0$.

In order to reduce the general case of discrete amenable groups to the case of countable ones, we shall need Lemma 2.26 below. For this we need the machinery of quasi-tiling developed by Ornstein and Weiss. The following lemma is contained in the proof of Theorem 6 in [35].

Lemma 2.25. *Given $1 > \varepsilon > 0$, if $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ are nonempty finite subsets of G such that F_{i+1} is $(F_iF_i^{-1}, \eta_i)$-invariant, $\eta_i|F_iF_i^{-1}| \leq \frac{\varepsilon^2}{4}$ for $i = 1, 2, \dots, k-1$, and $(1 - \frac{\varepsilon}{2})^k < \varepsilon$, then for any $(F_k, \frac{\varepsilon^2}{4})$-invariant finite nonempty subset F of G there are translates $\{F_ic_{ij}\}_{i,j}$ contained in F and subsets $E_{ij} \subseteq F_ic_{ij}$ such that $E_{ij} \cap E_{i'j'} = \emptyset$ for all $(i,j) \neq (i',j')$, $|E_{ij}|/|F_ic_{ij}| \geq 1 - \varepsilon$ for all (i,j), and $|\bigcup_{ij} F_ic_{ij}|/|F| \geq 1 - \varepsilon$.*

The following lemma is a direct consequence of Lemma 2.25. For any φ satisfying the conditions below, by Proposition 3.22 in [30], $\frac{\varphi(F)}{|F|}$ converges as F becomes more and more invariant. Note that every subgroup of G is amenable [36, Prop. 1.12].

Lemma 2.26. *If φ is a real-valued function which is defined on the set of finite subsets of G and satisfies*

1. *$0 \leq \varphi(A) < +\infty$ and $\varphi(\emptyset) = 0$,*
2. *$\varphi(A) \leq \varphi(B)$ for all $A \subseteq B$,*
3. *$\varphi(As) = \varphi(A)$ for all finite $A \subseteq G$ and $s \in G$,*
4. *$\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ if $A \cap B = \emptyset$,*

then the limit of $\frac{\varphi(F)}{|F|}$ as F becomes more and more invariant in G is the minimum of the corresponding limits of $\frac{\varphi(F)}{|F|}$ as F becomes more and more invariant in H for H running over the countable subgroups of G.

Theorem 2.27. *For every $k \geq 2$, a nondiagonal tuple in X^k is a μ-IE-tuple if and only if it is a μ-entropy tuple.*

Proof. The fact that a nondiagonal μ-IE-tuple is a μ-entropy tuple follows from Lemma 2.15. In the case that X is metrizable and G is countably infinite, Lemmas 2.24 and 2.15 combine to show that a μ-entropy tuple is a μ-IE-tuple. Suppose now that X is arbitrary. When G is finite, it is easily seen that the nondiagonal μ-IE-tuples and μ-entropy tuples are both precisely the nondiagonal tuples in $\mathrm{supp}(\mu)^k$. When G is countably infinite, write X as a projective limit of a net of metrizable spaces X_j equipped with compatible G-actions and induced Borel probability measures μ_j. Then by Proposition 2.16(5) the μ-IE-tuples are the projective limits of the μ_j-IE-tuples. Since the image of a measure entropy tuple under a factor map is clearly again a measure entropy tuple as long as its image is nondiagonal, we conclude from the metrizable case that every μ-entropy tuple is a μ-IE-tuple. Finally, when G is uncountably infinite, it follows from Lemma 2.26 that the set of μ-entropy tuples for (X, G) is equal to the intersection over the countable subgroups G' of G of the sets consisting of the μ-entropy tuples for (X, G'). It is also easily verified that the set of μ-IE-tuples for (X, G) contains the intersection over the countable subgroups G' of G of the sets consisting of the μ-IE-entropy tuples for (X, G'). We thus obtain the result. □

To prove the product formula for μ-IE-tuples we will use the Pinsker von Neumann algebra $\mathfrak{P}_X$, i.e., the G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ corresponding to the Pinsker σ-algebra (see the beginning of the next section). Denote by E_X the conditional expectation $L^\infty(X, \mu) \to \mathfrak{P}_X$. The following lemma appeared as Lemma 4.3 in [22]. Note that the assumptions in [22] that X is metrizable and $G = \mathbb{Z}$ are not needed here.

Lemma 2.28. *Let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a Borel cover of X. Then $\prod_{i=1}^k E_X(\chi_{U_i^c}) \neq 0$ if and only if $h_\mu(\mathcal{P}) > 0$ for every finite Borel partition $\mathcal{P}$ finer than $\mathcal{U}$ as a cover.*

Combining Lemma 2.28, Proposition 2.16(3), and Theorem 2.27, we obtain the following charaterization of μ-IE tuples.

Lemma 2.29. *A tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ is a μ-IE tuple if and only if for any Borel neighbourhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$, respectively, one has $\prod_{i=1}^k E_X(\chi_{U_i}) \neq 0$.*

Theorem 2.30. *Let (Y, G) be another topological G-system and ν a G-invariant Borel probability measure on Y. Then for all $k \geq 1$ we have $\mathrm{IE}^k_{\mu\times\nu}(X \times Y) = \mathrm{IE}^k_\mu(X) \times \mathrm{IE}^k_\nu(Y)$.*

Proof. By Proposition 2.16(5) we have $\mathrm{IE}^k_{\mu\times\nu}(X \times Y) \subseteq \mathrm{IE}^k_\mu(X) \times \mathrm{IE}^k_\nu(Y)$. Thus we just need to prove $\mathrm{IE}^k_\mu(X) \times \mathrm{IE}^k_\nu(Y) \subseteq \mathrm{IE}^k_{\mu\times\nu}(X \times Y)$.

Assume first that both X and Y are metrizable and G is countable. Then $\mathfrak{P}_{X\times Y} = \mathfrak{P}_X \otimes \mathfrak{P}_Y$ [7, Theorem 0.4(3)] (see also [14, Theorem 4] for the ergodic case) and hence $E_{X\times Y}(f \otimes g) = E_X(f) \otimes E_Y(g)$ for any $f \in L^\infty(X, \mu)$ and $g \in L^\infty(Y, \nu)$. Now the desired inclusion follows from Lemma 2.29.

The proof for the general case follows the argument in the proof of Theorem 2.27. □

In the case $G = \mathbb{Z}$, the product formula for measure entropy pairs was established in [11], while for general measure entropy tuples it is implicit in Theorem 8.1 of [22], whose proof we have essentially followed here granted the general tensor product formula for Pinsker von Neumann algebras. Notice that our IE-tuple viewpoint results in a particularly simple formula.

3. Combinatorial independence and the Pinsker algebra

Continuing within the realm of entropy, we will assume throughout the section that (X, G) is a topological dynamical system with G amenable and μ is a G-invariant Borel probability measure on X. Recall that the *Pinsker σ-algebra* is the G-invariant σ-subalgebra of $\mathscr{B}$ generated by all finite Borel partitions of X with zero entropy (or, equivalently, all two-element Borel partitions of X with zero entropy), and it defines the largest factor of the system with zero entropy (see Chapter 18 of [12]). The corresponding G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ will be denoted by $\mathfrak{P}_X$ and referred to as the *Pinsker von Neumann algebra*. In Theorem 3.7 we will give various local descriptions of the Pinsker von Neumann algebra in terms of combinatorial independence, ℓ_1 geometry, and c.p. approximation entropy.

The notion of c.p. (completely positive) approximation entropy was introduced by Voiculescu in [46] for *-automorphisms of hyperfinite von Neumann algebras preserving a faithful normal state (see [34] for a general reference on dynamical entropy in operator algebras). We will formulate here a version of the definition for amenable acting groups. So let M be a von Neumann algebra, σ a faithful normal state on M, and β a σ-preserving action of the discrete amenable group G on M by *-automorphisms. For a finite set $\Upsilon \subseteq M$ and $\delta > 0$ we write $\mathrm{CPA}_\sigma(\Upsilon, \delta)$ for the set of all triples (φ, ψ, B) where B is a finite-dimensional C^*-algebra and $\varphi : M \to B$ and $\psi : B \to M$ are unital completely positive maps such that $\sigma \circ \psi \circ \varphi = \sigma$ and $\|(\psi \circ \varphi)(a) - a\|_\sigma < \delta$ for all $a \in \Upsilon$. We then set

$$\mathrm{rcp}_\sigma(\Upsilon, \delta) = \inf\{\mathrm{rank}\, B : (\varphi, \psi, B) \in \mathrm{CPA}_\sigma(\Upsilon, \delta)\}$$

if the set on the right is nonempty, which is always the case if M is commutative or hyperfinite. Otherwise we put $\mathrm{rcp}_\sigma(\Upsilon, \delta) = \infty$. Write $\mathrm{hcpa}_\sigma(\beta, \Upsilon, \delta)$ for the limit supremum of $\frac{1}{|F|} \ln \mathrm{rcp}_\sigma(\bigcup_{s\in F} \alpha_s(\Upsilon), \delta)$ as F becomes more and more invariant, and define

$$\mathrm{hcpa}_\sigma(\beta, \Upsilon) = \sup_{\delta>0} \mathrm{hcpa}_\sigma(\beta, \Upsilon, \delta),$$
$$\mathrm{hcpa}_\sigma(\beta) = \sup_{\Upsilon} \mathrm{hcpa}_\sigma(\beta, \Upsilon),$$

where the last supremum is taken over all finite subsets Υ of M. We refer to $\mathrm{hcpa}_\sigma(\beta, \Upsilon)$ as the *c.p. approximation entropy* of β. When $G = \mathbb{Z}$ and M is commutative and has separable predual, this coincides with Voiculescu's original definition by the arguments leading to Corollary 3.8 in [46].

Question 3.1. Does the above definition always coincide with Voiculescu's when $G = \mathbb{Z}$?

By Corollary 3.8 in [46], when X is metrizable, $G = \mathbb{Z}$, and the action is ergodic, the c.p. approximation entropy of the induced action α on $L^\infty(X, \mu)$ agrees with the measure entropy $h_\mu(X)$. The arguments also work for general amenable G. It follows using the ergodic decomposition of entropy (see the paragraph before Lemma 2.24) that when X is metrizable the Pinsker von Neumann algebra is the largest G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ on which the c.p. approximation entropy is zero.

We next define geometric analogues of upper and lower measure independence density from Section 2. Let $f \in L^\infty(X, \mu)$. Let p be a projection in $L^\infty(X, \mu)$ and let $\lambda \geq 1$. We say that a set $J \subseteq G$ is an *ℓ_1-λ-isomorphism set for f relative to p* if $\{p\alpha^i(f) : i \in J\}$ is λ-equivalent to the standard basis of ℓ_1^J. For $\delta > 0$ denote by $\mathscr{P}(\mu, \delta)$ the set of projections $p \in L^\infty(X, \mu)$ such that $\mu(p) \geq 1 - \delta$. For every finite subset F of G, $\lambda \geq 1$, and $\delta > 0$ we define

$$\varphi_{f,\lambda,\delta}(F) = \min_{p\in\mathscr{P}(\mu,\delta)} \max \big\{|F \cap J| : J \text{ is an } \ell_1\text{-}\lambda\text{-isomorphism set for } f \text{ relative to } p\big\}.$$

Write $\overline{\mathrm{I}}_\mu(f, \lambda, \delta)$ for the limit supremum of $\frac{1}{|F|}\varphi_{f,\lambda,\delta}(F)$ as F becomes more and more invariant, and $\underline{\mathrm{I}}_\mu(f, \lambda, \delta)$ for the corresponding limit infimum. Set $\overline{\mathrm{I}}_\mu(f, \lambda) = \sup_{\delta>0} \overline{\mathrm{I}}_\mu(f, \lambda, \delta)$ and $\underline{\mathrm{I}}_\mu(f, \lambda) = \sup_{\delta>0} \underline{\mathrm{I}}_\mu(f, \lambda, \delta)$. Finally, we define $\overline{\mathrm{I}}_\mu(f) = \sup_{\lambda\geq 1} \overline{\mathrm{I}}_\mu(f, \lambda)$ and $\underline{\mathrm{I}}_\mu(f) = \sup_{\lambda\geq 1} \underline{\mathrm{I}}_\mu(f, \lambda)$, and refer to these quantities respectively as the *upper μ-ℓ_1-isomorphism density* and *lower μ-ℓ_1-isomorphism density* of f. On the topological side, for each $\lambda \geq 1$ the limit of

$$\frac{1}{|F|} \max \big\{|F \cap J| : \{\alpha^i(f) : i \in J\} \text{ is } \lambda\text{-equivalent to the standard basis of } \ell_1^J\big\}$$

as F becomes more and more invariant exists (see the end of Section 3 in [30]), and we refer to the supremum of these limits over all $\lambda \geq 1$ as the *ℓ_1-isomorphism density* of f.

Glasner and Weiss proved the next lemma for the real scalar case [16, Lemma 2.3]. The complex scalar version follows by considering the map $E \to (\ell_\infty^n)_\mathbb{R} \oplus_\infty (\ell_\infty^n)_\mathbb{R} = (\ell_\infty^{2n})_\mathbb{R}$ sending each $v \in E \subseteq \ell_\infty^n$ to the pair consisting of its real and imaginary parts.

Lemma 3.2. *For all $b > 0$ and $\delta > 0$ there exist $c > 0$ and $\varepsilon > 0$ such that, for all sufficiently large n, if E is a subset of the unit ball of ℓ_∞^n which is δ-separated and $|E| \geq e^{bn}$, then there are a $t \in [-1, 1]$ and a set $J \subseteq \{1, 2, \ldots, n\}$ for which*

(1) $|J| \geq cn$, *and*
(2) *either for every* $\sigma \in \{0,1\}^J$ *there is a* $v \in E$ *such that for all* $j \in J$

$$\begin{aligned} \mathrm{re}(v(j)) &\geq t + \varepsilon \quad &\text{if } \sigma(j) = 1, \text{ and} \\ \mathrm{re}(v(j)) &\leq t - \varepsilon \quad &\text{if } \sigma(j) = 0, \end{aligned}$$

or for every $\sigma \in \{0,1\}^J$ *there is a* $v \in E$ *such that for all* $j \in J$ *the above holds with* $\mathrm{re}(v(j))$ *replaced by* $\mathrm{im}(v(j))$.

The following is a consequence of Lemma 3.6 in [30].

Lemma 3.3. *There exists a* $c > 0$ *such that whenever* I *is a finite set and* $A_{i,1}$, $A_{i,2}$, *and* B_i *for* $i \in I$ *are subsets of a given set such that the collection* $\{(A_{i,1} \cup A_{i,2}, B_i) : i \in I\}$ *is independent, there are a set* $J \subseteq I$ *with* $|J| \geq c|I|$ *and a* $j \in \{1,2\}$ *for which the collection* $\{(A_{i,j}, B_i) : i \in J\}$ *is independent.*

Lemma 3.4. *For every* $\delta > 0$ *there exist* $c > 0$ *and* $\varepsilon > 0$ *such that, for every compact Hausdorff space* Y *and finite subset* Θ *of the unit ball of* $C(Y)$ *of sufficiently large cardinality, if the linear map* $\gamma : \ell_1^\Theta \to C(Y)$ *sending the standard basis of* ℓ_1^Θ *to* Θ *is an isomorphism with* $\|\gamma^{-1}\|^{-1} \geq \delta$, *then there exist closed disks* $B_1, B_2 \subseteq \mathbb{C}$ *of diameter at most* $\varepsilon/6$ *with* $\mathrm{dist}(B_1, B_2) \geq \varepsilon$ *and an* $I \subseteq \Theta$ *with* $|I| \geq c|\Theta|$ *such that the collection* $\{(f^{-1}(B_1), f^{-1}(B_2)) : f \in I\}$ *is independent.*

Proof. Let $\delta > 0$. Define a pseudometric d_Θ on Y by

$$d_\Theta(x,y) = \sup_{f\in\Theta} |f(x) - f(y)|$$

and pick a maximal $(\delta/4)$-separated subset Z of Y. Then the open balls $B(z, \delta/4)$ with radius $\delta/4$ and centre z for $z \in Z$ cover Y. A standard partition of unity argument (see the proof of Proposition 4.8 in [46]) yields the bound $\mathrm{rc}(\Theta, \delta/2) \leq |Z|$ for the contractive $(\delta/2)$-rank of Θ as defined in [29]. By Lemma 3.2 of [29] we have $\ln \mathrm{rc}(\Theta, \delta/2) \geq |\Theta| a \|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta/2)^2$ for some universal constant $a > 0$. Thus $|Z| \geq e^{|\Theta| a\|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1}-\delta/2)^2} \geq e^{|\Theta| a\delta^2/4}$. Evaluation of the functions in Θ on the points of Y yields a map ψ from Y to the unit ball of ℓ_∞^Θ such that $\psi(Z)$ is $(\delta/4)$-separated. By Lemma 3.2 there are $c > 0$ and $\varepsilon > 0$ depending only on a and δ such that there exist closed disks B_1 and B_2 contained in the unit disk of $\mathbb{C}$ with $\mathrm{dist}(B_1, B_2) \geq 4\varepsilon/3$ and an $I \subseteq \Theta$ with $|I| \geq c|\Theta|$ such that the collection $\{(f^{-1}(B_1), f^{-1}(B_2)) : f \in I\}$ is independent. Now for some $N \in \mathbb{N}$ depending on ε we can cover each of B_1 and B_2 with N disks of diameter at most $\varepsilon/6$. By repeated application of Lemma 3.3 we can then replace each of B_1 and B_2 with one of the smaller disks to obtain the result (with a smaller c). □

Lemma 3.5. *Let* $\delta > 0$ *and* $\lambda > 0$. *Let* $\Omega = \{f_1, \dots, f_n\}$ *be a subset of the unit ball of* $L^\infty(X, \mu)$ *and suppose that for all* $g_1, \dots, g_n$ *in the unit ball of* $L^\infty(X,\mu)$ *with* $\max_{1\leq i\leq n} \|g_i - f_i\|_\mu < \delta$ *there exists an* $I \subseteq \{1, \dots, n\}$ *of cardinality at least* dn *for which the linear map* $\ell_1^I \to \mathrm{span}\{g_i : i \in I\}$ *sending the standard basis element with index* $i \in I$ *to* g_i *has an inverse with norm at most* λ. *Then*

$$\ln \mathrm{rcp}_\mu(\Omega, \delta) \geq an$$

for some constant $a > 0$ *which depends only on* λ.

Proof. Let $(\varphi, \psi, B) \in \mathrm{CPA}_\mu(\Omega, \delta)$. Then there exists an $I \subseteq \{1, \dots, n\}$ of cardinality at least dn for which the linear map $\ell_1^I \to \mathrm{span}\{(\psi \circ \varphi)(f_i) : i \in I\}$ sending the standard basis element with index $i \in I$ to g_i has an inverse with norm at most λ. It follows using the operator norm contractivity of φ and ψ that for any scalars c_i for $i \in I$ we have

$$\Big\| \sum_{i \in I} c_i \varphi(f_i) \Big\| \geq \Big\| \sum_{i \in I} c_i (\psi \circ \varphi)(f_i) \Big\| \geq \lambda^{-1} \sum_{s \in I} |c_i|,$$

so that the subset $\{\varphi(f_i) : i \in I\}$ of B is λ-equivalent to the standard basis of ℓ_1^I. Lemma 3.1 of [28] then guarantees the existence of a constant $a > 0$ depending only on λ such that $\ln \mathrm{rank}(B) \geq an$, yielding the result. □

Lemma 3.6. *Let $\delta > 0$. Let $\Omega = \{f_1, \dots, f_n\}$ be a subset of the unit ball of $L^\infty(X, \mu)$ and for each $i = 1, \dots, n$ let $\mathcal{P}_i$ be a finite Borel partition of X such that* $\operatorname{ess\,sup}_{x,y \in P} |f_i(x) - f_i(y)| < \delta$ *for every $P \in \mathcal{P}_i$. Suppose that $H(\mathcal{P}) \leq n\delta^2$ where $\mathcal{P} = \bigvee_{i=1}^n \mathcal{P}_i$. Then*

$$\ln \mathrm{rcp}_\mu(\Omega, \sqrt{\delta^2 + 4\delta}) \leq 2n\delta$$

if n is sufficiently large as a function of δ.

Proof. For a finite Borel partition $\mathcal{Q}$ of X we write $I(\mathcal{Q})$ for the information function $-\sum_{Q \in \mathcal{Q}} \mathbf{1}_Q \ln \mu(Q)$. Then $H(\mathcal{P}) = \int_X I(\mathcal{P})\, d\mu$, and so by our assumption the set D on which the nonnegative function $I(\mathcal{P})/n$ takes values less than δ has measure at least $1 - \delta$. Then $\mu(P) \geq e^{-n\delta}$ for all $P \in \mathcal{P}$ such that $\mu(P \cap D) \neq \emptyset$. Let B be the linear span of $\{\mathbf{1}_{P \cap D} : P \in \mathcal{P} \text{ and } \mu(P \cap D) \neq \emptyset\} \cup \{\mathbf{1}_{D^c}\}$. Then B is a unital *-subalgebra of $L^\infty(X, \mu)$ and $\dim B \leq e^{n\delta} + 1$. Taking the μ-preserving conditional expectation $\varphi : L^\infty(X, \mu) \to B$ and the inclusion $\psi : B \to L^\infty(X, \mu)$ it is readily checked that $(\varphi, \psi, B) \in \mathrm{CPA}_\mu(\Omega, \sqrt{\delta^2 + 4\delta})$ so that $\mathrm{rcp}_\mu(\Omega, \sqrt{\delta^2 + 4\delta}) \leq e^{n\delta} + 1$, from which the desired conclusion follows. □

Regarding $L^\infty(X, \mu)$ as a unital commutative C^*-algebra, it is isomorphic by Gelfand theory to $C(\Omega)$ for some compact Hausdorff space Ω, which we can identify with the spectrum of $L^\infty(X, \mu)$ (i.e., the space of nonzero multiplicative linear functionals on $L^\infty(X, \mu)$) equipped with the relative weak* topology (see Chapter 1 of [8]). Accordingly we will view elements of $L^\infty(X, \mu)$ as continuous functions on Ω when appropriate. The action α of G on $L^\infty(X, \mu)$ gives rise to a topological dynamical system (Ω, G) with the action of G defined by $(s, \sigma) \mapsto \sigma \circ \alpha_{s^{-1}}$. Since μ defines a state on $L^\infty(X, \mu)$ it gives rise to a G-invariant Borel probability measure on Ω, which we will also denote by μ. For a projection $p \in L^\infty(X, \mu)$ we write Ω_p for the clopen subset of Ω whose characteristic function is p.

Theorem 3.7. *Let $f \in L^\infty(X, \mu)$. Let $\{F_n\}_{n \in \Lambda}$ be a Følner net in G. Then the following are equivalent:*

1. *$f \notin \mathfrak{P}_X$,*
2. *there is a μ-IE-pair $(\sigma_1, \sigma_2) \in \Omega \times \Omega$ such that $f(\sigma_1) \neq f(\sigma_2)$,*
3. *there are $d > 0$, $\delta > 0$, and $\lambda > 0$ such that, for all n greater than some $n_0 \in \Lambda$, whenever g_s for $s \in F_n$ are elements of $L^\infty(X, \mu)$ satisfying $\|g_s - \alpha_s(f)\|_\mu < \delta$ for every $s \in F_n$ there exists an $I \subseteq F_n$ of cardinality at least $d|F_n|$ for which the linear map $\ell_1^I \to \mathrm{span}\{g_s : s \in I\}$ sending the standard basis element with index $s \in I$ to g_s has an inverse with norm at most λ,*

(4) *the same as* (3) *with "for all n greater than some $n_0 \in \Lambda$" replaced by "for all n in a cofinal subset of Λ",*
(5) $\underline{\mathrm{I}}_\mu(f) > 0$*,*
(6) $\overline{\mathrm{I}}_\mu(f) > 0$*,*
(7) $\mathrm{hcpa}_\mu(\alpha, \{f\}) > 0$*,*
(8) $\mathrm{hcpa}_\mu(\beta) > 0$ *for the restriction β of α to the von Neumann subalgebra of $L^\infty(X, \mu)$ dynamically generated by f.*

When the action is ergodic and either X is metrizable or G is countable, we can add:

(9) *there is a $\delta > 0$ such that every $g \in L^\infty(X, \mu)$ satisfying $\|g - f\|_\mu < \delta$ has positive ℓ_1-isomorphism density with respect to the operator norm.*

When $f \in C(X)$ we can add:

(10) *$f \notin C(Y)$ whenever $\pi : X \to Y$ is a topological G-factor map such that $h_{\pi_*(\mu)}(Y) = 0$,*
(11) *there is a μ-IE-pair $(x_1, x_2) \in X \times X$ such that $f(x_1) \neq f(x_2)$.*

Proof. (1)⇒(2). Since the α-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ generated by f is also dynamically generated by the set of spectral projections of f over closed subsets of the complex plane, we can find a clopen set $Z \subseteq \Omega$ corresponding to a spectral projection of f over A for some set $A \subseteq \mathbb{C}$ such that the two-element clopen partition $\mathcal{Z} = \{Z, Z^c\}$ satisfies $h_\mu(\Omega, \mathcal{Z}) > 0$. Using Lemma 2.8 we can find a closed set $B \subseteq \mathbb{C}$ with $B \cap A = \emptyset$ such that the pair (Z, Z') has positive μ-independence density, where Z' is the subset of Ω supporting the spectral projection of f over B. By Proposition 2.16(1) there is a μ-IE-pair $(\sigma_1, \sigma_2) \in \Omega \times \Omega$ such that $\sigma_1 \in Z$ and $\sigma_2 \in Z'$. Then $f(\sigma_1) \in A$ while $f(\sigma_2) \in B$, establishing (2).

(2)⇒(3). Let $(\sigma_1, \sigma_2) \in \Omega \times \Omega$ be a μ-IE-pair such that $f(\sigma_1) \neq f(\sigma_2)$. Choose disjoint closed disks $B_1, B_2 \subseteq \mathbb{C}$ such that $\mathrm{diam}(B_1) = \mathrm{diam}(B_2) \leq \frac{1}{10}\mathrm{dist}(B_1, B_2)$, $f(\sigma_1) \in \mathrm{int}(B_1)$, and $f(\sigma_2) \in \mathrm{int}(B_2)$ and set $\varepsilon = \frac{1}{10}\mathrm{dist}(B_1, B_2)$. Choose clopen neighbourhoods A_1 and A_2 of σ_1 and σ_2, respectively, such that $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$. Write $\boldsymbol{A}$ for the pair (A_1, A_2). Since (σ_1, σ_2) is a μ-IE-pair there exists by Proposition 2.4 and Lemma 2.15 a $\delta > 0$ such that $\underline{\mathrm{I}}'_\mu(\boldsymbol{A}, \delta) > 0$. Take an $\eta > 0$ such that whenever h is an element of $L^\infty(X, \mu)$ for which $\|h\|_\mu < \eta$ the set $\{x \in X : |h(x)| \leq \varepsilon\}$ has measure at least $1 - \delta$.

Now let $n \in \Lambda$ and suppose that we are given $g_s \in L^\infty(X, \mu)$ for $s \in F_n$ such that $\|g_s - \alpha_s(f)\|_\mu < \eta$ for every $s \in F_n$. For each $s \in F_n$ set $D_s = \{\sigma \in \Omega : |g_s(\sigma) - \alpha_s(f)(\sigma)| \leq \varepsilon\}$, which has measure at least $1 - \delta$ by our choice of η, and for $s \in G \setminus F_n$ set $D_s = \Omega$. Put $d = \underline{\mathrm{I}}'_\mu(\boldsymbol{A}, \delta)/2$. Assuming that $n > n_0$ for a suitable $n_0 \in \Lambda$, there exist an independence set $I \subseteq F_n$ for $\boldsymbol{A}$ relative to the map $s \mapsto D_s$ such that $|I| \geq d|F_n|$. The standard Rosenthal-Dor argument [9] then shows that the linear map $\ell_1^I \to \mathrm{span}\{g_s : s \in I\}$ sending the standard basis element with index $s \in I$ to g_s has an inverse with norm at most ε^{-1}, yielding (3).

(3)⇒(4). Trivial.

(4)⇒(7). Apply Lemma 3.5.

(3)⇒(5). We may assume that $\|f\| = 1$. Let d, δ, and λ be as given by (3). Then for any $p \in \mathscr{P}(\mu, \delta^2)$ and $s \in G$ we have $\|p\alpha_s(f) - \alpha_s(f)\|_\mu \leq \|p - 1\|_\mu \|f\| \leq \delta$. It follows that $\varphi_{f,\lambda,\delta^2}(F_n) \geq d|F_n|$ for every $n \in \mathbb{N}$, and hence $\underline{\mathrm{I}}_\mu(f) \geq \underline{\mathrm{I}}_\mu(f, \lambda, \delta^2) \geq d > 0$.

(5)⇒(6). Trivial.

(6)⇒(4). We may assume that G is infinite and $\|f\| = 1$. By (6) there are a $\lambda \geq 1$ and a $\delta > 0$ such that $\overline{\mathrm{I}}_\mu(f, \lambda, \delta) > 0$. Then there is a $d > 0$ and a cofinal set $L \subseteq \Lambda$ such that $\varphi_{f,\lambda,\delta}(F_n) \geq d|F_n|$ for all $n \in L$. Let b be a positive number to be further specified below, and set $\delta' = \delta b$. Let $c > 0$ and $\varepsilon > 0$ be as given by Lemma 3.4 with respect to $\delta = \lambda^{-1}$. Take an $\eta > 0$ such that whenever h is an element of $L^\infty(X, \mu)$ for which $\|h\|_\mu < \eta$ the set $\{x \in X : |h(x)| \leq \varepsilon/12\}$ has measure at least $1 - \delta'$.

Now let $n \in L$, and suppose we are given $g_s \in L^\infty(X, \mu)$ for $s \in F_n$ such that $\|g_s - \alpha_s(f)\|_\mu < \eta$ for every $s \in F_n$. By our choice of η, for every $s \in F_n$ there is a projection $p_s \in \mathscr{P}(\mu, \delta')$ such that $\|p_s(g_s - \alpha_s(f))\| \leq \varepsilon/12$. Denote by $\mathcal{S}$ the set of all $\sigma \in \{1, 2\}^{F_n}$ such that $|\sigma^{-1}(2)| \leq b|F_n|$. Setting $p_{s,1} = p_s$ and $p_{s,2} = p_s^\perp$ we define the projection $r = \sum_{\sigma \in \mathcal{S}} \prod_{s \in F_n} p_{s,\sigma(s)}$. Then

$$\mu(r^\perp) b|F_n| \leq \sum_{s \in F_n} \mu(p_s^\perp) \leq |F_n|\delta'$$

and so $\mu(r^\perp) \leq b^{-1}\delta' = \delta$. Hence there is an $K \subseteq F_n$ with $|K| \geq d|F_n|$ such that K is an ℓ_1-λ-isomorphism set for f relative to r.

By our choice of c and ε, assuming that $|F_n|$ is sufficiently large we can find closed disks $B_1, B_2 \subseteq \mathbb{C}$ of diameter at most $\varepsilon/6$ with $\mathrm{dist}(B_1, B_2) \geq \varepsilon$ and a $J \subseteq K$ with $|J| \geq c|K|$ such that the collection

$$\big\{\big((\alpha_s(f)|_{\Omega_r})^{-1}(B_1), (\alpha_s(f)|_{\Omega_r})^{-1}(B_2)\big) : s \in J\big\}$$

of pairs of subsets of Ω_r is independent. Define the subsets $C_{s,1} = (g_s|_{\Omega_r})^{-1}(B_1')$ and $C_{s,2} = (g_s|_{\Omega_r})^{-1}(B_2')$ of Ω_r, where B_1' (resp. B_2') is the closed disk with the same centre as B_1 (resp. B_2) but with radius bigger by $\varepsilon/12$. Since $\max_{s \in J} \|p_s(g_s - \alpha_s(f))\| \leq \varepsilon/12$, for each $\sigma \in \{1, 2\}^J$ we can find by the definition of r a set $J_\sigma \subseteq J$ with $|J \setminus J_\sigma| \leq b|F_n|$ such that $\bigcap_{s \in J_\sigma} (\Omega_{p_s} \cap C_{s,\sigma(s)}) \neq \emptyset$, and we define $\rho_\sigma \in \{0, 1, 2\}^J$ by

$$\rho_\sigma(s) = \begin{cases} \sigma(s) & \text{if } s \in J_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\max_{\sigma \in \{1,2\}^J} |\rho_\sigma^{-1}(0)| \leq 2^{b|F_n|}$, for every $\rho \in \{0, 1, 2\}^J$ the number of $\sigma \in \{1, 2\}^J$ for which $\rho_\sigma = \rho$ is at most $2^{b|F_n|}$, and so the set $\mathcal{R} = \big\{\rho_\sigma : \sigma \in \{1, 2\}^J\big\}$ has cardinality at least $2^{|J|}/2^{d|F_n|} \geq 2^{(cd-b)|F_n|}$. It follows by Lemma 2.2 that for a small enough b not depending on n there exists a $t > 0$ for which we can find an $I \subseteq J$ with $|I| \geq t|J| \geq tcd|F_n|$ such that $\mathcal{R}|_I \supseteq \{1, 2\}^I$. Then the collection $\{(C_{s,1}, C_{s,2}) : s \in I\}$ is independent, and since $\mathrm{dist}(B_1', B_2') \geq 5\varepsilon/6 > 2\max(\mathrm{diam}(B_1'), \mathrm{diam}(B_2'))$ the standard Rosenthal-Dor argument [9] shows that the linear map $\ell_1^I \to \mathrm{span}\{g_s : s \in I\}$ sending the standard basis element with index $s \in I$ to g_s has an inverse with norm at most $10\varepsilon^{-1}$. We thus obtain (4).

(7)⇒(8). It suffices to note that if N is an G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ then for every finite subset $\Theta \subseteq N$ we have $\mathrm{hcpa}_{\mu|_N}(N, \Theta) = \mathrm{hcpa}_\mu(L^\infty(X, \mu), \Theta)$, i.e., for computing c.p. approximation entropy it doesn't matter whether Θ is considered as a subset of N or $L^\infty(X,\mu)$. This follows from the fact that there is a μ-preserving conditional expectation from $L^\infty(X,\mu)$ onto N [45, Prop. V.2.36]. See the proof of Proposition 3.5 in [46].

(8)⇒(1). Suppose that $f \in \mathfrak{P}_X$. Let Υ be a finite subset of the von Neumann subalgebra of $L^\infty(X,\mu)$ generated by f and let $\delta > 0$. Take a finite Borel partition $\mathcal{P}$ of X such that the characteristic functions of the atoms of $\mathcal{P}$ are spectral projections of f and $\sup_{g\in\Omega} \operatorname{ess\,sup}_{x,y\in P}|g(x) - g(y)| < \delta$ for each $P \in \mathcal{P}$. Then $h_\mu(X,\mathcal{P}) = 0$ by our assumption, and thus, since we may suppose G to be infinite (for otherwise the system has completely positive entropy), we obtain $\mathrm{hcpa}_\mu(\beta,\Upsilon,\sqrt{\delta^2+4\delta}) \leq 2\delta$ by Lemma 3.6. Hence (8) fails to hold. Thus (8) implies (1).

Assume now that G is countable and the action is free and ergodic and let us show that (9) is equivalent to the other conditions.

(3)⇒(9). Let d, δ, and λ be as given by (3). Let g be an element of $L^\infty(X,\mu)$ such that $\|g - f\|_\mu < \delta$. Then $\|\alpha_s(g) - \alpha_s(f)\|_\mu < \delta$ for all $s \in G$, and so for every $n \in \mathbb{N}$ there is an $I \subseteq F_n$ of cardinality at least $d|F_n|$ for which $\{\alpha_s(g) : s \in I\}$ is $\|g\|\lambda$-equivalent in the operator norm to the standard basis of ℓ_1^I. Thus g has positive ℓ_1-isomorphism density.

(9)⇒(8). Suppose that G is countable. We will first treat the case that the action of G on X is free. Suppose contrary to (8) that $\mathrm{hcpa}_\mu(\beta) = 0$. Since α is free and ergodic so is β, and since G is countable the von Neumann subalgebra of $L^\infty(X,\mu)$ dynamically generated by f has separable predual. We can thus apply the Jewett-Krieger theorem for free ergodic measure-preserving actions of countable discrete amenable groups on Lebesgue spaces (see [40], which shows the finite entropy case; the general case was announced in [48] but remains unpublished) to obtain a topological G-system (Y,G) with a unique invariant Borel probability measure ν such that β can be realized as the action of G on $L^\infty(Y,\nu)$ arising from the action of G on Y. Now let $\delta > 0$ be as given by (9). Take a function $g \in C(Y) \subseteq L^\infty(Y,\nu)$ such that $\|g-f\|_\mu < \delta$. Since the system (Y,G) has zero topological entropy by the variational principle [33], it follows by Theorem 5.3 of [29] (which is stated for $\mathbb{Z}$-systems but is readily seen to cover actions of general amenable groups) that the function g has zero ℓ_1-isomorphism density, contradicting our choice of δ. We thus obtain (9)⇒(8) in the case that the action is free.

Suppose now that the action of G on X is not free. Take a free weakly mixing measure-preserving acion of G on a Lebesgue space $(Z,\mathscr{Z},\omega)$ (e.g., a Bernoulli shift). Then the product action on $X \times Z$ is free and ergodic. Write E for the conditional expectation of $L^\infty(X\times Z,\mu\times\omega)$ onto $L^\infty(X,\mu)$. With $\delta > 0$ as given by (9), for every $g \in L^\infty(X\times Z,\mu\times\omega)$ such that $\|E(g) - f\|_\mu < \delta$ the function $E(g)$ has positive ℓ_1-isomorphism density, which implies that g has positive ℓ_1-isomorphism density since E is contractive and G-equivariant. Thus the function $f \otimes \mathbf{1}$ in $L^\infty(X \times Z, \mu \times \omega)$ also satisfies (9) for the same δ. By the previous paragraph we obtain (8) for $f \otimes \mathbf{1}$. But this is equivalent to (8) for f itself, yielding (9)⇒(8) when G is countable.

Suppose that G is uncountable and X is metrizable. In this case we will actually show (9)⇒(7). For every $s \in G$ write $\mathcal{E}_s$ for the orthogonal complement in $L^2(X,\mu)$ of the subspace of vectors fixed by s. Then the span of $\bigcup_{s\in G}\mathcal{E}_s$ is dense in $L^2(X,\mu) \ominus \mathbb{C}\mathbf{1}$ by ergodicity, and since $L^2(X,\mu)$ is separable there is a countable set $J \subseteq G$ such that the span of $\bigcup_{s\in J}\mathcal{E}_s$ is dense in $L^2(X,\mu) \ominus \mathbb{C}\mathbf{1}$. It follows that the subgroup H generated by

J does not fix any vectors in $L^2(X,\mu) \ominus \mathbb{C}\mathbf{1}$. This means that the action of H on X is ergodic, as is the action of any subgroup of G containing H. By Lemma 2.26 condition (9) holds for the action of every subgroup of G containing H, and thus for the action of a countable such subgroup we get (9)⇒(8) by the two previous paragraphs and hence (9)⇒(7). But if (7) fails for the action of G then it fails for the action of every subgroup of G containing some fixed countable subgroup W of G and in particular for the action of the countable subgroup generated by H and W, yielding a contradiction.

Finally, we suppose that $f \in C(X)$ and demonstrate the equivalence of (11) and (12) with the other conditions.

(2)⇒(11). The inclusion $C(\mathrm{supp}(\mu)) \subseteq L^\infty(X,\mu)$ gives rise at the spectral level to a topological G-factor map $\Omega \to \mathrm{supp}(\mu)$, and so the implication follows from Proposition 2.16(5).

(11)⇒(10). Use Proposition 2.16(5).

(10)⇒(11). Suppose that $f(x_1) = f(x_2)$ for every $(x_1,x_2) \in \mathrm{IE}_2^\mu(X)$. Set $E = \{(x,y) \in X \times X : f(x) = f(y)\}$. Then E is a closed equivalence relation on X. Thus $\bigcap_{s\in G} sE$ is a G-invariant closed equivalence relation on X and hence gives rise to a topological G-factor Y of X. In particular, $f \in C(Y)$. Denote the factor map $X \to Y$ by π. Our assumption says that $\mathrm{IE}_2^\mu(X) \subseteq E$. Since $\mathrm{IE}_2^\mu(X)$ is G-invariant, $\mathrm{IE}_2^\mu(X) \subseteq \bigcap_{s\in G} sE$. This means that $(\pi \times \pi)(\mathrm{IE}_2^\mu(X)) \subseteq \triangle_Y$. By (2) and (5) of Proposition 2.16, $h_{\pi_*(\mu)}(Y) = 0$.

(11)⇒(3). Apply the same argument as for (2)⇒(3). □

Theorem 3.7 shows that for general X the Pinsker von Neumann algebra is the largest G-invariant von Neumann subalgebra of $L^\infty(X,\mu)$ on which the c.p. approximation entropy is zero.

Remark 3.8. One interesting consequence of Theorem 3.7 is the following. In the case that G is countable, if a weakly mixing measure-preserving action of G on a Lebesgue space $(Y,\mathscr{Y},\nu)$ does not have completely positive entropy, then it has a metrizable topological model (Z,G) for which the set $\mathrm{IE}^k(Z)$ of topological IE-tuples has zero ν^k-measure for each $k \geq 2$. Indeed weak mixing implies that the product action of G on Y^k is ergodic with respect to ν^k, so that for a topological model (Z,G) and $k \geq 2$ the set $\mathrm{IE}^k(Z)$ has ν^k-measure either zero or one. If for every metrizable topological model (Z,G) we had $\nu^k(\mathrm{IE}^k(Z)) = 1$ for some $k \geq 2$, it would follow that every element of $L^\infty(Y,\nu)$ has positive ℓ_1-isomorphism density, since such an element is a continuous function for some metrizable topological model by the countability of G and hence separates a topological IE-pair. But then $(Y,\mathscr{Y},\nu,G)$ would have completely positive entropy by Theorem 3.7. Actually the weak mixing assumption can be weakened to the requirement that there be no sets of measure strictly between zero and one with finite G-orbit.

We also point out that, in a related vein, if the topological system (X,G) does not have completely positive entropy, then for a G-invariant Borel probability measure on X the set $\mathrm{IE}^k(X)$ has zero product measure for each $k \geq 2$, unless some nontrivial quotient of (X,G) has points with positive induced measure. The reason is that if $\mathrm{IE}^k(X)$ for some $k \geq 2$ has positive product measure then so does $\mathrm{IE}^k(Y)$ with respect to the induced measure for every quotient (Y,G) of (X,G), and if every point in such a quotient (Y,G) has zero induced measure then the diagonal in Y^k has zero product measure and hence does not contain $\mathrm{IE}^k(Y)$, implying that (Y,G) has positive topological entropy. In particular, we see that if (X,G) is minimal and does not have completely positive entropy and X is connected (and hence has no nontrivial finite quotients) then for every G-invariant Borel probability measure on X the set $\mathrm{IE}^k(X)$ has zero product measure for each $k \geq 2$.

At the extreme end of completely positive entropy where the Pinsker von Neumann algebra reduces to the scalars, the picture topologizes and we have the following result. Recall that a topological system is said to have *completely positive entropy* if every nontrivial factor has positive topological entropy, *uniformly positive entropy* if every nondiagonal element of $X \times X$ is an entropy pair, and *uniformly positive entropy of all orders* if for each $k \geq 2$ every nondiagonal element of X^k is an entropy tuple (see [12, Chap. 19] and [22]).

Theorem 3.9. *Suppose that X is metrizable or G is countable. Let $\mathbf{\Omega} = (\Omega, G)$ be the topological dynamical system associated to $\boldsymbol{X} = (X, \mathscr{B}, \mu, G)$ as above. Then the following are equivalent:*

(1) *$\boldsymbol{X}$ has completely positive entropy,*
(2) *every nonscalar element of $L^\infty(X,\mu)$ has positive ℓ_1-isomorphism density,*
(3) *$\mathbf{\Omega}$ has completely positive entropy,*
(4) *$\mathbf{\Omega}$ has uniformly positive entropy,*
(5) *$\mathbf{\Omega}$ has uniformly positive entropy of all orders.*

Proof. (1)⇒(5). Every Borel partition of Ω is μ-equivalent to a clopen partition and thus every nontrivial such partition has positive entropy by (1). It follows that, for each $k \geq 2$, every nondiagonal tuple in Ω^k is a μ-entropy tuple and hence a μ-IE-tuple by Theorem 2.27. Since μ-IE-tuples are obviously IE-tuples and the latter are easily seen to be entropy tuples when they are nondiagonal, we obtain (5).

(5)⇒(4)⇒(3). These implications hold for any topological G-system, the first being trivial and the second being a consequence of the properties of entropy for open covers with respect to taking extensions.

(3)⇒(2). Apply Corollary 5.5 of [29] as extended to actions of discrete amenable groups.

(2)⇒(1). By (2) there do not exist any nonscalar G-invariant projections in $L^\infty(X,\mu)$, i.e., the system $\boldsymbol{X}$ is ergodic. We can thus apply (9)⇒(1) of Theorem 3.7. □

For $G = \mathbb{Z}$ the equivalence of (1), (3), (4), and (5) in Theorem 3.9 can also be obtained from Section 3 of [16].

One might wonder whether a similar type of topologization occurs at the other extreme of zero entropy. Glasner and Weiss showed however in [15] that every free ergodic $\mathbb{Z}$-system has a minimal topological model with uniformly positive entropy.

Using Theorem 3.7 and viewing joinings as equivariant unital positive maps, we can give a linear-geometric proof of the disjointness of zero entropy systems from completely positive entropy systems, which for measure-preserving actions of discrete amenable groups on Lebesgue spaces was established in [14] (see also Chapter 6 of [12]). Recall that a *joining* between two measure-preserving G-systems $(Y, \mathscr{Y}, \nu, G)$ and $(Z, \mathscr{Z}, \omega, G)$ is a G-invariant probability measure on $(Y \times Z, \mathscr{Y} \otimes \mathscr{Z})$ with ν and ω as marginals. The two systems are said to be *disjoint* if $\nu \times \omega$ is the only joining between them.

Proposition 3.10. *Let $(Y,\mathscr{Y},\nu,G)$ and $(Z,\mathscr{Z},\omega,G)$ be measure-preserving G-systems. Let $\varphi : L^\infty(Y,\nu) \to L^\infty(Z,\omega)$ be a G-equivariant unital positive linear map such that $\omega\circ\varphi = \nu$. Then $\varphi(\mathfrak{P}_X) \subseteq \mathfrak{P}_Y$.*

Proof. Since φ is unital and positive it is operator norm contractive and for every $f \in L^\infty(Y,\nu)$ we have

$$\|\varphi(f)\|_\omega = \omega(\varphi(f)^*\varphi(f))^{1/2} \leq \omega(\varphi(f^*f))^{1/2} = \nu(f^*f)^{1/2} = \|f\|_\nu,$$

that is, φ is also contractive for the norms $\|\cdot\|_\nu$ and $\|\cdot\|_\omega$. Thus if condition (3) in Theorem 3.7 holds for a given $f \in L^\infty(Z,\omega)$ with witnessing constants d, δ, and λ then it also holds for every element of $\varphi^{-1}(\{f\})$ with the same witnessing constants. The equivalence (1)⇔(3) in Theorem 3.7 now yields the proposition. □

A joining η between two measure-preserving systems $\boldsymbol{Y} = (Y,\mathscr{Y},\nu,G)$ and $\boldsymbol{Z} = (Z,\mathscr{Z},\omega,G)$ gives rise as follows to a G-equivariant unital positive linear map $\varphi : L^\infty(Y,\nu) \to L^\infty(Z,\omega)$ such that $\omega\circ\varphi = \nu$ (this is a special case of a construction for correspondences between von Neumann algebras [38]). Define the operator $S : L^2(Z,\omega) \to L^2(Y\times Z,\eta)$ by $(S\xi)(y,z) = \xi(z)$ for all $\xi \in L^2(Z,\omega)$ and $(y,z) \in Y\times Z$ and the representation $\pi : L^\infty(Y,\nu) \to \mathcal{B}(L^2(Y\times Z,\eta))$ by $(\pi(f)\zeta)(y,z) = f(y)\zeta(y,z)$ for all $f \in L^\infty(Y,\nu)$, $\zeta \in L^2(Y\times Z,\eta)$, and $(y,z) \in Y\times Z$. Then for $f \in L^\infty(Y,\nu)$ we set $\varphi(f) = S^*\pi(f)S$. It is easily checked that $S^*\pi(f)S$ commutes with every element of the commutant $L^\infty(Z,\omega)'$, so that $\varphi(f) \in L^\infty(Z,\omega)'' = L^\infty(Z,\omega)$. Now define the representation $\rho : L^\infty(Z,\omega) \to \mathcal{B}(L^2(Y\times Z,\eta))$ by $(\rho(g)\zeta)(y,z) = g(z)\zeta(y,z)$ for all $g \in L^\infty(Z,\omega)$, $\zeta \in L^2(Y\times Z,\eta)$, and $(y,z) \in Y\times Z$. Then for $f \in L^\infty(Y,\nu)$ and $g \in L^\infty(Z,\omega)$ we have, with $\mathbf{1}$ denoting the unit in the appropriate L^∞ algebra,

$$\begin{aligned}\eta(\pi(f)\rho(g)) &= \langle \pi(f)\rho(g), \mathbf{1}\otimes\mathbf{1}\rangle_\eta = \langle \pi(f)\rho(g)S\mathbf{1}, S\mathbf{1}\rangle_\eta \\ &= \langle \pi(f)Sg\mathbf{1}, S\mathbf{1}\rangle_\eta = \langle S^*\pi(f)Sg\mathbf{1}, \mathbf{1}\rangle_\omega \\ &= \omega(\varphi(f)g).\end{aligned}$$

In the case that the image of φ is the scalars, we see that η gives rise to the product state $\varphi\otimes\omega$ on $L^\infty(Y,\nu)\otimes L^\infty(Z,\omega)$ under composition with the representation $f\otimes g \mapsto \pi(f)\rho(g)$, and furthermore $\varphi = \nu$ by the assumption on the marginals in the definition of joining.

Corollary 3.11. *Let $\boldsymbol{Y} = (Y,\mathscr{Y},\nu,G)$ and $\boldsymbol{Z} = (Z,\mathscr{Z},\omega,G)$ be measure-preserving G-systems. Suppose that $\boldsymbol{Y}$ has zero entropy and $\boldsymbol{Z}$ has completely positive entropy. Then $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are disjoint.*

Proof. As above, a joining η between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ gives rise to a G-equivariant unital positive linear map $\varphi : L^\infty(Y,\nu) \to L^\infty(Z,\omega)$ such that $\omega\circ\varphi = \nu$. By Proposition 3.10 the image of such a map φ must be the scalars. Hence there is only the one joining $\nu\times\omega$. □

4. Measure IN-tuples

In this section (X,G) is an arbitrary topological dynamical system and μ a G-invariant Borel probability measure on X. We will define μ-IN-tuples and establish some properties in analogy with μ-IE-tuples. Here the role of measure entropy is played by measure sequence entropy. The combinatorial phenomena responsible for the properties of μ-IE-tuples in Proposition 2.16 apply equally well to the sequence entropy framework, and so it will essentially be a matter of recording the analogues of various lemmas from Section 2. We will also show that nondiagonal μ-IN-tuples are the same as μ-sequence entropy tuples and derive the measure IN-tuple product formula.

For $\delta > 0$ we say that a finite tuple $\boldsymbol{A}$ of subsets of X has *δ-μ-independence density over arbitrarily large finite sets* if there exists a $c > 0$ such that for every $M > 0$ there is a finite set $F \subseteq G$ of cardinality at least M which possesses the property that for every $D \in \mathscr{B}'(X, \delta)$ there is an independence set $I \subseteq F$ relative to D with $|I| \geq c|F|$. We say that $\boldsymbol{A}$ has *positive sequential μ-independence density* if for some $\delta > 0$ it has δ-μ-independence density over arbitrarily large finite sets.

Arguing as in the proof of Lemma 2.6 yields:

Lemma 4.1. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X which has positive sequential μ-independence density. Suppose that $A_1 = A_{1,1} \cup A_{1,2}$. Then at least one of the tuples $\boldsymbol{A}_1 = (A_{1,1}, A_2, \dots, A_k)$ and $\boldsymbol{A}_2 = (A_{1,2}, A_2, \dots, A_k)$ has positive sequential μ-independence density.*

In [30] we defined a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ to be an IN-tuple (or an IN-pair in the case $k = 2$) if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the G-orbit of the tuple $(U_1, \dots, U_k)$ has arbitrarily large finite independent subcollections. Here is the measure-theoretic analogue:

Definition 4.2. We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ a *μ-IN-tuple* (or *μ-IN-pair* in the case $k = 2$) if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the tuple $(U_1, \dots, U_k)$ has positive sequential μ-independence density. We denote the set of μ-IN-tuples of length k by $\mathrm{IN}_k^\mu(X)$.

Obviously every μ-IN-tuple is an IN-tuple.

The following analogue of Lemma 2.8 follows immediately from Lemma 2.7.

Lemma 4.3. *Let $\mathcal{P} = \{P_1, P_2\}$ be a two-element Borel partition of X such that $h_\mu(\mathcal{P}; \mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G. Then there exists $\varepsilon > 0$ such that whenever $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ are Borel sets with $\mu(P_1 \setminus A_1), \mu(P_2 \setminus A_2) < \varepsilon$ the pair $\boldsymbol{A} = (A_1, A_2)$ has positive sequential μ-independence density.*

Fix a sequence $\mathfrak{s} = \{s_j\}_{j \in \mathbb{N}}$ in G. Recalling the notation $\varphi_{\boldsymbol{A},\delta}$ and $\varphi'_{\boldsymbol{A},\delta}$ from Subsection 2.1, for $\delta > 0$ we set

$$\begin{aligned}
\overline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta; \mathfrak{s}) &= \limsup_{n \to \infty} \frac{1}{n} \varphi_{\boldsymbol{A},\delta}(\{s_1, \dots, s_n\}), \\
\overline{\mathrm{I}}'_\mu(\boldsymbol{A}, \delta; \mathfrak{s}) &= \limsup_{n \to \infty} \frac{1}{n} \varphi'_{\boldsymbol{A},\delta}(\{s_1, \dots, s_n\}), \\
\overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s}) &= \sup_{\delta > 0} \overline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta; \mathfrak{s}).
\end{aligned}$$

By Lemma 2.3, we have

$$a(k)\overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s}) \leq \sup_{\delta > 0} \overline{\mathrm{I}}'_\mu(\boldsymbol{A}, \delta; \mathfrak{s}) \leq \overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s})$$

where $a(k)$ is as defined in Subsection 2.1. Clearly $\boldsymbol{A}$ has positive sequential μ-independence density if and only if $\overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G.

Let $\mathcal{U}$ be a finite Borel cover of X. Recall that $H(\mathcal{U})$ denotes the infimum of the entropies $H(\mathcal{P})$ over all finite Borel partitions $\mathcal{P}$ of X that refine $\mathcal{U}$. For $\delta > 0$ we set

$$\begin{aligned}
\overline{h}_{c,\mu}(\mathcal{U}, \delta; \mathfrak{s}) &= \limsup_{n\to\infty} \frac{1}{n} \ln N_\delta\bigg(\bigvee_{j=1}^{n} s_j^{-1}\mathcal{U} \bigg), \\
\overline{h}_{c,\mu}(\mathcal{U}; \mathfrak{s}) &= \sup_{\delta>0} \overline{h}_{c,\mu}(\mathcal{U}, \delta; \mathfrak{s}), \\
h_\mu^-(\mathcal{U}; \mathfrak{s}) &= \limsup_{n\to\infty} \frac{1}{n} H\bigg(\bigvee_{j=1}^{n} s_j^{-1}\mathcal{U} \bigg), \\
h_\mu^+(\mathcal{U}; \mathfrak{s}) &= \inf_{\mathcal{P}\succeq\mathcal{U}} h_\mu(\mathcal{P}; \mathfrak{s}),
\end{aligned}$$

where the last infimum is taken over finite Borel partitions refining $\mathcal{U}$. Both $h_\mu^-(\mathcal{U}; \mathfrak{s})$ and $h_\mu^+(\mathcal{U}; \mathfrak{s})$ appeared in [21] for the case of $G = \mathbb{Z}$. We have $h_\mu^-(\mathcal{U}; \mathfrak{s}) \leq h_\mu^+(\mathcal{U}; \mathfrak{s})$ trivially.

The next lemma is the analogue of Lemma 2.12 and follows directly from Lemma 2.11.

Lemma 4.4. *Let $\pi : X \to Y$ be a factor of X. For any finite Borel cover $\mathcal{U}$ of Y, one has*

$$h_\mu^-(\pi^{-1}\mathcal{U}; \mathfrak{s}) = h_{\pi_*(\mu)}^-(\mathcal{U}; \mathfrak{s}).$$

The argument in the proof of Lemma 2.13 can also be used to show:

Lemma 4.5. *We have $\delta \cdot \overline{h}_{c,\mu}(\mathcal{U}, \delta; \mathfrak{s}) \leq h_\mu^-(\mathcal{U}; \mathfrak{s}) \leq \overline{h}_{c,\mu}(\mathcal{U}; \mathfrak{s})$.*

Next we come to the analogue of Lemma 2.15.

Lemma 4.6. *For a finite Borel cover $\mathcal{U}$ of X, the quantities $h_\mu^-(\mathcal{U}; \mathfrak{s})$ and $\overline{h}_{c,\mu}(\mathcal{U}; \mathfrak{s})$ are either both zero or both nonzero. If the complements in X of the members of $\mathcal{U}$ are pairwise disjoint and $\boldsymbol{A}$ is a tuple consisting of these complements, then we may also add the third quantity $\overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s})$ to the list.*

Proof. The first assertion is a consequence of Lemma 2.13. For a tuple $\boldsymbol{A}$ as in the lemma statement, Lemma 3.3 of [30] and Lemma 2.14 show that $\overline{h}_{c,\mu}(\mathcal{U}; \mathfrak{s}) > 0$ if and only if $\overline{\mathrm{I}}_\mu(\boldsymbol{A}; \mathfrak{s}) > 0$. □

Proposition 4.7. *The following hold:*

1. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of closed subsets of X which has positive sequential μ-independence density. Then there exists a μ-IN-tuple $(x_1, \dots, x_k)$ with $x_j \in A_j$ for $j = 1, \dots, k$.*
2. *$\mathrm{IN}_2^\mu(X) \setminus \Delta_2(X)$ is nonempty if and only if the system $(X, \mathscr{B}, \mu, G)$ is nonnull.*
3. *$\mathrm{IN}_1^\mu(X) = \mathrm{supp}(\mu)$ when G is an infinite group.*
4. *$\mathrm{IN}_k^\mu(X)$ is a closed G-invariant subset of X^k.*
5. *Let $\pi : X \to Y$ be a topological G-factor map. Then $\pi^k(\mathrm{IN}_k^\mu(X)) = \mathrm{IN}_k^{\pi_*(\mu)}(Y)$.*

Proof. (1) Apply Lemma 4.1 and a compactness argument.

(2) As is well known and easy to show, (X, μ) is nonnull if and only if there is a two-element Borel partition of X with positive sequence entropy with respect to some sequence in G. We thus obtain the "if" part by (1) and Lemma 4.3. For the "only if" part apply Lemma 4.6.

(3) This follows from Lemma 2.9.
(4) Trivial.
(5) This follows from (1), (3), (4) and Lemmas 4.4 and 4.6. □

The concept of measure sequence entropy tuple originates in [21], which deals with the case $G = \mathbb{Z}$. The definition works equally well for general G. Thus for $k \geq 2$ we say that a nondiagonal tuple $(x_1, \dots, x_k) \in X^k$ is a *sequence entropy tuple for* μ if whenever $U_1, \dots, U_l$ are pairwise disjoint Borel neighbourhoods of the distinct points in the list $x_1, \dots, x_k$, every Borel partition of X refining the cover $\{U_1^c, \dots, U_l^c\}$ has positive measure sequence entropy with respect to some sequence in G. To show that nondiagonal μ-IN-tuples are the same as μ-sequence entropy tuples, it suffices by Lemma 4.6 to prove that if $\mathcal{U}$ is a cover of X consisting of the complements of neighbourhoods of the points in a μ-sequence entropy tuple then $h_\mu^-(\mathcal{U}; \mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G. For $G = \mathbb{Z}$ this was done by Huang, Maass, and Ye in Theorem 3.5 of [21]. Their methods readily extend to the general case, as we will now indicate.

Given a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$, the Hilbert space $\mathcal{H}$ orthogonally decomposes into two G-invariant closed subspaces $\mathcal{H}_{\rm wm}$ and $\mathcal{H}_{\rm cpct}$ such that π is weakly mixing on $\mathcal{H}_{\rm wm}$ and the G-orbit of every vector in $\mathcal{H}_{\rm cpct}$ has compact closure [18]. For our μ-preserving action of G on X, considering its associated unitary representation of G on $L^2(X, \mu)$ there exists by Theorem 7.1 of [49] a G-invariant von Neumann subalgebra $\mathfrak{D}_X \subseteq L^\infty(X, \mu)$ such that $L^2(X, \mu)_{\rm cpct} = L^2(\mathfrak{D}_X, \mu|_{\mathfrak{D}_X})$. The following lemma generalizes part of Theorem 2.3 of [21] with essentially the same proof. In [21] X is assumed to be metrizable, but that is not necessary here.

Lemma 4.8. *Let $\mathcal{P}$ be a finite Borel partition of X. Then there is a sequence $\mathfrak{s}$ in G such that $h_\mu(\mathcal{P}; \mathfrak{s}) \geq H(\mathcal{P}|\mathfrak{D}_X)$.*

Proof. First we show that, given a finite Borel partition $\mathcal{Q}$ of X and an $\varepsilon > 0$, the set of all $s \in G$ such that $H(s^{-1}\mathcal{P}|\mathcal{Q}) \geq H(\mathcal{P}|\mathfrak{D}_X) - \varepsilon$ is thickly syndetic. Write $\mathcal{P} = \{P_1, \dots, P_k\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_l\}$ and denote by E the μ-preserving conditional expectation onto $\mathfrak{D}_X$. Since $\mathbf{1}_A - E(\mathbf{1}_A) \in L^2(X, \mu)_{\rm wm}$ for every Borel set $A \subseteq X$ and thick syndeticity is preserved under taking finite intersections, for each $\eta > 0$ the set of all $s \in G$ such that $\sup_{1 \leq i \leq k, 1 \leq j \leq l} |\langle U_s(\mathbf{1}_{P_i} - E(\mathbf{1}_{P_i})), \mathbf{1}_{Q_j}\rangle| < \eta$ is thickly syndetic. It follows that for all s in some thickly syndetic set we have, using the concavity of the function $x \mapsto -x \ln x$,

$$\begin{aligned} H(s^{-1}\mathcal{P}|\mathcal{Q}) + \varepsilon &\geq \sum_{i=1}^k \sum_{j=1}^l -\langle U_s E(\mathbf{1}_{P_i}), \mathbf{1}_{Q_j}\rangle \ln\left(\frac{\langle U_s E(\mathbf{1}_{P_i}), \mathbf{1}_{Q_j}\rangle}{\mu(Q_j)}\right) \\ &\geq \sum_{i=1}^k \int_X -U_s E(\mathbf{1}_{P_i}) \ln(U_s E(\mathbf{1}_{P_i}))\, d\mu \\ &= H(\mathcal{P}|\mathfrak{D}_X), \end{aligned}$$

as desired.

We can now recursively construct a sequence $\mathfrak{s} = \{s_1 = e, s_2, s_3, \dots\}$ in G such that $H(s_n^{-1}\mathcal{P}|\bigvee_{i=1}^{n-1} s_i^{-1}\mathcal{P}) \geq H(\mathcal{P}|\mathfrak{D}_X) - 2^{-n}$ for each $n > 1$. Using the identity $H(\bigvee_{i=1}^n s_i^{-1}\mathcal{P}) =$

$H(\bigvee_{i=1}^{n-1} s_i^{-1}\mathcal{P}) + H(s_n^{-1}\mathcal{P}|\bigvee_{i=1}^{n-1} s_i^{-1}\mathcal{P})$ we then get

$$h_\mu(\mathcal{P};\mathfrak{s}) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} H(s_k^{-1}\mathcal{P}|\bigvee_{i=1}^{k-1} s_i^{-1}\mathcal{P}) \geq H(\mathcal{P}|\mathfrak{D}_X).$$

□

Using Lemma 4.8 we can now argue as in the proof of Theorem 3.5 of [21] to deduce that $h_\mu^-(\mathcal{U};\mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G whenever $\mathcal{U}$ is a cover of X whose elements are the complements of neighbourhoods of the points in a μ-sequence entropy tuple (it can be checked that the metrizability hypothesis on X in [21] is not necessary in this case). In [21] the authors use the fact that $\mathfrak{D}_X$-measurable partitions have zero measure sequence entropy for all sequences, which for $G = \mathbb{Z}$ and metrizable X is contained in [31]. In our more general setting we can appeal to Theorem 5.5 from the next section. We thus obtain the desired result:

Theorem 4.9. *For every $k \geq 2$, a nondiagonal tuple in X^k is a μ-IN-tuple if and only if it is a μ-sequence entropy tuple.*

To establish the product formula for μ-IN-tuples we will make use of the maximal null von Neumann algebra $\mathfrak{N}_X \subseteq L^\infty(X,\mu)$, which corresponds to the largest factor of the system with zero sequence entropy for all sequences (see the beginning of the next section). Denote by E'_X the conditional expectation $L^\infty(X,\mu) \to \mathfrak{N}_X$. The following lemma is the analogue of Lemma 4.10 and appeared as Lemma 3.3 in [21]. Note that the assumptions in [21] that X is metrizable and $G = \mathbb{Z}$ are not needed here.

Lemma 4.10. *Let $\mathcal{U} = \{U_1,\dots,U_k\}$ be a Borel cover of X. Then $\prod_{i=1}^k E'_X(\chi_{U_i^c}) \neq 0$ if and only if for every finite Borel partition $\mathcal{P}$ finer than $\mathcal{U}$ as a cover one has $h_\mu(\mathcal{P};\mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G.*

Combining Lemma 4.10, Proposition 4.7(3), and Theorem 4.9, we obtain the following analogue of Lemma 2.29.

Lemma 4.11. *When G is infinite, a tuple $\boldsymbol{x} = (x_1,\dots,x_k) \in X^k$ is a μ-IN tuple if and only if for any Borel neighbourhoods $U_1,\dots,U_k$ of $x_1,\dots,x_k$, respectively, one has $\prod_{i=1}^k E'_X(\chi_{U_i}) \neq 0$.*

The following is the analogue of Theorem 2.30.

Theorem 4.12. *Let (Y,G) be another topological G-system and ν a G-invariant Borel probability measure on Y. Then for all $k \geq 1$ we have $\mathrm{IN}^k_{\mu\times\nu}(X\times Y) = \mathrm{IN}^k_\mu(X)\times\mathrm{IN}^k_\nu(Y)$.*

Proof. When G is finite, both sides are empty. So we may assume that G is infinite. By Proposition 4.7(5) we have $\mathrm{IN}^k_{\mu\times\nu}(X\times Y) \subseteq \mathrm{IN}^k_\mu(X)\times\mathrm{IN}^k_\nu(Y)$. Thus we just need to prove $\mathrm{IN}^k_\mu(X)\times\mathrm{IN}^k_\nu(Y) \subseteq \mathrm{IN}^k_{\mu\times\nu}(X\times Y)$.

Since the tensor product of a weakly mixing unitary representation of G and any other unitary representation of G is weakly mixing, we have $L^2(X\times Y,\mu\times\nu)_{\mathrm{cpct}} = L^2(X,\mu)_{\mathrm{cpct}}\otimes L^2(Y,\nu)_{\mathrm{cpct}}$. It follows that $\mathfrak{D}_{X\times Y} = \mathfrak{D}_X\otimes\mathfrak{D}_Y$. By Theorem 5.5 from the next section we have $\mathfrak{N}_X = \mathfrak{D}_X$. Thus $\mathfrak{N}_{X\times Y} = \mathfrak{N}_X\otimes\mathfrak{N}_Y$ and hence $E'_{X\times Y}(f\otimes g) = E'_X(f)\otimes E'_Y(g)$ for any $f \in L^\infty(X,\mu)$ and $g \in L^\infty(Y,\nu)$. Now the desired inclusion follows from Lemma 4.11. □

In the case $G = \mathbb{Z}$, the product formula for measure sequence entropy tuples is implicit in Theorem 4.5 of [21], and we have essentially applied the argument from there granted the fact that for general G the maximal null factor is the same as the maximal isometric factor, as shown by Theorem 5.5.

5. Combinatorial independence and the maximal null factor

We will continue to assume that (X, G) is an arbitrary topological dynamical system and μ is a G-invariant Borel probability measure on X. In analogy with the Pinsker σ-algebra in the context of entropy, the G-invariant σ-subalgebra of $\mathscr{B}$ generated by all finite Borel partitions of X with zero sequence entropy for all sequences (or, equivalently, all two-element Borel partitions of X with zero sequence entropy for all sequences) defines the largest factor of the system with zero sequence entropy for all sequences (see [21]). The corresponding G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ will be denoted by $\mathfrak{N}_X$ and referred to as the *maximal null von Neumann algebra*. The system $(X, \mathscr{B}, \mu, G)$ is said to be *null* if $\mathfrak{N}_X = L^\infty(X, \mu)$ (i.e., if it has zero measure sequence entropy for all sequences) and *completely nonnull* if $\mathfrak{N}_X = \mathbb{C}$. Kushnirenko showed that an ergodic $\mathbb{Z}$-action on a Lebesgue space is isometric if and only if $\mathfrak{N}_X = L^\infty(X, \mu)$ [31]. As Theorem 5.5 will demonstrate more generally, $\mathfrak{N}_X$ always coincides with $\mathfrak{D}_X$, as defined prior to Lemma 4.8.

Our main goal in this section is to establish Theorem 5.5, which gives various local descriptions of the maximal null factor in analogy with Theorem 3.7. To a large extent the same arguments apply and we will simply refer to the appropriate places in the proof Theorem 3.7. On the other hand, several conditions appear in Theorem 5.5 which have no analogue in the entropy setting, reflecting the fact that there is a particularly strong dichotomy between nullness and nonnullness. This dichotomy hinges on the orthogonal decomposition of $L_2(X, \mu)$ into the G-invariant closed subspaces $L_2(X, \mu)_{\rm wm}$ and $L_2(X, \mu)_{\rm cpct}$ (as described prior to Lemma 4.8) and the relationship between compact orbit closures and finite-dimensional subrepresentations recorded below in Proposition 5.3.

To define the sequence analogue of c.p. approximation entropy, let M be a von Neumann algebra, σ a faithful normal state on M, and β a σ-preserving action of the discrete group G on M by *-automorphisms. Let $\mathfrak{s} = \{s_n\}_n$ be a sequence in G. Recall the quantities $\mathrm{rcp}_\sigma(\cdot, \cdot)$ from the beginning of Section 3. For a finite set $\Upsilon \subseteq M$ and $\delta > 0$ we set

$$\mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta, \Upsilon, \delta) = \limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{rcp}_\sigma\left(\bigcup_{i=1}^{n} \beta_{s_i}(\Upsilon), \delta \right)$$

and define

$$\mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta, \Upsilon) = \sup_{\delta>0} \mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta, \Upsilon, \delta),$$
$$\mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta) = \sup_{\Upsilon} \mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta, \Upsilon)$$

where the last supremum is taken over all finite subsets Υ of M. We call $\mathrm{hcpa}^{\mathfrak{s}}_\sigma(\beta, \Upsilon)$ the *sequence c.p. approximation entropy* of β.

In analogy with the upper μ-ℓ_1-isomorphism density from Section 3, given a sequence $\mathfrak{s} = \{s_n\}_n$ in G, $f \in L^\infty(X, \mu)$, $\lambda \geq 1$, and $\delta > 0$ we set

$$\bar{\mathrm{I}}_\mu(f, \lambda, \delta; \mathfrak{s}) = \limsup_{n\to\infty} \frac{1}{n} \varphi_{f,\lambda,\delta}(\{s_1, \dots, s_n\})$$

and define

$$\bar{\mathrm{I}}_\mu(f, \lambda; \mathfrak{s}) = \sup_{\delta>0} \bar{\mathrm{I}}_\mu(f, \lambda, \delta; \mathfrak{s}),$$
$$\bar{\mathrm{I}}_\mu(f; \mathfrak{s}) = \sup_{\lambda\geq 1} \bar{\mathrm{I}}_\mu(f, \lambda; \mathfrak{s}).$$

We could also define the lower version but this is less significant for our applications, in which we would always be able to pass to a subsequence.

To establish (10)⇒(5) in Theorem 5.5 we will need the relationship between relatively compact orbits and finite-dimensional invariant subspaces given by Proposition 5.3, which is presumably well known. For this we record a couple of lemmas.

Lemma 5.1. *Suppose that G acts on a Banach space V by isometries. Then the action factors through a compact Hausdorff group (for a strongly continuous action on V and a homomorphism from G into this group) if and only if the norm closure of the orbit of each vector is compact.*

Proof. The "only if" part is obvious. Suppose that the action is compact. Denote by E the closure of the image of G in the space $\mathcal{B}(V)$ of bounded linear operators on V with respect to the strong operator topology. Then E is prescisely the closure of $\{(sv)_{v\in V} : s \in G\}$ in $\prod_{v\in V} \overline{Gv}$. Thus E is a compact Hausdorff space. Note that multiplication on the unit ball of $\mathcal{B}(V)$ is jointly continuous for the strong operator topology. It follows easily that E is a compact Hausdorff group of isometric operators on V and that the action of E on V is strongly continuous. This yields the "if" part. □

A *compactification* of G is a pair (Γ, φ) where Γ is a compact Hausdorff group and φ is a homomorphism from G to Γ with dense image. The Bohr compactification $\overline{G}$ of G is the spectrum of the space of almost periodic bounded functions on G and has the universal property that every compactification of G factors through it (see [1]).

Lemma 5.2. *Suppose that G acts on a von Neumann algebra M by *-automorphisms. Let σ be a G-invariant faithful normal state on M such that the induced unitary representation of G on $L^2(M, \sigma)$ has the property that the norm closure of the orbit of each vector is compact. Then the action factors through an ultraweakly continuous action of $\overline{G}$ on M.*

Proof. Denote the unitary on $L^2(M, \sigma)$ corresponding to $s \in G$ by U_s. By Lemma 5.1 the unitary representation $s \mapsto U_s$ of G factors through a strongly continuous unitary representation of $\overline{G}$. Denote the unitary on $L^2(M, \sigma)$ corresponding to $t \in \overline{G}$ by U_t. Note that the action of $s \in G$ on M is conjugation by U_s. It follows that the conjugation by U_t for each $t \in \overline{G}$ preserves M. □

For any ultraweakly continuous action of a locally compact group Γ on a von Neumann algebra as automorphisms, there is a Γ-invariant ultraweakly dense unital C^*-subalgebra of the von Neumann algebra on which the action of Γ is strongly continuous [37, Lemma

7.5.1]. For any strongly continuous action of a compact group on a Banach space as isometries, the subspace of elements whose orbit spans a finite-dimensional subspace is dense [5, Theorem III.5.7]. Thus we have:

Proposition 5.3. *Under the hypotheses of Lemma 5.2, there are a $\overline{G}$-invariant ultraweakly dense unital C^*-subalgebra A of M on which the action of $\overline{G}$ is strongly continuous and a norm dense *-subalgebra B of A such that the orbit of every element in B spans a finite-dimensional subspace.*

The following lemma is a local version of Theorem 5.2 of [19] and is a consequence of the proof given there in conjunction with the Rosenthal-Dor ℓ_1 theorem, which asserts that a bounded sequence in a Banach space has either a weakly Cauchy subsequence or a subsequence equivalent to the standard basis of ℓ_1 [41, 9]. Indeed if f is as in the lemma statement, then given a sequence $\{g_j\}_{j=1}^{\infty}$ in the L^2 closure of $\{\alpha_s(f) : s \in G\}$ we take for each j an $s_j \in G$ with $\|\alpha_{s_j}(f) - g_j\|_2 < 1/j$ and use the Rosenthal-Dor theorem to find an $h \in L^\infty(X,\mu) \cong C(\Omega)$ and $1 \leq j_1 < j_2 < \dots$ such that $\lim_{i\to\infty} \alpha_{s_{j_i}}(f)(\sigma) = h(\sigma)$ for all $\sigma \in \Omega$, where Ω is the spectrum of $L^\infty(X,\mu)$ (see the discussion before Theorem 3.7). Replacing the topological model X by Ω produces the same L^2 completion, so that $\lim_{i\to\infty} \|\alpha_{s_{j_i}}(f) - h\|_2 = 0$ and thus $\lim_{i\to\infty} \|g_{j_i} - h\|_2 = 0$, yielding the sequential compactness and hence compactness of the closure of $\{\alpha_s(f) : s \in G\}$ in $L^2(X,\mu)$.

Lemma 5.4. *Let f be a function in $L^\infty(X,\mu)$ whose G-orbit does not contain an infinite subset equivalent to the standard basis of ℓ_1. Then $f \in L^2(X,\mu)_{\rm cpct}$.*

The converse of lemma 5.4 is false. Indeed by [15] every free ergodic $\mathbb{Z}$-system has a minimal topological model with uniformly positive entropy, which means in particular that there are L^∞ functions whose G-orbit has a positive density subset equivalent to the standard basis of ℓ_1.

In the following theorem (Ω, G) is the topological G-system associated to $(X, \mathscr{B}, G, \mu)$ described before Theorem 3.7.

Theorem 5.5. *Let $f \in L^\infty(X,\mu)$. Then the following are equivalent:*

(1) $f \notin \mathfrak{N}_X$,
(2) *there is a μ-IN-pair $(\sigma_1,\sigma_2) \in \Omega\times\Omega$ such that $f(\sigma_1) \neq f(\sigma_2)$,*
(3) *there are $d > 0$, $\delta > 0$, and $\lambda > 0$ such that for any $M > 0$ there is some finite subset $F \subseteq G$ with $|F| \geq M$ such that whenever g_s for $s \in F$ are elements of $L^\infty(X,\mu)$ satisfying $\|g_s - \alpha_s(f)\|_\mu < \delta$ for every $s \in F$ there exists an $I \subseteq F$ of cardinality at least $d|F|$ for which the linear map $\ell_1^I \to {\rm span}\{g_s : s \in I\}$ sending the standard basis element with index $s \in I$ to g_s has an inverse with norm at most λ,*
(4) $\overline{\rm I}_\mu(f;\mathfrak{s}) > 0$ *for some sequence $\mathfrak{s}$ in G,*
(5) $f \notin L^2(X,\mu)_{\rm cpct}$,
(6) ${\rm hcpa}_\mu^{\mathfrak{s}}(\alpha,\{f\}) > 0$ *for some sequence $\mathfrak{s}$ in G,*
(7) ${\rm hcpa}_\mu^{\mathfrak{s}}(\beta) > 0$ *for some sequence $\mathfrak{s}$ in G where β is the restriction of α to the von Neumann subalgebra of M dynamically generated by f.*
(8) *there is a $\delta > 0$ such that every $g \in L^\infty(X,\mu)$ satisfying $\|g - f\|_\mu < \delta$ has an infinite ℓ_1-isomorphism set,*

(9) *there is a $\delta > 0$ such that every $g \in L^\infty(X, \mu)$ satisfying $\|g-f\|_\mu < \delta$ has arbitrarily large λ-ℓ_1-isomorphism sets for some $\lambda > 0$,*
(10) *there is a $\delta > 0$ such that every $g \in L^\infty(X, \mu)$ satisfying $\|g - f\|_\mu < \delta$ has noncompact orbit closure in the operator norm.*

When $f \in C(X)$ we can add:

(11) *$f \notin C(Y)$ whenever $\pi : X \to Y$ is a topological G-factor map such that $\pi_*(\mu)$ is null,*
(12) *there is a μ-IN-pair $(x_1, x_2) \in X \times X$ such that $f(x_1) \neq f(x_2)$.*

Proof. (1)⇒(2) Argue as for (1)⇒(2) in Theorem 3.7 using Lemma 4.3 and Proposition 4.7(1) instead of Lemma 2.8 and Proposition 2.16(1).

(2)⇒(3). Apply the same argument as for (2)⇒(3) in Theorem 3.7, replacing $\underline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta)$ by $\underline{\mathrm{I}}_\mu(\boldsymbol{A}, \delta; \mathfrak{s})$ for a suitable sequence $\mathfrak{s}$ in G.

(3)⇔(4). Use the arguments for (6)⇒(4) and (3)⇒(6) in the proof of Theorem 3.7.

(3)⇒(6). Argue as for (4)⇒(7) in Theorem 3.7.

(6)⇒(7). As in the case of complete positive approximation entropy, if N is an G-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ and $\mathfrak{s}$ is a sequence in G then for every finite subset $\Theta \subseteq N$ we have $\mathrm{hcpa}^{\mathfrak{s}}_{\mu|_N}(N, \Theta) = \mathrm{hcpa}^{\mathfrak{s}}_\mu(M, \Theta)$, which follows from the fact that there is a μ-preserving conditional expectation from $L^\infty(X, \mu)$ onto N [45, Prop. V.2.36] (cf. Proposition 3.5 in [46]).

(7)⇒(1). This can be deduced from Lemma 3.6 in the same way that (8)⇒(1) of Theorem 3.7 was.

(6)⇒(5). Suppose that $f \in L^2(X, \mu)_{\mathrm{cpct}}$. Let $\delta > 0$. Then the G-orbit $\{\alpha_s(f) : s \in G\}$ contains a finite δ-net Ω for the L^2-norm. Take a finite Borel partition $\mathcal{P}$ of X such that $\sup_{g\in\Omega} \operatorname{ess\,sup}_{x,y\in P} |g(x)-g(y)| < \delta$ for each $P \in \mathcal{P}$. Let B be the *-subalgebra of $L^\infty(X, \mu)$ generated by $\mathcal{P}$ and let φ be the μ-preserving condition expectation of $L^\infty(X, \mu)$ onto B. Now for every $s \in G$ we can find a $g \in \Omega$ such that $\|\alpha_s(f) - g\|_\mu \leq \delta$ so that

$$\|\varphi(\alpha_s(f)) - \alpha_s(f)\|_\mu \leq \|\varphi(\alpha_s(f) - g)\|_\mu + \|\varphi(g) - g\|_\mu + \|g - \alpha_s(f)\|_\mu < 3\delta.$$

Taking the inclusion $\psi : B \hookrightarrow L^\infty(X, \mu)$ it follows that for every finite set $F \subseteq G$ we have $(\varphi, \psi, B) \in \mathrm{CPA}_\mu(\{\alpha_s(f) : s \in F\}, 3\delta)$ and hence $\mathrm{rcp}_\mu(\{\alpha_s(f) : s \in F\}, 3\delta) \leq \dim B$. Since δ is arbitrary we conclude that $\mathrm{hcpa}^{\mathfrak{s}}_\mu(\alpha, \{f\}) = 0$ for every sequence $\mathfrak{s}$ in G.

(5)⇒(1). Since $f \notin L^2(X, \mu)_{\mathrm{cpct}}$ the restriction of α to the von Neumann algebra N dynamically generated by f has nonzero weak mixing component at the unitary level, and so there exists a finite partition $\mathcal{P}$ of X that is N-measurable but not $\mathfrak{D}_X$-measurable (where $\mathfrak{D}_X$ is as defined prior to Lemma 4.8). By Lemma 4.8 there is a sequence $\mathfrak{s}$ in G such that $h^{\mathfrak{s}}_\mu(X, \mathcal{P}) \geq H(\mathcal{P}|\mathfrak{D}_X) > 0$, from which we infer that $f \notin \mathfrak{N}_X$.

(5)⇒(8). This follows by observing that if $\{g_k\}_{k\in\mathbb{N}}$ were a sequence in $L^\infty(X, \mu)$ converging to f in the μ-norm such that each g_k lacks an infinite ℓ_1-isomorphism set, then we would have $g_k \in L^2(X, \mu)_{\mathrm{cpct}}$ for each k by Lemma 5.4 and hence $f \in L^2(X, \mu)_{\mathrm{cpct}}$.

(8)⇒(9). Trivial.

(9)⇒(10). It is easy to see that if an element g of $L^\infty(X, \mu)$ has arbitrarily large λ-ℓ_1-isomorphism sets for some $\lambda > 0$ then its G-orbit fails to have a finite ε-net for some $\varepsilon > 0$ depending on λ and $\|g\|$.

(10)⇒(5). Suppose contrary to (5) that $f \in L^2(X,\mu)_{\mathrm{cpct}}$. Then the restriction of α to the von Neumann algebra N dynamically generated by f has the property that the norm closure of the G-orbit of each vector in $L^2(N,\mu)$ is compact. By Proposition 5.3 this contradicts (10).

Suppose now that $f \in C(X)$. To prove (2)⇒(12), observe that the inclusion $C(\mathrm{supp}(\mu)) \subseteq L^\infty(X,\mu)$ gives rise to a topological G-factor map $\Omega \to \mathrm{supp}(\mu)$, so that we can apply Proposition 4.7(5). For (12)⇒(3) apply the same argument as for (2)⇒(3). For (11)⇒(12) argue as for (11)⇒(12) in Theorem 3.7, this time using Proposition 4.7. Finally, for (12)⇒(11) use Proposition 4.7(5). □

As pointed out at the beginning of the section and as used in the proof of Theorem 4.9, Theorem 5.5 shows that a measure-preserving system is isometric if and only if it is null, which in the case of a $\mathbb{Z}$-action on a Lebesgue space is a result of Kushnirenko [31]. Note that Theorem 5.5 does not depend in any way on Theorem 4.9. In conjunction with Theorem 3.7, Theorem 5.5 gives a geometric explanation for the well-known fact that isometric measure-preserving systems have zero entropy.

Condition (8) in Theorem 5.5 is the analogue of tameness from topological dynamics [13, 30]. Its equivalence with the other conditions shows that tameness as distinct from nullness is a specifically topological-dynamical phenomenon. This equivalence relies in part, via Lemma 5.4, on the local argument used by Huang in the case $G = \mathbb{Z}$ to prove that if X is metrizable and the system (X,G) is tame then every G-invariant Borel probability measure on X is measure null [19, Theorem 5.2]. The following standard type of example illustrates that the converse of Huang's result fails in an extreme way.

Example 5.6. By Lemma 7.2 of [30], when G is Abelian, every nontrivial metrizable weakly mixing system (X,G) is completely untame. We will show how to construct a weakly mixing uniquely ergodic subshift $(X,\mathbb{Z})$ with the invariant measure supported at a fixed point. We indicate first how to construct weakly mixing subshifts $(X,\mathbb{Z})$ with $X \subseteq \{0,1\}^{\mathbb{Z}}$. We shall construct two elements p and q in $\{0,1\}^{\mathbb{Z}}$ so that (p,q) is a transitive point for $X \times X$ where X is the orbit closure of p, and determine two increasing sequences $0 = a_1 < a_2 < \dots$ and $0 = a'_1 < a'_2 < \dots$ of nonnegative integers with $a_n \leq a'_n < a_{n+1}$ for all n. Set $p(k) = q(k) = 0$ unless $a_n \leq k \leq a'_n$ for some n. Set $p(0) = 1$ and $q(0) = 0$. Suppose that we have determined $a_1, \dots, a_m$ and $a'_1, \dots, a'_m$ and $p(k)$ and $q(k)$ for all $k \leq a'_m$. Take a_{m+1} to be any integer bigger than $\max(m, a'_m)$. If $m+1 \equiv 1 \mod 3$, we take $a'_{m+1} = a_{m+1} + 2m$ and set q to be 0 on the interval $[a_{m+1}, a'_{m+1}]$ while setting p on $[a_{m+1}, a'_{m+1}]$ to be the shift of q on $[-m,m]$. If $m+1 \equiv 2 \mod 3$, we take $a'_{m+1} = a_{m+1} + 2m$ and set p to be 0 on the interval $[a_{m+1}, a'_{m+1}]$ while setting q on $[a_{m+1}, a'_{m+1}]$ to be the shift of p on $[-m,m]$. If $m+1 \equiv 0 \mod 3$, consider the set S consisting of the sequences of values of p over the finite subintervals of $(-\infty, a'_m]$. Consider pairs of elements in S of the same length which don't appear as the sequence of values of (p,q) on some finite subinterval of $(-\infty, a'_m]$. Choose one such pair (f,g) with the smallest length d. Set $a'_{m+1} = a_{m+1} + d - 1$ and set p and q to be f and g, respectively, on the interval $[a_{m+1}, a'_{m+1}]$. Then it is clear that (p,q) is a transitive point for $X \times X$ where X is the orbit closure of p.

In general, note that if U is an open subset of X such that there is an infinite subset H of G for which the sets hU for $h \in H$ are pairwise disjoint, then $\mu(U) = 0$ for any invariant Borel probability measure μ on X. Denote by Y the complement of the union of all such U. Then every invariant Borel probability measure μ of X is supported on Y. We claim that in the construction above, by choosing a_{m+1} large enough at each step, we can arrange for Y to consist of only the point 0. Then $(X, \mathbb{Z})$ is uniquely ergodic and the invariant measure is supported at 0. Note that Y is always an invariant closed subset of X. Let V be the subset of X consisting of elements with value 1 at 0. It suffices to find an infinite subset $H = \{h_1, h_2, \dots\}$ of $\mathbb{Z}$ such that the sets hV for $h \in H$ are pairwise disjoint. Set $h_1 = 0$. Suppose that we have determined $a_1, \dots, a_m$ and $a'_1, \dots, a'_m$ and $h_1, \dots, h_m$ and $p(k)$ and $q(k)$ for all $k \le a'_m$. Take $h_{m+1} > h_m + a'_m - a_1$ and $a_{m+1} > a'_m + h_{m+1} - h_1$.

The following theorem addresses the extreme case of complete nonnullness, where we see the same kind of topologization as in the entropy setting of Theorem 3.9. For the definitions of the topological-dynamical properties of complete nonnullness, complete untameness, uniform nonnullness of all orders, and uniform untameness of all orders, see Sections 5 and 6 of [30].

Theorem 5.7. *Let $\boldsymbol{X} = (X, \mathscr{X}, \mu, G)$ be a measure-preserving dynamical system. Let $\boldsymbol{\Omega} = (\Omega, G)$ be the associated topological dynamical system on the spectrum Ω of $L^\infty(X, \mu)$. Then the following are equivalent:*

1. *$\boldsymbol{X}$ is weakly mixing,*
2. *$\boldsymbol{X}$ is completely nonnull,*
3. *for every nonscalar $f \in L^\infty(X, \mu)$ there is a $\lambda \ge 1$ such that for every $m \in \mathbb{N}$ there exists a set $I \subseteq G$ of cardinality m such that $\{\alpha_s(f) : s \in I\}$ is λ-equivalent to the standard basis of ℓ_1^m,*
4. *every nonscalar element of $L^\infty(X, \mu)$ has an infinite ℓ_1-isomorphism set,*
5. *$\boldsymbol{\Omega}$ is completely nonnull,*
6. *$\boldsymbol{\Omega}$ is completely untame,*
7. *$\boldsymbol{\Omega}$ is uniformly nonnull of all orders,*
8. *$\boldsymbol{\Omega}$ is uniformly untame of all orders.*

Proof. (1)⇒(8). Use Theorems 8.2, 8.6, and 9.10 of [30].

(8)⇒(7)⇒(5) and (8)⇒(6). These implications hold for any topological system (see Sections 5 and 6 of [30]).

(6)⇒(4). Apply Propositions 6.4 and 6.6 of [30].

(5)⇒(3). Apply Proposition 5.4 and Theorem 5.8 of [29].

(4)⇒(2), (3)⇒(2), and (2)⇒(1). Apply Theorem 5.5. □

In analogy with Proposition 3.10, if $(Y, \mathscr{Y}, \nu, G)$ and $(Z, \mathscr{Z}, \omega, G)$ are measure-preserving G-systems and $\varphi : L^\infty(Y, \nu) \to L^\infty(Z, \omega)$ is a G-equivariant unital positive linear map such that $\omega \circ \varphi = \nu$, then $\varphi(\mathfrak{N}_X) \subseteq \mathfrak{N}_Y$. One can deduce this using the characterization of functions in the maximal null von Neumann algebra in terms of either ℓ_1-isomorphism sets or compact orbit closures in L^2. In particular we see that isometric systems are disjoint from weakly mixing systems. Of course it is well known more generally that distal systems are disjoint from weak mixing systems (see Chapter 6 of [12]).

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