MORITA EQUIVALENCE OF SMOOTH NONCOMMUTATIVE TORI

GEORGE A. ELLIOTT AND HANFENG LI

Abstract. We show that in the generic case the smooth noncommutative tori associated to two $n \times n$ real skew-symmetric matrices are Morita equivalent if and only if the matrices are in the same orbit of the natural $\mathrm{SO}(n, n|\,\mathbb{Z})$ action.

1. Introduction

Let $n \geq 2$ and denote by $\mathcal{T}_n$ the space of $n \times n$ real skew-symmetric matrices. For each $\theta \in \mathcal{T}_n$ the corresponding n-dimensional noncommutative torus A_θ is defined as the universal C*-algebra generated by unitaries $U_1, \cdots, U_n$ satisfying the relation

$$U_k U_j = e(\theta_{kj}) U_j U_k,$$

where $e(t) = e^{2\pi i t}$. Noncommutative tori are one of the canonical examples in noncommutative differential geometry [34, 10].

One may also consider the smooth version A_θ^∞ of a noncommutative torus, which is the algebra of formal series

$$\sum c_{j_1, \cdots, j_n} U_1^{j_1} \cdots U_n^{j_n},$$

where the coefficient function $\mathbb{Z}^n \ni (j_1, \cdots, j_n) \mapsto c_{j_1, \cdots, j_n}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$. This is the space of smooth elements of A_θ for the canonical action of $\mathbb{T}^n$ on A_θ.

A notion of Morita equivalence of C*-algebras (as an analogue of Morita equivalence of unital rings [1, Chapter 6]) was introduced by Rieffel in [31, 33]. This is now often called Rieffel-Morita equivalence. It is known that two unital C*-algebras are Morita equivalent as unital $\mathbb{C}$-algebras if and only if they are Rieffel-Morita equivalent [2, Theorem 1.8]. Rieffel-Morita equivalent C*-algebras share a lot in common such as equivalent categories of Hilbert C*-modules, isomorphic K-groups, etc., and hence are usually thought of as having the same geometry.

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In [36] Schwarz introduced the notion of complete Morita equivalence of smooth noncommutative tori, which includes Rieffel-Morita equivalence of the corresponding C*-algebras, but is stronger, and has important application in M(atrix) theory [36, 23].

A natural question is to classify noncommutative tori and their smooth counterparts up to the various notions of Morita equivalence. Such results have important application to physics [11, 36]. For $n = 2$ this was done by Rieffel [32]. (For the earlier problem of isomorphism, see below.) In this case it does not matter what kind of Morita equivalence we are referring to: there is a (densely defined) action of the group $\mathrm{GL}(2, \mathbb{Z})$ on $\mathcal{T}_2$, and two matrices in $\mathcal{T}_2$ give rise to Morita equivalent noncommutative tori or smooth noncommutative tori if and only if they are in the same orbit of this action, and also if and only if the ordered K_0-groups of the algebras are isomorphic. The higher dimensional case is much more complicated and there are examples showing that the smooth counterparts of two Rieffel-Morita equivalent noncommutative tori may fail to be completely Morita equivalent [35].

In [35] Rieffel and Schwarz found a (densely defined) action of the group $\mathrm{SO}(n, n|\,\mathbb{Z})$ on $\mathcal{T}_n$ generalizing the above $\mathrm{GL}(2, \mathbb{Z})$ action. Recall that $\mathrm{O}(n, n|\,\mathbb{R})$ denotes the group of linear transformations of the vector space $\mathbb{R}^{2n}$ preserving the quadratic form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$, and that $\mathrm{SO}(n, n|\,\mathbb{Z})$ refers to the subgroup of $\mathrm{O}(n, n|\,\mathbb{R})$ consisting of matrices with integer entries and determinant 1.

Following [35], let us write the elements of $\mathrm{O}(n, n|\,\mathbb{R})$ in 2×2 block form:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then A, B, C, and D are arbitrary $n \times n$ matrices satisfying

$$A^tC + C^tA = 0 = B^tD + D^tB, \quad A^tD + C^tB = I. \tag{1}$$

The action of $\mathrm{SO}(n, n|\,\mathbb{Z})$ is then defined as

$$g\theta = (A\theta + B)(C\theta + D)^{-1}, \tag{2}$$

whenever $C\theta + D$ is invertible. There is a dense subset of $\mathcal{T}_n$ on which the action of every $g \in \mathrm{SO}(n, n|\,\mathbb{Z})$ is defined [35, page 291].

After the work of Rieffel, Schwarz, and the second named author in [35, 36, 24] (see also [39]) it is now known that two matrices in $\mathcal{T}_n$ give completely Morita equivalent smooth noncommutative tori (in the sense of [36]) if and only if they are in the same orbit of the $\mathrm{SO}(n, n|\,\mathbb{Z})$ action.

Phillips has been able to show that two simple noncommutative tori are Rieffel-Morita equivalent if and only if their ordered K_0-groups are isomorphic [29, Remark 7.9]. Using the result in [24] and Phillips's result, recently we have completed the classification of noncommutative tori up to Rieffel-Morita equivalence [17]. In general, two noncommutative tori are Rieffel-Morita equivalent if and only if they have isomorphic ordered K_0-groups and centers.

It remains to classify smooth noncommutative tori up to Morita equivalence as unital algebras. We shall consider a natural subset $\mathcal{T}_n' \subseteq \mathcal{T}_n$ which can be described both algebraically in terms of the properties of the algebra A_θ^∞ (see Notation 4.1 and Corollary 4.10) and number theoretically (see Proposition 4.11), and has the property that the complement $\mathcal{T}_n \setminus \mathcal{T}_n'$ has Lebesgue measure zero (Proposition 4.3). The main result of this paper is the Morita equivalence classification of the algebras arising from the subset $\mathcal{T}_n'$:

Theorem 1.1. *(1) The set $\mathcal{T}_n'$ is closed under Morita equivalence of the associated smooth noncommutative tori; in other words, if the algebras A_θ^∞ and $A_{\theta'}^\infty$ are Morita equivalent and $\theta \in \mathcal{T}_n'$, then $\theta' \in \mathcal{T}_n'$.*

(2) Two matrices in $\mathcal{T}_n'$ give rise to Morita equivalent smooth noncommutative tori if and only if they are in the same orbit of the $\mathrm{SO}(n, n|\,\mathbb{Z})$ *action.*

Denote by $\mathcal{T}_n^\flat$ the subset of $\mathcal{T}_n$ consisting of θ's such that A_θ is simple. The weaker form of the part (1) of Theorem 1.1 with $\mathcal{T}_n'$ replaced by $\mathcal{T}_n' \cap \mathcal{T}_n^\flat$ is a consequence of [28] (see the discussion after the proof of Proposition 4.11).

Consider the subset $\tilde{\mathcal{T}}_3$ of $\mathcal{T}_3$ consisting of θ's such that the seven numbers consisting of 1, θ_{12}, θ_{13}, θ_{23}, together with all products of any two of these four, are linearly independent over the rational numbers. In [35] Rieffel and Schwarz showed that for any $\theta \in \tilde{\mathcal{T}}_3$, the matrices θ and $-\theta$ are not in the same orbit of the $\mathrm{SO}(n, n|\,\mathbb{Z})$ action, although, by the work of Q. Lin and the first named author on the structure of 3-dimensional simple noncommutative tori culminating in [25], the C*-algebras A_θ and $A_{-\theta}$ are isomorphic. It is easy to see that the complement $\mathcal{T}_3 \setminus \tilde{\mathcal{T}}_3$ has Lebesgue measure zero. Our Theorem 1.1 (together with Proposition 4.3) shows that the complement $\mathcal{T}_3 \setminus (\mathcal{T}_3' \cap \tilde{\mathcal{T}}_3)$ also has Lebesgue measure zero, and for any $\theta \in \mathcal{T}_3' \cap \tilde{\mathcal{T}}_3$, the algebras A_θ^∞ and $A_{-\theta}^\infty$ are not Morita equivalent.

A related question is the classification of noncommutative tori and their smooth counterparts up to isomorphism. The case $n = 2$ was done by Pimsner, Rieffel and Voiculescu [30, 32] and the simple C*-algebra case for $n > 2$ was also done by Phillips [29, Theorem 7.6] (see [29, Section 5] for more of the history). There have been various results for the smooth algebra case with $n > 2$ [15, 13, 7]. In particular, Cuntz, Goodman, Jorgensen, and the first named author showed that two matrices in $\mathcal{T}'_n \cap \mathcal{T}^\flat_n$ give isomorphic smooth noncommutative tori if and only if the associated skew-symmetric bicharacters of $\mathbb{Z}^n$ are isomorphic [13]. This result is essentially a special case of Theorem 1.1, and the proof of it obtained in this way (see Remark 5.8) is new.

Schwarz proved the "only if" part of Theorem 1.1(2) in the context of complete Morita equivalence [36, Section 5]. His proof is based on the Chern character [8, 16], which is essentially a topological algebra invariant. In order to show that his argument still works in our situation, we have to show that a purely algebraic Morita equivalence between smooth noncommutative tori is automatically "topological" in a suitable sense. For this purpose and also for the proof of Theorem 1.1(1), in Sections 2 and 3 we show that any algebraic isomorphism between two "smooth algebras" (see Remark 2.5 below) is continuous and any derivation of a "smooth algebra" is continuous. These are the noncommutative analogues of the following well-known facts in classical differential geometry: any algebraic isomorphism between the smooth function algebras of two smooth manifolds corresponds to a diffeomorphism between the manifolds, and any derivation of the smooth function algebra corresponds to a (complexified) smooth vector field on the manifold. We introduce the set $\mathcal{T}'_n$ and prove Theorem 1.1(1) in Section 4. Theorem 1.1(2) will be proved in Section 5.

Throughout this paper A will be a C*-algebra, and A^∞ will be a dense sub-$*$-algebra of A closed under the holomorphic functional calculus (after the adjunction of a unit) and equipped with a Fréchet space topology stronger than the C*-algebra norm topology. Unless otherwise specified, the topology considered on A^∞ will always be this Fréchet topology.

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2. Continuity of Algebra Isomorphisms

In this section we shall prove Proposition 2.2 and Theorem 2.3, which indicate that the topology of A^∞ necessarily behaves well with respect to the algebra structure.

Lemma 2.1. *Let $\varphi : A^\infty \to A^\infty$ be an $\mathbb{R}$-linear map. If $A^\infty \xrightarrow{\varphi} A$ is continuous, then so is $A^\infty \xrightarrow{\varphi} A^\infty$.*

Proof. We shall use the closed graph theorem [3, Corollary 48.6] to prove the continuity of $A^\infty \xrightarrow{\varphi} A^\infty$. Since $A^\infty \xrightarrow{\varphi} A$ is continuous, the graph of φ is closed with respect to the norm topology in the second coordinate. It is then also closed with respect to the Fréchet topology in the second coordinate. It follows that $A^\infty \xrightarrow{\varphi} A^\infty$ is continuous. □

Proposition 2.2. *In A^∞ the $*$-operation is continuous, and the multiplication is jointly continuous. In other words, A^∞ is a topological $*$-algebra.*

Proof. By Lemma 2.1, the $*$-operation is continuous and the multiplication is separately continuous in A^∞. Proposition 2.2 follows because of the fact that if the multiplication of an algebra equipped with a Fréchet topology is separately continuous then it is jointly continuous [40, Proposition VII.1]. □

The following theorem was proved by Gardner in the case $A^\infty = A$ [18, Proposition 4.1], and was given as Lemma 4 of [13] in the case of smooth noncommutative tori.

Theorem 2.3. *Every algebra isomorphism $\varphi : A_1^\infty \to A_2^\infty$ is an isomorphism of topological algebras.*

Proof. Our proof is a modification of that for [18, Proposition 4.1]. It suffices to show that φ^{-1} is continuous. For $a \in A_1^\infty$ let $r(a)$ denote the spectral radius of a, in A_1^∞ and also in A_1 [37, Lemma 1.2].

First, for any $a \in A_1^\infty$ we have

$$\|a\|^2 = \|aa^*\| = r(aa^*) = r(\varphi(a)\varphi(a^*)) \leq \|\varphi(a)\varphi(a^*)\| \leq \|\varphi(a)\| \cdot \|\varphi(a^*)\|.$$

Next, we use the closed graph theorem (cf. above) to show that $\varphi \circ * \circ \varphi^{-1}$ is continuous on A_2^∞. Let $\{a_m\}_{m\in\mathbb{N}} \subseteq A_1^\infty$ be such that $\varphi(a_m) \to 0$ and $\varphi(a_m^*) \to \varphi(b)$ for some $b \in A_1^\infty$. By the preceding inequality we have

$$\|a_m\|^2 \leq \|\varphi(a_m)\| \cdot \|\varphi(a_m^*)\| \to 0 \cdot \|\varphi(b)\| = 0,$$
$$\|a_m^* - b\|^2 \leq \|\varphi(a_m^*) - \varphi(b)\| \cdot \|\varphi(a_m) - \varphi(b^*)\| \to 0 \cdot \|\varphi(b^*)\| = 0.$$

Therefore $b = 0$. This shows that the graph of $\varphi \circ * \circ \varphi^{-1}$ is closed.

Finally, let us use the closed graph theorem to show that φ^{-1} is continuous. Let $\{a_m\}_{m\in\mathbb{N}} \subseteq A_1^\infty$ be such that $\varphi(a_m) \to 0$ and $a_m \to b$ for some $b \in A_1^\infty$. By continuity of $\varphi\circ*\circ\varphi^{-1}$ we have $\varphi(a_m^*) \to 0$. By the inequality derived in the second paragraph it follows that $\|a_m\|^2 \to 0$. Therefore $b = 0$. This shows that the graph of φ^{-1} is closed. □

Question 2.4. *Is every algebra isomorphism $\varphi : A_1^\infty \to A_2^\infty$ continuous with respect to the C^*-algebra norm topology?*

Remark 2.5. Let us say that a Fréchet topology on an algebra $\mathcal{A}$ is a *smooth topology* if there is a C*-algebra A and a continuous injective homomorphism $\varphi : \mathcal{A} \hookrightarrow A$ such that $\varphi(\mathcal{A})$ is a dense sub-$*$-algebra of

A closed under the holomorphic functional calculus. Theorem 2.3 can be restated as that *every algebra admits at most one smooth topology.*

Example 2.6. Let $p \in M_n(A^\infty)$ be a projection. Then $pM_n(A^\infty)p$ is a dense sub-$*$-algebra of $pM_n(A)p$, and the relative topology on $pM_n(A^\infty)p$ is a Fréchet topology stronger than the C*-algebra norm topology. By [37, Corollary 2.3] the subalgebra $M_n(A^\infty) \subseteq M_n(A)$ is closed under the holomorphic functional calculus. It follows easily that the subalgebra $pM_n(A^\infty)p \subseteq pM_n(A)p$ is closed under the holomorphic functional calculus.

3. Continuity of Derivations

Throughout this section we shall assume further that A^∞ is closed under the smooth functional calculus. By this we mean that, after the adjunction of a unit, for any $a \in (A^\infty)_{\mathrm{sa}}$ and $f \in C^\infty(\mathbb{R})$ we have $f(a) \in A^\infty$. Our goal in this section is to prove Theorem 3.4.

Example 3.1. Let B be a C*-algebra, and let Δ be a set of closed, densely defined $*$-derivations of B. For $k = 1, 2, \cdots, \infty$, define

$$B_k := \{b \in B : j \leq k+1,\ \delta_1, \cdots, \delta_j \in \Delta \Rightarrow b \in \text{ the domain of } \delta_1 \cdots \delta_j\}.$$

It is routine to check that B_k is a sub-$*$-algebra of B and has the Fréchet topology determined by the seminorms

$$b \mapsto \|\delta_1 \cdots \delta_j b\|, \quad j < k+1, \quad \delta_1, \cdots, \delta_j \in \Delta.$$

By [6, Lemma 3.2], if B is unital, then B_k is closed under the smooth functional calculus. In particular, A^∞_θ is closed under the smooth functional calculus.

Example 3.2. Let G be a discrete group equipped with a length function l, that is, a nonnegative real valued function on G such that $l(1_G) = 0$, $l(g^{-1}) = l(g)$, and $l(gh) \leq l(g) + l(h)$ for all g and h in G. Consider the self-adjoint and closed unbounded linear operator $D_l : l^2(G) \to l^2(G)$ defined by $(D_l\xi)(g) = l(g)\xi(g)$ for $g \in G$. Then $\delta_l(a) = i[D_l, a]$ defines a closed, unbounded $*$-derivation from $\mathscr{B}(l^2(G))$ into itself. By Example 3.1 the intersection of the domains of δ_l^k for all $k \in \mathbb{N}$ is a Fréchet sub-$*$-algebra of $\mathscr{B}(l^2(G))$ closed under the smooth functional calculus. Denote by $S^l(G)$ the intersection of this algebra with the reduced group algebra $C^*_r(G)$. Then $S^l(G)$ is a dense Fréchet sub-$*$-algebra of $C^*_r(G)$ closed under the smooth functional calculus and containing $\mathbb{C}G$. This construction is due to Connes and Moscovici [12, page 384]. Recall that G is said to be *rapidly decaying* if there exists a length function l on G such that the intersection of the domains of D_l^k for all $k \in \mathbb{N}$, which we shall denote by $H_l^\infty(G)$, is contained in

$C_r^*(G)$ [12, 20]. In such a case, it is a result of Ji [21, Theorem 1.3] that $H_l^\infty(G)$ coincides with $S^l(G)$, and the Fréchet topology is also induced by the seminorms $\|D_l^k(\cdot)\|_{l^2}$ for $0 \le k < \infty$. Consequently, $H_l^\infty(G)$ is closed under the smooth functional calculus if it is contained in $C_r^*(G)$.

Question 3.3. *Is $M_m(A^\infty)$ closed under the smooth functional calculus for all $m \in \mathbb{N}$?*

The following theorem was proved by Sakai in the case $A^\infty = A$ and by Bratteli et al. in the case $A^\infty = A_\theta^\infty$ [6, Corollary 5.3.C2]) (cf. also [27, Theorem 1], [6, Theorem 3.1], and the proof of [13, Lemma 4]). (In fact we shall only use the result in the case $A^\infty = A_\theta^\infty$—in contrast to Theorem 2.3 which is needed in a more general setting.)

Theorem 3.4. *Every derivation $\delta : A^\infty \to A^\infty$ is continuous.*

Theorem 3.4 is useful for determining the derivations of various smooth (twisted) group algebras. As an example, let us determine the derivations of the smooth group algebra of the 3-dimensional discrete Heisenberg group H_3. This group is the multiplicative group

$$\{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}\}.$$

It is also the universal group generated by two elements U and V such that $W = VUV^{-1}U^{-1}$ is central. It is amenable [14, page 200]. Note that it is finitely generated. The Fréchet space $H_l^\infty(H_3)$ does not depend on the choice of generators if we use the word length function corresponding to finitely many generators. So we shall denote it by $H^\infty(H_3)$. Using the word length function associated to the generators U and V, one checks easily that

$$H^\infty(H_3) = \{\sum_{p,q,r \in \mathbb{Z}} a_{p,q,r} U^p V^q W^r\}, \tag{3}$$

where $\{a_{p,q,r}\}$ is in the Schwartz space $\mathcal{S}(\mathbb{Z}^3)$, and the Fréchet topology on $H^\infty(H_3)$ is just the canonical Fréchet topology on $\mathcal{S}(\mathbb{Z}^3)$. Note that H_3 has polynomial growth, and thus $H^\infty(H_3)$ is contained in $C^*(H_3) = C_r^*(H_3)$ [20, Theorem 3.1.7]. By Example 3.2 and Theorem 3.4 we know that every derivation of $H^\infty(H_3)$ into itself is continuous, and hence is determined by the restriction on $\mathbb{C}H_3$. Define derivations ∂_U and ∂_V on $\mathbb{C}H_3$ by

$$\partial_U(U) = U, \quad \partial_U(V) = 0, \quad \partial_U(W) = 0,$$
$$\partial_V(U) = 0, \quad \partial_V(V) = V, \quad \partial_V(W) = 0,$$

and extend them continuously to $H^\infty(H_3)$ using (3). We shall denote these extensions also by ∂_U and ∂_V. It is a result of Hadfield that every derivation $\delta : \mathbb{C}H_3 \to H^\infty(H_3)$ can be written uniquely as $\delta = z_U\partial_U + z_V\partial_V + \tilde{\delta}$ for some z_U, z_V in the center of $H^\infty(H_3)$ and some $\tilde{\delta}$ as the restriction of some inner derivation of $H^\infty(H_3)$ [19, Theorem 6.4]. It is also known that the center of $H^\infty(H_3)$ is just the smooth algebra generated by W, i.e., $\{\sum_{r\in\mathbb{Z}} a_rW^r : \{a_r\} \in \mathcal{S}(\mathbb{Z})\}$ [19, Lemma 6.2]. Thus we get

Corollary 3.5. *Every derivation $\delta : H^\infty(H_3) \to H^\infty(H_3)$ can be written uniquely as $\delta = z_U\partial_U + z_V\partial_V + \tilde{\delta}$ for some z_U, z_V in the center of $H^\infty(H_3)$ and some inner derivation $\tilde{\delta}$ of $H^\infty(H_3)$.*

In view of Lemma 2.1, to prove Theorem 3.4 it suffices to prove

Lemma 3.6. *Every derivation $\delta : A^\infty \to A$ is continuous.*

The proof of Lemma 3.6 is similar to that of [27, Theorem 1] and [6, Theorem 3.1]. For the convenience of the reader, we repeat the main arguments in our present setting below (in which the seminorms may not be submultiplicative).

Lemma 3.7. *Let I be a closed (two-sided) ideal of A such that A/I is infinite-dimensional. Then there exists $b \in (A^\infty)_{\mathrm{sa}}$ such that the image of b in A/I has infinite spectrum.*

Proof. Without loss of generality, we may assume that A^∞ is unital. Consider the quotient map $\varphi : A \to A/I$. Note that $\mathcal{A} := \varphi(A^\infty)$ is infinite-dimensional. Assume that every element in $\mathcal{A}_{\mathrm{sa}} = \varphi((A^\infty)_{\mathrm{sa}})$ has finite spectrum (in A/I). We assert that there exist nonzero projections $P_1, \cdots, P_m, \cdots$ in $\mathcal{A}_{\mathrm{sa}}$ such that $P_jP_k = 0$ for all $j \neq k$. Assume that we have constructed $P_1, \cdots, P_j$ for some $j \geq 0$ with the additional property that $Q_j\mathcal{A}Q_j$ is infinite-dimensional, where $Q_j = 1 - \sum_{s=1}^{j} P_s$. Choose an element b in $(Q_j\mathcal{A}Q_j)_{\mathrm{sa}} \setminus \mathbb{R}Q_j$. Since b has finite spectrum (in A/I) we can find a nonzero projection $P \in Q_j\mathcal{A}Q_j$ with $P \neq Q_j$. Since $Q_j\mathcal{A}Q_j$ is infinite-dimensional it is easy to see that either $P\mathcal{A}P$ or $(Q_j - P)\mathcal{A}(Q_j - P)$ has to be infinite-dimensional. (Recall that A/I is a C*-algebra.) We may now choose P_{j+1} to be one of P and $Q_j - P$ in such a way that $Q_{j+1}\mathcal{A}Q_{j+1}$ is infinite-dimensional, where $Q_{j+1} = 1 - \sum_{s=1}^{j+1} P_s$. This finishes the induction step.

Since A^∞ is a Fréchet space we can find a complete translation-invariant metric d on A^∞ giving the topology of A^∞ [3, Corollary 13.5]. Choose $b_m \in (A^\infty)_{\mathrm{sa}}$ such that $\varphi(b_m) = P_m$. Choose $\lambda_m \in \mathbb{R} \setminus \{0\}$ such that $d(\lambda_mb_m, 0) \leq 2^{-m}$. Note that $d(\sum_{n=k+1}^{m} \lambda_nb_n, 0) \leq 2^{-k}$ for all

$k < m$. So the series $\sum_{n=1}^{\infty} \lambda_n b_n$ converges to some b in A^∞. Since the convergence holds in particular in A, it follows that $\varphi(b) = \sum_{n=1}^{\infty} \lambda_n P_n$, the convergence being with respect to the norm topology on A/I. In particular, we have that $\lambda_m \to 0$. Since also $\lambda_m \neq 0$ we see that $\varphi(b)$ has infinite spectrum (in A/I). This contradicts the assumption to the contrary, which is therefore false. In other words, $\mathcal{A}_{\mathrm{sa}} = \varphi((A^\infty)_{\mathrm{sa}})$ contains an element with infinite spectrum (in A/I). □

Lemma 3.8. *Let $\delta : A^\infty \to A$ be a derivation. Set $\mathcal{I} = \{b \in A^\infty : a \in A^\infty \mapsto \delta(ab) \in A$ is continuous$\}$ and denote by I the closure of $\mathcal{I}$ in A. Then I is a closed (two-sided) ideal of A, and A/I is finite-dimensional.*

Proof. Clearly $\mathcal{I}$ is an ideal of A^∞. So I is an ideal of A. Assume that A/I is infinite-dimensional. By Lemma 3.7 we can find a self-adjoint element b of A^∞ such that the image of b in A/I has infinite spectrum (in A/I). Choosing suitable $f_m \in C^\infty(\mathbb{R})$ and setting $b_m = f_m(b)$ we obtain $\{b_m\}_{m\in\mathbb{N}} \subseteq A^\infty$ such that $b_j^2 \notin I$ for all $j \in \mathbb{N}$ and $b_j b_k = 0$ for all $j \neq k$. We may assume that $\|b_j\|, \|\delta(b_j)\| \leq 1$ for all $j \in \mathbb{N}$.

Let d be as in the proof of Lemma 3.7. Since $b_m^2 \notin \mathcal{I}$ there exists some $\varepsilon_m > 0$ such that for any $\varepsilon > 0$ we can find some $a' \in A_\infty$ with $d(a', 0) < \varepsilon$ and $\|\delta(a' b_m^2)\| \geq \varepsilon_m$. The multiplication in A^∞ is continuous by Proposition 2.2. Thus there exists some $\varepsilon > 0$ such that $d((m/\varepsilon_m) b' b_m, 0) \leq 2^{-m}$ for any $b' \in A^\infty$ with $d(b', 0) < \varepsilon$. Take an a' as above for this ε and set $a_m = (m/\varepsilon_m) a'$. Then $d(a_m b_m, 0) \leq 2^{-m}$ and $\|\delta(a_m b_m^2)\| \geq m$. Note that $d(\sum_{n=k+1}^{m} a_n b_n, 0) \leq 2^{-k}$ for all $k < m$. So the series $\sum_{n=1}^{\infty} a_n b_n$ converges in A^∞, say to a. Then (with a second use of Proposition 2.2)

$$\|\delta(a)\| + \|a\| \geq \|\delta(a) b_m\| + \|a \delta(b_m)\| \geq \|\delta(a b_m)\| = \|\delta(a_m b_m^2)\| \geq m,$$

which is a contradiction. The assumption that A/I is infinite-dimensional is therefore not tenable. We must conclude that A/I is finite-dimensional. □

4. The Generic Set

In this section we shall define the set $\mathcal{T}_n'$ and prove the part (1) of Theorem 1.1.

Denote by $\mathrm{Der}(A_\theta^\infty)$ the linear space of derivations $\delta : A_\theta^\infty \to A_\theta^\infty$. Set $\mathbb{R}^n = L$. We shall think of $\mathbb{Z}^n$ as the standard lattice in L^* (so that $\mathrm{Hom}(G, \mathbb{R})$ in [6] is just our L), and shall regard θ as an element of $\bigwedge^2 L$. Let us write $L \otimes_{\mathbb{R}} \mathbb{C} = L^{\mathbb{C}}$. Recall that $e(t) = e^{2\pi i t}$. One may also describe the C^*-algebra A_θ as the universal C*-algebra generated by unitaries $\{U_x\}_{x\in\mathbb{Z}^n}$ satisfying the relations

$$U_x U_y = \sigma_\theta(x,y)U_{x+y}, \tag{4}$$

where $\sigma_\theta(x,y) = e((x\cdot\theta y)/2)$. In this description the smooth algebra A_θ^∞ becomes $\mathcal{S}(\mathbb{Z}^n,\sigma_\theta)$, the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ equipped with the multiplication induced by (4). There is a canonical action of the Lie algebra $L^{\mathbb{C}}$ as derivations of A_θ^∞, which is induced by the canonical action of $\mathbb{T}^n$ on A_θ and is given explicitly by

$$\delta_X(U_x) = 2\pi i\,\langle X, x\rangle\,U_x$$

for $X \in L^{\mathbb{C}}$ and $x \in \mathbb{Z}^n$.

Notation 4.1. Let $e_1,\cdots,e_n$ be a basis of $\mathbb{Z}^n$. Denote by $\mathcal{T}_n'$ the subset of $\mathcal{T}_n$ consisting of those θ's such that every $\delta \in \mathrm{Der}(A_\theta^\infty)$ can be written as $\sum_{j=1}^n a_j\delta_{e_j} + \tilde{\delta}$ for some $a_1,\cdots,a_n$ in the center of A_θ^∞ and some inner derivation $\tilde{\delta}$.

Remark 4.2. For any $\theta\in\mathcal{T}_n$ and $\delta\in\mathrm{Der}(A_\theta^\infty)$ there is at most one way of writing δ as $\sum_{j=1}^n a_j\delta_{e_j} + \tilde{\delta}$ for some $a_1,\cdots,a_n$ in the center of A_θ^∞ and some inner derivation $\tilde{\delta}$ (see Proposition 4.7).

One may identify $\mathcal{T}_n$ with $\mathbb{R}^{\frac{n(n-1)}{2}}$ in a natural way. We may therefore talk about Lebesgue measure on $\mathcal{T}_n$.

Proposition 4.3. *The Lebesgue measure of $\mathcal{T}_n\setminus\mathcal{T}_n'$ is* 0.

Let $\rho_\theta : \mathbb{Z}^n\wedge\mathbb{Z}^n \to \mathbb{T}$ denote the bicharacter of $\mathbb{Z}^n$ corresponding to θ, i.e., $\rho_\theta(x\wedge y) = e(x\cdot\theta y)$. Recall that A_θ is simple if ρ_θ is nondegenerate in the sense that if $\rho_\theta(g\wedge h)=1$ for some $g\in\mathbb{Z}^n$ and all $h\in\mathbb{Z}^n$, then $g=0$ [38, Theorem 3.7]. (In fact, the converse is also true, though we don't need this fact here.) If ρ_θ is nondegenerate, and for every $0\neq g\in\mathbb{Z}^n$ the function $h\mapsto|\rho_\theta(g\wedge h)-1|^{-1}$ for $\rho_\theta(g\wedge h)\neq 1$ grows at most polynomially, then $\theta\in\mathcal{T}_n'$ [6, page 185]. So Proposition 4.3 follows from

Lemma 4.4. *Denote by $\mathcal{T}_n''$ the set of $\theta\in\mathcal{T}_n$ such that ρ_θ is nondegenerate and for every $0\neq g\in\mathbb{Z}^n$ the function $h\mapsto|\rho_\theta(g\wedge h)-1|^{-1}$ for $\rho_\theta(g\wedge h)\neq 1$ grows at most polynomially. Then $\mathcal{T}_n\setminus\mathcal{T}_n''$ has Lebesgue measure* 0.

Proof. If $\theta-\theta'\in M_n(\mathbb{Z})$, then $\rho_\theta=\rho_{\theta'}$. Hence

$$\mathcal{T}_n\setminus\mathcal{T}_n'' = \bigcup_{\eta\in M_n(\mathbb{Z})\cap\mathcal{T}_n} (\eta\ +\ \tilde{\mathcal{T}}_n\setminus\mathcal{T}_n''),$$

where $\tilde{\mathcal{T}}_n$ consists of $\theta = (\theta_{jk}) \in \mathcal{T}_n$ with $0 \le \theta_{jk} < 1$ for all $1 \le j < k \le n$. Denote Lebesgue measure on $\mathcal{T}_n$ by μ. It suffices to show that $\mu(\tilde{\mathcal{T}}_n \setminus \mathcal{T}''_n) = 0$. If there is some polynomial f in $\frac{n(n-1)}{2}$ variables such that

$$1 < |\sum_{1\le j<k\le n} \theta_{jk} m_{jk} - t| f(\vec{m})$$

for all $t \in \mathbb{Z}$ and $0 \ne \vec{m} = (m_{jk})_{1\le j<k\le n} \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, then both ρ_θ is nondegenerate and the required growth condition is satisfied—in other words, $\theta \in \mathcal{T}''_n$. Set $F(\vec{m}) = \prod_{1\le j<k\le n}(m_{jk}^2+1)^2$, and consider the set

$$Z_{s,\vec{m},t} = \{\theta \in \tilde{\mathcal{T}}_n : 1 \ge |\sum_{1\le j<k\le n} \theta_{jk} m_{jk} - t| s F(\vec{m})\}$$

for every $s \in \mathbb{N}$, $0 \ne \vec{m} \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, and $t \in \mathbb{Z}$. Set $\bigcup_{\vec{m},t} Z_{s,\vec{m},t} = W_s$. Then every θ in $\tilde{\mathcal{T}}_n \setminus \bigcap_{s\in\mathbb{N}} W_s$ satisfies the above condition. Therefore, it suffices to show that $\bigcap_{s\in\mathbb{N}} W_s$ has measure 0. Set $\bigcup_{\substack{\vec{m},t\\ m_{12}\ne 0}} Z_{s,\vec{m},t} = W'_s$. Then $\mu(W_s) \le \frac{n(n-1)}{2}\mu(W'_s)$ because of the symmetry between the θ_{jk}'s for $1 \le j < k \le n$. Integrating the characteristic function of $Z_{s,\vec{m},t}$ over θ_{12} first and then over the other θ_{jk}'s for $1 \le j < k \le n$, $(j,k) \ne (1,2)$, we get that

$$\mu(Z_{s,\vec{m},t}) \le 2s^{-1}(F(\vec{m}))^{-1}|m_{12}^{-1}|$$

for $m_{12} \ne 0$ and $|t| \le |\vec{m}| := \sum_{1\le j<k\le n}|m_{jk}|$, while

$$Z_{s,\vec{m},t} = \emptyset$$

for $|t| > |\vec{m}|$. It follows that

$$\begin{aligned}\mu(W'_s) \le 2s^{-1}\sum_{\substack{\vec{m}\\ m_{12}\ne 0}} (F(\vec{m}))^{-1}|m_{12}^{-1}|\cdot|\vec{m}| \quad &\le \quad 2s^{-1}\Big(\sum_{v\in\mathbb{Z}}\frac{1}{v^2+1}\Big)^{-\frac{n(n-1)}{2}} \\ &\to \quad 0 \quad \text{as } s \to \infty.\end{aligned}$$

Consequently, $\mu(\bigcap_{s\in\mathbb{N}} W_s) = 0$. □

We shall now give two other characterizations of $\mathcal{T}'_n$, one in Corollary 4.10, in terms of the properties of the algebra, and one in Proposition 4.11, in terms of the number-theoretical properties of θ. We need the following well-known fact.

Lemma 4.5. *An element $a = \sum_{h\in\mathbb{Z}^n} a_h U_h$ is in the center of A_θ^∞ if and only if the support of the coefficients a_h is contained in the subgroup $H = \{h \in \mathbb{Z}^n : \rho_\theta(g \wedge h) = 1$ for all $g \in \mathbb{Z}^n\}$. In particular, the center of A_θ^∞ is $\mathbb{C}$ if and only if ρ_θ is nondegenerate in the sense that $H = \{0\}$.*

Proof. The element a is in the center of A_θ^∞ exactly if $U_g a = aU_g$ for all $g \in \mathbb{Z}^n$. Note that $U_g a(U_g)^{-1} = \sum_{h\in\mathbb{Z}^n} a_h \rho_\theta(g \wedge h)U_h$. Consequently, $U_g a = aU_g$ for all $g \in \mathbb{Z}^n$ if and only if $a_h = 0$ for all $h \in \mathbb{Z}^n \setminus H$. $\square$

Recall that the topology considered on A_θ^∞ will always be the Fréchet topology, unless otherwise specified.

Definition 4.6. Let us say that a derivation $\delta \in \mathrm{Der}(A_\theta^\infty)$ is *approximately inner* if there is a sequence $\{a_m\}_{m\in\mathbb{N}} \subseteq A_\theta^\infty$ such that $[a_m, a] \to \delta(a)$ (in the Fréchet topology) for every $a \in A_\theta^\infty$. Denote by $\mathrm{ADer}(A_\theta^\infty)$ the linear space of approximately inner derivations of A_θ^∞.

By [6, Corollary 5.3.D2], every $\delta \in \mathrm{Der}(A_\theta^\infty)$ can be written uniquely as $\sum_{j=1}^n a_j\delta_{e_j} + \tilde{\delta}$ for some $a_1, \cdots, a_n$ in the center of A_θ^∞ and some $\tilde{\delta} \in \mathrm{Der}(A_\theta^\infty)$ such that there is a sequence $\{b_m\}_{m\in\mathbb{N}} \subseteq A_\theta^\infty$ with $\|[b_m, a] - \tilde{\delta}(a)\| \to 0$ for every $a \in A_\theta^\infty$. In fact, we can require $[b_m, a] \to \tilde{\delta}(a)$ in the Fréchet topology:

Proposition 4.7. *Every $\delta \in \mathrm{Der}(A_\theta^\infty)$ can be written uniquely as $\sum_{j=1}^n a_j\delta_{e_j} + \tilde{\delta}$ for some $a_1, \cdots, a_n$ in the center of A_θ^∞ and some $\tilde{\delta} \in \mathrm{ADer}(A_\theta^\infty)$. The bicharacter ρ_θ is nondegenerate if and only if every $\delta \in \mathrm{Der}(A_\theta^\infty)$ can be written uniquely as $\delta_X + \tilde{\delta}$ for some $X \in L^{\mathbb{C}}$ and some $\tilde{\delta} \in \mathrm{ADer}(A_\theta^\infty)$.*

Denote by A_θ^F the linear span of $\{U_x\}_{x\in\mathbb{Z}^n}$. This is a dense sub-$*$-algebra of A_θ. In the proof of [6, Corollary 5.3.D2], which itself is based on the proof of [6, Theorem 2.1], one sees easily that in the present case actually the sequence $\{b_m\}_{m\in\mathbb{N}}$ can be chosen in such a way that $[b_m, a] \to \tilde{\delta}(a)$ in the Fréchet topology for every $a \in A_\theta^F$. Thus Proposition 4.7 follows from the following lemma and Lemma 4.5.

Lemma 4.8. *Let $\delta, \delta_1, \delta_2, \cdots \in \mathrm{Der}(A_\theta^\infty)$ be such that $\delta_m(a) \to \delta(a)$ for every $a \in A_\theta^F \subseteq A_\theta^\infty$. Then $\delta_m(a) \to \delta(a)$ for every $a \in A_\theta^\infty$.*

Proof. The proof is similar to that of [6, Corollary 3.3.4]. Let $\vec{j} = (j_1, \cdots, j_k)$ with $1 \leq j_1, \cdots, j_k \leq n$ and $k \geq 0$. We say that $\vec{w} = (w_1, \cdots, w_s)$ is a subtuple of $\vec{j}$ if $s \leq k$ and there is a strictly increasing map $f : \{1, \cdots, s\} \to \{1, \cdots, k\}$ such that $w_m = j_{f(m)}$. Set $\|a\|_{\vec{j}} = \sup \|\delta_{w_s} \cdots \delta_{w_1}(a)\|$, where the sup runs over all subtuples $\vec{w}$ of $\vec{j}$, for $a \in A_\theta^\infty$. It suffices to show that $\|\delta(a) - \delta_m(a)\|_{\vec{j}} \to 0$ for every $a \in A_\theta^\infty$ and $\vec{j}$. For each $g = (q_1, \cdots, q_n) \in \mathbb{Z}^n$ set $|g| = \sum_{s=1}^s |q_s|$. Set $M = \sup\{\|\delta_m(U_t^s)\|_{\vec{j}} : t = 1, \cdots, n; s = \pm 1; m = 1, 2, \cdots\}$. Using the derivation property of δ_m we have $\|\delta_m(U_1^{q_1} \cdots U_n^{q_n})\|_{\vec{j}} \leq (2\pi)^k M|g|^{k+1}$ for every $g = (q_1, \cdots, q_n) \in \mathbb{Z}^n$. For any finite subset $Z \subseteq \mathbb{Z}^n$ and $a =$

$\sum_{g\in\mathbb{Z}^n} a_g U_g \in A_\theta^\infty$ set $a_Z = \sum_{g\in Z} a_g U_g \in A_\theta^F$. By Theorem 3.4 every derivation on A_θ^∞ is continuous. Thus $\|\delta(a-a_Z)\|_{\vec{j}} \to 0$ as Z goes to $\mathbb{Z}^n$. Also $\|\delta_m(a-a_Z)\|_{\vec{j}} \le \sum_{g\in\mathbb{Z}^n\setminus Z}(2\pi)^k M|g|^{k+1}|a_g|$. So $\|\delta_m(a-a_Z)\|_{\vec{j}} \to 0$ uniformly as Z goes to $\mathbb{Z}^n$. By assumption $\|\delta(a_Z) - \delta_m(a_Z)\|_{\vec{j}} \to 0$ as $m \to \infty$. Thus $\|\delta(a) - \delta_m(a)\|_{\vec{j}} \to 0$ as $m \to \infty$. □

Corollary 4.9. *A derivation $\delta \in \mathrm{Der}(A_\theta^\infty)$ is approximately inner if and only if there is a sequence $\{b_m\}_{m\in\mathbb{N}} \subseteq A_\theta^\infty$ with $\|[b_m, a]-\delta(a)\| \to 0$ for every $a \in A_\theta^\infty$.*

Combining Lemma 4.5 and Proposition 4.7 we get

Corollary 4.10. *The set $\mathcal{T}_n'$ consists of those θ's such that every $\delta \in \mathrm{ADer}(A_\theta^\infty)$ is inner.*

Set $F(g) = \max_{1\le j\le n} |\rho_\theta(g \wedge e_j) - 1|$ for $g \in \mathbb{Z}^n$. Then F vanishes exactly on the subgroup H in Lemma 4.5. Denote by F^{-1} the function on $\mathbb{Z}^n$ taking on the value $F(g)^{-1}$ at $g \notin H$ and the value 0 at $g \in H$.

Proposition 4.11. *The set $\mathcal{T}_n'$ consists of those θ's such that the function F^{-1} grows at most polynomially.*

Proof. In view of Corollary 4.10 it suffices to show that every $\delta \in \mathrm{ADer}(A_\theta^\infty)$ is inner if and only if the function F^{-1} grows at most polynomially.

By Proposition 4.7, Theorem 3.4, [6, Corollary 5.3.E2], and the proof of [6, Theorem 5.1] (see also the first paragraph of the proof of [6, Theorem 2.1]) the derivations $\delta \in \mathrm{ADer}(A_\theta^\infty)$ are in bijective correspondence with those $\mathbb{C}$-valued functions Q on $\mathbb{Z}^n$ such that Q vanishes on H and the function $c_h : g \mapsto Q(g)(\rho_\theta(g \wedge h) - 1)$ on $\mathbb{Z}^n$ is in the Schwarz space $\mathcal{S}(\mathbb{Z}^n)$ for every $h \in \mathbb{Z}^n$. Actually $\delta(U_h) = \sum_g c_h(g)U_hU_g$. Furthermore, δ is inner if and only if $Q \in \mathcal{S}(\mathbb{Z}^n)$. In this case, $\delta(\cdot) = [\sum_g Q(g)U_g, \cdot]$. Clearly, $c_h \in \mathcal{S}(\mathbb{Z}^n)$ for every $h \in \mathbb{Z}^n$ if and only if $c_{e_j} \in \mathcal{S}(\mathbb{Z}^n)$ for every $1 \le j \le n$, and if and only if the function $g \mapsto Q(g)F(g)$ is in $\mathcal{S}(\mathbb{Z}^n)$. In other words, the derivations $\delta \in \mathrm{ADer}(A_\theta^\infty)$ are in bijective correspondence with those $Q : \mathbb{Z}^n \to \mathbb{C}$ such that Q vanishes on H and the function $g \mapsto Q(g)F(g)$ is in $\mathcal{S}(\mathbb{Z}^n)$. Therefore, every $\delta \in \mathrm{ADer}(A_\theta^\infty)$ is inner if and only if the pointwise multiplication by F^{-1} sends $\mathcal{S}(\mathbb{Z}^n)$ into itself. Using the closed graph theorem [3, Corollary 48.6] it is easy to see that if the pointwise multiplication by F^{-1} sends $\mathcal{S}(\mathbb{Z}^n)$ into itself, then this map is continuous and hence F^{-1} grows at most polynomially. Conversely, if F^{-1} grows at most polynomially, then obviously the pointwise multiplication by F^{-1} sends $\mathcal{S}(\mathbb{Z}^n)$ into itself. Therefore every $\delta \in \mathrm{ADer}(A_\theta^\infty)$ is inner if and only if F^{-1} grows at most polynomially. □

Denote by $\mathcal{T}_n^\flat$ the subset of $\mathcal{T}_n$ consisting of θ's such that ρ_θ is nondegenerate. Let us indicate how to deduce the weaker form of the part (1) of Theorem 1.1 with $\mathcal{T}_n'$ replaced by $\mathcal{T}_n' \cap \mathcal{T}_n^\flat$ from [28] using the topological or algebraic Hochschild cohomology of A_θ^∞. Recall that if two unital algebras are Morita equivalent, then their algebraic Hochschild cohomology is isomorphic [26]. By Theorem 2.3 and Example 2.6 if two unital smooth algebras are Morita equivalent, then their topological Hochschild cohomology is also isomorphic. Nest calculated the topological Hochschild cohomology $\mathrm{H}^*_{\mathrm{top}}(A_\theta^\infty, (A_\theta^\infty)^*_{\mathrm{top}})$ of (the Fréchet algebra) A_θ^∞ with coefficients in the topological dual $(A_\theta^\infty)^*_{\mathrm{top}}$ in [28, Theorem 4.1] (the 2-dimensional case was calculated earlier by Connes in [9]). From [28, Theorem 4.1] it is easy to see that ρ_θ is nondegenerate and θ satisfies the condition in Proposition 4.11 if and only if $\mathrm{H}^*_{\mathrm{top}}(A_\theta^\infty, (A_\theta^\infty)^*_{\mathrm{top}})$ is finite-dimensional in every degree. Using the simple projective resolution of A_θ^∞ as an A_θ^∞-bimodule in [28, Section 3] one also finds that this happens if and only if the algebraic Hochschild cohomology $\mathrm{H}^*_{\mathrm{alg}}(A_\theta^\infty, (A_\theta^\infty)^*_{\mathrm{alg}})$ is finite-dimensional in every degree. Thus the above weak form of the part (1) of Theorem 1.1 follows from considering either the topological or algebraic Hochschild cohomology of A_θ^∞.

In the 2-dimensional case, when ρ_θ is nondegenerate, the condition in Proposition 4.11 was called a *diophantine condition* by Connes [9, page 349].

In [5] Boca introduced a certain subset of $\mathcal{T}_n$ the complement of which has Lebesgue measure 0 and which is also described number theoretically. His set is contained in $\mathcal{T}_n^\flat$. We do not know whether his set is the same as $\mathcal{T}_n' \cap \mathcal{T}_n^\flat$ or not.

To prove the part (1) of Theorem 1.1 we start with some general facts about the comparison of derivation spaces for Morita equivalent algebras. Let $\mathcal{A}$ be a unital algebra. Let E be a finitely generated projective right $\mathcal{A}$-module and set $\mathrm{End}(E_{\mathcal{A}}) = \mathcal{B}$. If we take an isomorphism of right $\mathcal{A}$-modules $E \to p(k\mathcal{A})$ for some projection $p \in M_k(\mathcal{A})$, where $k\mathcal{A}$ is the direct sum of k copies of $\mathcal{A}$ as right $\mathcal{A}$-modules with vectors written as columns, then we have an induced isomorphism $\mathcal{B} \to pM_k(\mathcal{A})p$.

Let $\delta \in \mathrm{Der}(\mathcal{A})$. Recall that a *connection* [8] for $(E_{\mathcal{A}}, \delta)$ is a linear map $\nabla : E \to E$ satisfying the Leibnitz rule

$$\nabla(fa) = \nabla(f)a + f\delta(a) \tag{5}$$

for all $f \in E$ and $a \in \mathcal{A}$. Let us say that a pair $(\delta', \delta) \in \mathrm{Der}(\mathcal{B}) \times \mathrm{Der}(\mathcal{A})$ is *compatible* if there is a linear map $\nabla : E \to E$ which is a connection for both $({}_{\mathcal{B}}E, \delta')$ and $(E_{\mathcal{A}}, \delta)$. One checks easily that for every $\delta \in \mathrm{Der}(\mathcal{A})$ there exists $\delta' \in \mathrm{Der}(\mathcal{B})$ such that the pair (δ', δ) is compatible, and δ' is unique up to adding an inner derivation. Explicitly, identifying E and $\mathcal{B}$ with $p(k\mathcal{A})$ and $pM_k(\mathcal{A})p$ respectively as above, and extending δ to $k\mathcal{A}$ and $M_k(\mathcal{A})$ componentwise, one may choose ∇ and δ' as defined by $\nabla(u) = p(\delta(u))$ for $u \in p(k\mathcal{A})$ and $\delta'(b) = p\delta(b)p$ for $b \in pM_k(\mathcal{A})p$ respectively.

Lemma 4.12. *If δ is inner and the pair (δ', δ) is compatible, then δ' is also inner.*

Proof. Say $\delta(\cdot) = [\cdot, a]$ for some $a \in \mathcal{A}$. The pair $(0, \delta)$ is compatible with respect to the connection $\nabla_a : f \mapsto fa$. Say the pair (δ', δ) is compatible with respect to the connection ∇. Then the pair $(\delta', 0)$ is compatible with respect to the connection $\nabla - \nabla_a$. Therefore, δ' is inner. □

Assume further that $\mathcal{A} = A^\infty$ is equipped with a smooth topology. By Example 2.6 and Theorem 2.3 we know that $\mathcal{B} = B^\infty$ also admits a unique smooth topology. Thus, the above isomorphism $B^\infty \to pM_k(A^\infty)p$ is a homeomorphism.

Lemma 4.13. *If $\delta \in \mathrm{ADer}(A^\infty)$ and the pair (δ', δ) is compatible, then $\delta' \in \mathrm{ADer}(B^\infty)$.*

Proof. We may assume that $E = p(kA^\infty)$ and $B^\infty = pM_k(A^\infty)p$ for some projection $p \in M_k(A^\infty)$. Choose a sequence of inner derivations $\delta_m \in \mathrm{Der}(A^\infty)$ such that $\delta_m(a) \to \delta(a)$ for every $a \in A^\infty$. Extend δ_m and δ to $M_k(A^\infty)$ componentwise. Consider the maps $\tilde{\delta} : b \to p\delta(b)p$ and $\tilde{\delta}_m : b \to p\delta_m(b)p$ on B^∞. Then $\tilde{\delta}$ and $\tilde{\delta}_m$ are derivations of B^∞. The pair $(\tilde{\delta}, \delta)$ is compatible, with respect to the Grassmann connection $\nabla : u \mapsto p(\delta(u))$. Similarly, the pair $(\tilde{\delta}_m, \delta_m)$ is compatible, with respect to the connection $\nabla_m : u \mapsto p(\delta_m(u))$. Notice that $\tilde{\delta}_m(b) \to \tilde{\delta}(b)$ for every $b \in B^\infty$. Lemma 4.13 now follows from Lemma 4.12. □

There are several equivalent ways of defining Morita equivalence of algebras (see for instance [1, Section 22]). Recall that a right $\mathcal{A}$-module $E_{\mathcal{A}}$ of a unital algebra $\mathcal{A}$ is a *generator* if $\mathcal{A}_{\mathcal{A}}$ is a direct summand of $rE_{\mathcal{A}}$ for some $r \in \mathbb{N}$. We shall say that two unital algebras $\mathcal{B}$ and $\mathcal{A}$ are *Morita equivalent* if there exists a bimodule ${}_{\mathcal{B}}E_{\mathcal{A}}$—a Morita equivalence bimodule—such that $E_{\mathcal{A}}$ and ${}_{\mathcal{B}}E$ are finitely generated projective modules and also generators, and, furthermore, $\mathcal{B} = \mathrm{End}(E_{\mathcal{A}})$, $\mathcal{A} = \mathrm{End}({}_{\mathcal{B}}E)$ [1, Theorem 22.2].

Now the part (1) of Theorem 1.1 follows from Corollary 4.10, Lemma 4.12 and Lemma 4.13.

The above proof employs the Fréchet topology on A_θ^∞. We give below a more algebraic proof. We are grateful to Ryszard Nest for suggesting using the Morita invariance of the module structure of $\mathrm{H}^1(\mathcal{A}, \mathcal{A})$ over the center of $\mathcal{A}$ for a unital algebra $\mathcal{A}$.

The part (1) of Theorem 1.1 follows directly from two facts. Denote by $Z(\mathcal{A})$ the center of a unital algebra $\mathcal{A}$. Given a Morita equivalence bimodule ${}_{\mathcal{B}}E_{\mathcal{A}}$ between two unital algebras $\mathcal{B}$ and $\mathcal{A}$, we may identify both $Z(\mathcal{B})$ and $Z(\mathcal{A})$ with $\mathrm{End}(E_{\mathcal{A}}) \cap \mathrm{End}({}_{\mathcal{B}}E)$ inside $\mathrm{Hom}_{\mathbb{C}}(E)$. Note that $\mathrm{Der}(\mathcal{A})$ has a natural $Z(\mathcal{A})$-module structure given by $(a\delta)(x) = a(\delta(x))$ for all $a \in Z(\mathcal{A})$, $x \in \mathcal{A}$, and $\delta \in \mathrm{Der}(\mathcal{A})$. Clearly the space of inner derivations is a submodule. Denote by $\mathrm{Out}(\mathcal{A})$ the quotient $Z(\mathcal{A})$-module. By Lemma 4.12 we have a natural linear isomorphism $\mathrm{Out}(\mathcal{A}) \to \mathrm{Out}(\mathcal{B})$. The first fact we need is that this linear isomorphism is (clearly) an isomorphism of $Z(\mathcal{A})(= Z(\mathcal{B}))$-modules. The second fact is that $\theta \in \mathcal{T}_n$ if and only if $\mathrm{Out}(A^\infty_\theta)$ is generated by n elements as a $Z(A^\infty_\theta)$-module. This follows from the decomposition of $\mathrm{Der}(A^\infty_\theta)$ quoted after Definition 4.6.

One may also deduce the second fact as follows. Recall that the first (algebraic) Hochschild cohomology $\mathrm{H}^1(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ with coefficients in $\mathcal{A}$ is exactly $\mathrm{Out}(\mathcal{A})$ [26, page 38]. Using the simple projective resolution of A^∞_θ as an A^∞_θ-bimodule in [28, Section 3] one can calculate $\mathrm{H}^*(A^\infty_\theta, A^\infty_\theta)$ and find that θ satisfies the condition in Proposition 4.11 if and only if $\mathrm{H}^1(A^\infty_\theta, A^\infty_\theta)$ is generated by n elements as a $Z(A^\infty_\theta)$-module.

Combining Lemma 4.5, Proposition 4.7, and Lemma 4.13 we also get

Proposition 4.14. *Suppose that ρ_θ and $\rho_{\theta'}$ are nondegenerate, and that E is a finitely generated projective right A^∞_θ-module with $\mathrm{End}(E_{A^\infty_\theta}) = A^\infty_{\theta'}$. Then there is a unique linear map $\varphi : L^{\mathbb{C}} \to L^{\mathbb{C}}$ such that for any $X \in L^{\mathbb{C}}$ and $\tilde{\delta} \in \mathrm{ADer}(A^\infty_\theta)$ there exists some $\tilde{\delta}' \in \mathrm{ADer}(A^\infty_{\theta'})$ such that the pair $(\delta_{\varphi(X)} + \tilde{\delta}', \delta_X + \tilde{\delta})$ is compatible. If, furthermore, the bimodule ${}_{A^\infty_{\theta'}}E_{A^\infty_\theta}$ is a Morita equivalence bimodule, then φ is an isomorphism.*

5. Proof of Theorem 1.1

In this section we prove the part (2) of Theorem 1.1.

We recall first the theory of curvature introduced by Connes in [8]. Let E be a finitely generated projective right A^∞_θ-module. If $X \in L^{\mathbb{C}} \mapsto \nabla_X \in \mathrm{Hom}_{\mathbb{C}}(E)$ is a linear map such that ∇_X is a connection of $(E_{A^\infty_\theta}, \delta_X)$ for every $X \in L^{\mathbb{C}}$, one may consider the *curvature* $[\nabla_X, \nabla_Y]$ which is easily seen to be in $\mathrm{End}(E_{A^\infty_\theta})$. We say that ∇ has *constant curvature* if $[\nabla_X, \nabla_Y] \in \mathbb{C}(= \mathbb{C} \cdot \mathrm{id}_E)$ for all $X, Y \in L^{\mathbb{C}}$.

Since complete Morita equivalence in the sense of Schwarz [36] explicitly implies Morita equivalence (see [17, Subsection 2.1]), the "if" part of the statement follows from [24, Theorem 1.2] (which deals with complete Morita equivalence).

Recall that $\mathcal{T}_n^\flat$ is the subset of $\mathcal{T}_n$ consisting of θ's such that ρ_θ is nondegenerate. To prove the "only if" part of the statement for all n, we shall reduce it first to the case of matrices in $\mathcal{T}_n' \cap \mathcal{T}_n^\flat$. For this purpose, we need the following lemma, in which we consider also $\mathcal{T}_0$ for convenience.

Lemma 5.1. *Suppose that $\theta \in \mathcal{T}_n$ is of the form*

$$\theta = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix},$$

where $\tilde{\theta}$ belongs to $\mathcal{T}_k$ for some $0 \leq k \leq n$ and $\rho_{\tilde{\theta}}$ is nondegenerate. Then for any maximal two-sided ideal I of A_θ^∞, A_θ^∞/I is isomorphic to $A_{\tilde{\theta}}^\infty$.

Proof. Since A_θ^∞ is closed under the smooth functional calculus (Example 3.1), by Theorem 13 of [22] and the remark following it there is a bijective correspondence between the lattice of two-sided ideals of A_θ^∞ closed with respect to the relative C*-algebra topology and the lattice of closed two-sided ideals of A_θ which in one direction consists in taking the intersection of an ideal with A_θ^∞ and in the other direction in taking the closure in A_θ. Since I is maximal, it is closed in the relative C*-algebra topology. It follows that the closed two-sided ideal K of A_θ corresponding to I is maximal. Note that $A_\theta = C(\mathbb{T}^{n-k}) \otimes A_{\tilde{\theta}}$ and that the center of A_θ is $C(\mathbb{T}^{n-k})$. Since $\rho_{\tilde{\theta}}$ is nondegenerate, $A_{\tilde{\theta}}$ is simple [38, Theorem 3.7]. It follows that K is equal to the kernel of the homomorphism $A_\theta \to A_{\tilde{\theta}}$ given by the evaluation at some point of $\mathbb{T}^{n-k}$. Then we may identify A_θ/K with $A_{\tilde{\theta}}$. It follows that A_θ^∞/I, which is contained in A_θ/K, is just $A_{\tilde{\theta}}^\infty$. $\square$

Suppose that θ', $\theta \in \mathcal{T}_n'$, and that $A_{\theta'}^\infty$ and A_θ^∞ are Morita equivalent, By [17, Proposition 3.3], we can find θ_1', $\theta_1 \in \mathcal{T}_n$ such that θ' and θ are in the same orbit of the $\mathrm{SO}(n, n|\mathbb{Z})$ action as θ_1' and θ_1 respectively and such that

$$\theta_1' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta}' \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix},$$

where $\tilde{\theta}'$ and $\tilde{\theta}$ belong to $\mathcal{T}_{k'}$ and $\mathcal{T}_k$ respectively for some $0 \leq k', k \leq n$ and $\rho_{\tilde{\theta}'}$ and $\rho_{\tilde{\theta}}$ are nondegenerate. By [24, Theorem 1.2], $A_{\theta'}^\infty$ and A_θ^∞ are completely Morita equivalent to $A_{\theta_1'}^\infty$ and $A_{\theta_1}^\infty$ respectively. Then $A_{\theta_1'}^\infty$ and $A_{\theta_1}^\infty$ are Morita equivalent. Since Morita equivalence between unital algebras (or rings) preserves the center [1, Proposition 21.10], by Lemma 4.5 we have $n - k' = n - k$. Therefore, $k' = k$. There is a natural bijection between the lattices of two-sided ideals of Morita equivalent unital algebras (or rings), and the corresponding quotient algebras are also Morita equivalent [1, Proposition 21.11]. It follows from Lemma 5.1 that $A^\infty_{\tilde{\theta}'}$ and $A^\infty_{\tilde{\theta}}$ are Morita equivalent. By the part (1) of Theorem 1.1, θ'_1 and θ_1 are also in $\mathcal{T}'_n$. From Proposition 4.11 we see that $\tilde{\theta}'$ and $\tilde{\theta}$ are both in $\mathcal{T}'_k$. If the "only if" part of the part (2) of Theorem 1.1 holds for all n with $\mathcal{T}'_n$ replaced by $\mathcal{T}'_n \cap \mathcal{T}^\flat_n$, then we can conclude that $\tilde{\theta}'$ and $\tilde{\theta}$ are in the same orbit of the $\mathrm{SO}(k, k|\mathbb{Z})$ action. It follows that θ'_1 and θ_1 are in the same orbit of the $\mathrm{SO}(n, n|\mathbb{Z})$ action. Consequently, θ' and θ are in the same orbit of the $\mathrm{SO}(n, n|\mathbb{Z})$ action.

Now what remains is to prove the "only if" part of the statement with $\mathcal{T}'_n$ replaced by $\mathcal{T}'_n \cap \mathcal{T}^\flat_n$. This follows from Theorems 5.2 and 5.3 below.

Theorem 5.2. *Suppose that $A^\infty_{\theta'}$ and A^∞_θ are Morita equivalent with respect to a Morita equivalence bimodule $E = {}_{A^\infty_{\theta'}}E_{A^\infty_\theta}$ with the following property: there are a $\mathbb{C}$-linear isomorphism $\varphi : L^{\mathbb{C}} \to L^{\mathbb{C}}$ and a linear map ∇ from $L^{\mathbb{C}}$ to $\mathrm{Hom}_{\mathbb{C}}(E)$, with ∇_X a connection for both $(E_{A^\infty_\theta}, \delta_X)$ and $({}_{A^\infty_{\theta'}}E, \delta_{\varphi(X)})$ for every $X \in L^{\mathbb{C}}$, such that ∇ has constant curvature (in both $\mathrm{End}(E_{A^\infty_\theta})$ and $\mathrm{End}({}_{A^\infty_{\theta'}}E)$). Then θ' and θ are in the same orbit of the $\mathrm{SO}(n, n|\,\mathbb{Z})$ action.*

Theorem 5.3. *Suppose that $\theta', \theta \in \mathcal{T}'_n \cap \mathcal{T}^\flat_n$, and that $A^\infty_{\theta'}$ and A^∞_θ are Morita equivalent, and let ${}_{A^\infty_{\theta'}}E_{A^\infty_\theta}$ be a Morita equivalence bimodule. Then there exist φ and ∇ satisfying the conditions of Theorem 5.2. (N.B: E and ∇ are not necessarily Hermitian—see the proof of Theorem 5.2 below.)*

Proof of Theorem 5.3. Let φ be as in Proposition 4.14. Let $e_1, \cdots, e_n$ be a basis of $L^{\mathbb{C}}$. Then for each $1 \leq k \leq n$ there is some $\delta'_k \in \mathrm{ADer}(A^\infty_{\theta'})$ such that $(\delta_{\varphi(e_k)} + \delta'_k, \delta_{e_k})$ is compatible. Since $\theta' \in \mathcal{T}'_n$, by Corollary 4.10 the derivation δ'_k is inner. Then $(\delta_{\varphi(e_k)}, \delta_{e_k})$ is compatible. Let ∇_{e_k} be a connection for both $({}_{A^\infty_{\theta'}}E, \delta_{\varphi(e_k)})$ and $(E_{A^\infty_\theta}, \delta_{e_k})$. Let $\nabla_{\sum_{1\leq k\leq n}\lambda_k e_k} = \sum_{1\leq k\leq n} \lambda_k \nabla_{e_k}$ for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Then for every $X \in L^{\mathbb{C}}$, ∇_X is a connection for both $({}_{A^\infty_{\theta'}}E, \delta_{\varphi(X)})$ and $(E_{A^\infty_\theta}, \delta_X)$. Consequently, $[\nabla_X, \nabla_Y] \in \mathrm{Hom}_{\mathbb{C}}(E)$ is in both $A^\infty_{\theta'}$ and A^∞_θ. Therefore, $[\nabla_X, \nabla_Y]$ lies in the center of A^∞_θ. By Lemma 4.5 the center of A^∞_θ is $\mathbb{C}$. Consequently, $[\nabla_X, \nabla_Y] \in \mathbb{C}$. □

Theorem 5.2 is an extension of Schwarz's result of [36, Section 5], in which he proved Theorem 5.2 under the addition hypotheses that E is a Hilbert bimodule, φ maps L to L, and ∇ is a Hermitian connection.

We shall essentially follow Schwarz's argument. In order to show that his argument still works without these additional hypotheses, we have to make some preparations.

Let $E_{A_\theta^\infty}$ be a finitely generated projective right A_θ^∞-module. Let τ be a trace on A_θ^∞. Denote by τ also the unique extension of τ to a trace of $M_k(A_\theta^\infty)$ for each $k \in \mathbb{N}$. Choosing an isomorphism $E \to p(kA_\theta^\infty)$ for some idempotent $p \in M_k(A_\theta^\infty)$, we get a trace τ' on $\mathrm{End}(E_{A_\theta^\infty})$, via the natural isomorphism of this algebra with $pM_k(A_\theta^\infty)p$. It is not difficult to see that τ' does not depend on the choice of p or the isomorphism $E \to p(kA_\theta^\infty)$.

Lemma 5.4. *Let $\delta \in \mathrm{Der}(A_\theta^\infty)$ be such that $\tau \circ \delta = 0$. Then $\tau' \circ \delta' = 0$ for any $\delta' \in \mathrm{Der}(\mathrm{End}(E_{A_\theta^\infty}))$ such that the pair (δ', δ) is compatible.*

Proof. We may assume that $E = p(kA_\theta^\infty)$ and $\mathrm{End}(E_{A_\theta^\infty}) = pM_k(A_\theta^\infty)p$ for some idempotent $p \in M_k(A_\theta^\infty)$. Extend δ to $M_k(A_\theta^\infty)$ componentwise. Let $\tilde{\delta}$ be the derivation of $pM_k(A_\theta^\infty)p$ defined as $\tilde{\delta}(b) = p\delta(b)p$. Then the pair $(\tilde{\delta}, \delta)$ is compatible. For each $b \in pM_k(A_\theta^\infty)p$ pick $u_j, v_j \in kA_\theta^\infty$ for $1 \le j \le k$ such that $b = \sum_{1\le j\le k} u_j v_j^t$. Then $b = pbp = \sum_{1\le j \le k}(pu_j)(v_j^t p)$, so we may assume that $pu_j = u_j$ and $v_j^t p = v_j^t$. Now

$$\begin{aligned}\tau(\tilde{\delta}(b)) &= \tau(\sum_j (p\delta(u_j)v_j^t p + pu_j\delta(v_j^t)p)) = \tau(\sum_j (v_j^t p\delta(u_j) + \delta(v_j^t)pu_j)) \\ &= \tau(\sum_j (v_j^t\delta(u_j) + \delta(v_j^t)u_j)) = \tau(\sum_j \delta(v_j^t u_j)) = 0.\end{aligned}$$

□

Denote by τ_θ the canonical trace on A_θ^∞ defined by

$$\tau_\theta(\sum_{h\in\mathbb{Z}^n} c_h U_h) = c_0.$$

Notice that, up to multiplication by a scalar, τ_θ is the unique continuous linear functional γ on A_θ^∞ satisfying $\gamma \circ \delta_X = 0$ for all $X \in L^{\mathbb{C}}$. In the proof of the next lemma, which is trivial in the case E a Hilbert bimodule, we make crucial use of Theorem 2.3.

Lemma 5.5. *Let $A_{\theta'}^\infty$, E, A_θ^∞, φ, and ∇ be as in Theorem 5.2. Denote by τ' the induced trace on $A_{\theta'}^\infty = \mathrm{End}(E_{A_\theta^\infty})$ obtained by the construction described above beginning with the trace τ_θ on A_θ^∞. Then $\tau' = \lambda\tau_{\theta'}$ for some $0 \ne \lambda \in \mathbb{C}$.*

Proof. By [37, Corollary 2.3] the subalgebra $M_k(A_\theta^\infty) \subseteq M_k(A_\theta)$ is closed under the holomorphic functional calculus for any $k \in \mathbb{N}$. Therefore, by [4, Proposition 4.6.2], every idempotent in $M_k(A_\theta^\infty)$ is similar to a self-adjoint one, i.e. a projection. Consequently, in the definition of τ' we may choose the idempotent $p \in M_k(A_\theta^\infty)$ to be a projection. By Example 2.6, $pM_n(A_\theta^\infty)p$ is closed under the holomorphic functional calculus and has the Fréchet topology as the restriction of that on $M_k(A_\theta^\infty)$. By Theorem 2.3, applied to $A_{\theta'}^\infty$ and the sub-$*$-algebra $(pM_k(A_\theta)p)^\infty := pM_k(A_\theta^\infty)p$ of $pM_k(A_\theta)p$, the natural isomorphism $A_{\theta'}^\infty = \mathrm{End}(E_{A_\theta^\infty}) \to pM_k(A_\theta^\infty)p$ is a homeomorphism. Therefore, the trace τ' is continuous on $A_{\theta'}^\infty$ (as τ_θ is obviously continuous on $pM_k(A_\theta^\infty)p$). By Lemma 5.4 we have $\tau' \circ \delta_X = 0$ for all $X \in L^{\mathbb{C}}$. Thus, (by the remark above) $\tau' = \lambda\tau_{\theta'}$ for some $\lambda \in \mathbb{C}$.

Using either that p is full (since $E_{A_\theta^\infty}$ is a generator) or that τ_θ is faithful one sees that $\tau_\theta(p) > 0$. Therefore $\lambda \neq 0$. $\square$

Let A^∞ (resp. B^∞) be a dense sub-$*$-algebra of a C*-algebra A (resp. B) closed under the holomorphic functional calculus and equipped with a Fréchet topology stronger than the C*-algebra norm topology.

Lemma 5.6. *The algebra $C^\infty(\mathbb{T}, A^\infty)$ is a dense sub-$*$-algebra of the C*-algebra $C(\mathbb{T}, A)$ closed under the holomorphic functional calculus and has a natural Fréchet topology stronger than the C*-algebra norm topology. If ${}_{B^\infty}E_{A^\infty}$ is a Morita equivalence bimodule, then $C^\infty(\mathbb{T}, B^\infty)$ and $C^\infty(\mathbb{T}, A^\infty)$ are Morita equivalent with respect to the equivalence bimodule ${}_{C^\infty(\mathbb{T},B^\infty)}C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T},A^\infty)}$.*

Proof. Clearly $C^\infty(\mathbb{T}, A^\infty)$ is a sub-$*$-algebra of $C(\mathbb{T}, A)$. Since it contains the algebraic tensor product $A^\infty \otimes C^\infty(\mathbb{T})$, we see that $C^\infty(\mathbb{T}, A^\infty)$ is dense in $C(\mathbb{T}, A)$. Endow $C^\infty(\mathbb{T}, A^\infty)$ with the topology of uniform convergence on $\mathbb{T}$ of the functions and of their derivatives up to s for every $s \in \mathbb{N}$. Clearly this is a metrizable locally convex topology stronger than the C*-algebra norm topology. We will show that this topology is complete. Let $\{f_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $C^\infty(\mathbb{T}, A^\infty)$. Then the s-th derivatives $f_m^{(s)}$ converge uniformly to some continuous function $g_s : \mathbb{T} \to A^\infty$. Notice that $f_m^{(s)}(e^{iw}) - f_m^{(s)}(e^{iv}) = \int_v^w f_m^{(s+1)}(e^{it})\, dt$. Taking limits we get $g_s(e^{iw}) - g_s(e^{iv}) = \int_v^w g_{s+1}(e^{it})\, dt$. Consequently, $g_s' = g_{s+1}$. Thus $g_0 \in C^\infty(\mathbb{T}, A^\infty)$. It follows easily that $f_m \to g_0$ in $C^\infty(\mathbb{T}, A^\infty)$ as $m \to \infty$. So $C^\infty(\mathbb{T}, A^\infty)$ is complete. Using the identity $a_1^{-1} - a_2^{-1} = a_1^{-1}(a_2 - a_1)a_2^{-1}$ it is easy to see that for any $f \in C^\infty(\mathbb{T}, A^\infty)$, if $f(t)$ is invertible in A^∞ for every $t \in \mathbb{T}$, then $f^{-1} \in C^\infty(\mathbb{T}, A^\infty)$. By [37, Lemma 1.2], $f(t)$ is invertible in A^∞ if and only if it is invertible in A. Therefore, for any $f \in C^\infty(\mathbb{T}, A^\infty)$, if it is invertible in $C(\mathbb{T}, A)$

then it is invertible in $C^\infty(\mathbb{T}, A^\infty)$. By [37, Lemma 1.2], $C^\infty(\mathbb{T}, A^\infty)$ is closed under the holomorphic functional calculus.

Let ${}_{B^\infty}E_{A^\infty}$ be a Morita equivalence bimodule. Identify E with $p(kA^\infty)$ for some projection $p \in M_k(A^\infty)$. Define $P \in C^\infty(\mathbb{T}, M_k(A^\infty)) = M_k(C^\infty(\mathbb{T}, A^\infty))$ to be the constant function with value p everywhere. Then

$$P^2 = P, \quad P(kC^\infty(\mathbb{T}, A^\infty)) = C^\infty(\mathbb{T}, E),$$

and

$$\begin{aligned} \mathrm{End}(C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T}, A^\infty)}) &= PM_k(C^\infty(\mathbb{T}, A^\infty))P \\ &= C^\infty(\mathbb{T}, pM_k(A^\infty)p) = C^\infty(\mathbb{T}, B^\infty). \end{aligned}$$

Since ${}_{B^\infty}E_{A^\infty}$ is a Morita equivalence bimodule, the right module E_{A^∞} is a generator, which means that $A^\infty_{A^\infty}$ is a direct summand of rE_{A^∞} for some $r \in \mathbb{N}$. Equivalently, there exist $\phi_j \in \mathrm{Hom}(E_{A^\infty}, A^\infty)$ and $u_j \in E$ for $1 \le j \le r$ such that $\sum_{1 \le j \le r} \phi_j(u_j) = 1_{A^\infty}$. Denote by $\Phi_j : C(\mathbb{T}, E) \to C(\mathbb{T}, A^\infty)$ the map consisting of ϕ_j acting in fibres. Clearly, $\Phi_j(C^\infty(\mathbb{T}, E)) \subseteq C^\infty(\mathbb{T}, A^\infty)$. Let $\mathcal{U}_j \in C^\infty(\mathbb{T}, E)$ denote the constant function with value u_j everywhere. Then $\sum_{1 \le j \le r} \Phi_j(\mathcal{U}_j) = 1_{C^\infty(\mathbb{T}, A^\infty)}$. Therefore the right module $C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T}, A^\infty)}$ is a generator. Similarly,

$$\mathrm{End}({}_{C^\infty(\mathbb{T}, B^\infty)}C^\infty(\mathbb{T}, E)) = C^\infty(\mathbb{T}, A^\infty),$$

and ${}_{C^\infty(\mathbb{T}, B^\infty)}C^\infty(\mathbb{T}, E)$ is a finitely generated projective module and a generator. Hence ${}_{C^\infty(\mathbb{T}, B^\infty)}C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T}, A^\infty)}$ is a Morita equivalence bimodule. $\square$

Proposition 5.7. *If ${}_{B^\infty}E_{A^\infty}$ is a Morita equivalence bimodule, then there are natural group isomorphisms*

$$\mathrm{K}_0(B) \oplus \mathrm{K}_1(B) \to \mathrm{K}_0(C^\infty(\mathbb{T}, B^\infty)) \to \mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty)) \to \mathrm{K}_0(A) \oplus \mathrm{K}_1(A).$$

Proof. By Bott periodicity we have a natural isomorphism

$$\mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \to \mathrm{K}_0(C(\mathbb{T}, A)).$$

By Lemma 5.6 the algebra $C^\infty(\mathbb{T}, A^\infty)$ is closed under the holomorphic functional calculus, so we have a natural isomorphism

$$\mathrm{K}_0(C(\mathbb{T}, A)) \to \mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty)).$$

Finally, the Morita equivalence bimodule ${}_{C^\infty(\mathbb{T}, B^\infty)}C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T}, A^\infty)}$ gives us a natural isomorphism

$$\mathrm{K}_0(C^\infty(\mathbb{T}, B^\infty)) \to \mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty)).$$

$\square$

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Consider the Fock space $\mathcal{F}^* = \Lambda((L^{\mathbb{C}})^*)$. Then one can identify $\mathrm{K}_0(A_\theta)$ and $\mathrm{K}_1(A_\theta)$ with $\Lambda^{\mathrm{even}}(\mathbb{Z}^n)$ and $\Lambda^{\mathrm{odd}}(\mathbb{Z}^n)$ respectively [16, Theorem 2.2]. By Proposition 5.7 we have a group isomorphism $\mathrm{K}_0(A_{\theta'}) \oplus \mathrm{K}_1(A_{\theta'}) \to \mathrm{K}_0(A_\theta) \oplus \mathrm{K}_1(A_\theta)$, which we shall think of as $\psi : \Lambda(\mathbb{Z}^n) \to \Lambda(\mathbb{Z}^n)$. Of course $\psi(\Lambda^{\mathrm{even}}(\mathbb{Z}^n)) = \Lambda^{\mathrm{even}}(\mathbb{Z}^n)$. Notice that L^* acts on $\mathcal{F}^*$ via multiplication. Also L acts on $\mathcal{F}^*$ via contraction. Let $a^1, \cdots, a^n$ and $b_1, \cdots, b_n$ denote the standard bases of L^* and L respectively. Denote $(a^1, \cdots, a^n)$ and $(b_1, \cdots, b_n)$ by $\vec{a}$ and $\vec{b}$ respectively. Denote by $\mathscr{A}$ the matrix of φ with respect to $\vec{b}$. Denote by Φ the $n \times n$ matrix $\frac{1}{2\pi i}([\nabla_{b_j}, \nabla_{b_k}])$. Using the Chern character which was defined in [8] and calculated for noncommutative tori in [16], Schwarz showed that in the case of complete Morita equivalence (not assumed here) the matrix

$$
\begin{aligned}
g &= \begin{pmatrix} S & R \\ N & M \end{pmatrix} \\
&:= \begin{pmatrix} \mathscr{A}^{-1} + \theta\Phi\mathscr{A}^{-1} & -\mathscr{A}^{-1}\theta' - \theta\Phi\mathscr{A}^{-1}\theta' + \theta\mathscr{A}^t \\ \Phi\mathscr{A}^{-1} & -\Phi\mathscr{A}^{-1}\theta' + \mathscr{A}^t \end{pmatrix}
\end{aligned} \tag{6}
$$

is in $\mathrm{SO}(n, n|\mathbb{R})$ and there is a linear operator V on $\mathcal{F}^*$ extending $\psi|_{\Lambda^{\mathrm{even}}(\mathbb{Z}^n)}$ such that

$$V(\vec{b}, \vec{a})V^{-1} = (\vec{b}, \vec{a})g. \tag{7}$$

Our equations (6) and (7) are exactly the equations (49), (50) and (53) of [36], in slightly different form. From the equation (6) above, Schwarz deduced

$$\theta = (S\theta' + R)(N\theta' + M)^{-1}, \tag{8}$$

which is our desired conclusion, except for the assertion that the matrix g belongs to $M_{2n}(\mathbb{Z})$ (and hence to $\mathrm{SO}(n, n|\,\mathbb{Z})$). Note that although in the definition of the Chern character in [8] Connes required the connections to be Hermitian, all the arguments there hold for arbitrary connections. Using Lemma 5.5 one checks that Schwarz's argument to get (7) and (8) still works in our situation (in which neither the connection nor the Morita equivalence are Hermitian) except that now we can only say that V acts on $\mathcal{F}^*$ and g is in $\mathrm{SO}(n, n|\,\mathbb{C})$; in other words, g might not be in $M_{2n}(\mathbb{R})$ a priori. In the complete Morita equivalence case, referring to the fact that V maps $\Lambda^{\mathrm{even}}(\mathbb{Z}^n)$ into itself and satisfies (7) with $g \in \mathrm{SO}(n, n|\,\mathbb{R})$, Schwarz concluded that $g \in M_{2n}(\mathbb{Z})$ for the case $n > 2$ (this is not true for $n = 2$), so that $g \in \mathrm{SO}(n, n|\,\mathbb{Z})$, as desired. We have not been able to understand this part of the argument, and so we shall follow another route: we assert that actually V extends all of ψ and hence maps all of $\Lambda(\mathbb{Z}^n)$ onto itself (not just $\Lambda^{\mathrm{even}}(\mathbb{Z}^n)$).

This follows from applying Schwarz's argument to the Morita equivalence bimodule ${}_{C^\infty(\mathbb{T},A^\infty_{\theta'})}C^\infty(\mathbb{T},E)_{C^\infty(\mathbb{T},A^\infty_\theta)}$ (combining even and odd degrees) of Lemma 5.6 instead of to ${}_{A^\infty_{\theta'}}E_{A^\infty_\theta}$. For the convenience of the reader, we sketch the argument here.

We recall the definition of the Chern character first. Consider the trivial Lie algebra $L^{\mathbb{C}} \oplus \mathbb{C}$. It acts on $C^\infty(\mathbb{T}, A^\infty_{\theta'})$ as derivations δ by extending the action of $L^{\mathbb{C}}$ on $A^\infty_{\theta'}$, with $L^{\mathbb{C}}$ acting on $C^\infty(\mathbb{T})$ trivially, and the action of the canonical unit vector of $\mathbb{C}$ being the differentiation with respect to the anti-clockwise unit vector field on $\mathbb{T}$. Tensoring $\tau_{\theta'}$ with the Lebesgue integral on $C^\infty(\mathbb{T})$, we obtain an $L^{\mathbb{C}} \oplus \mathbb{C}$-invariant trace on $C^\infty(\mathbb{T}, A^\infty_{\theta'})$, which we still denote by $\tau_{\theta'}$. For a finitely generated projective right $C^\infty(\mathbb{T}, A^\infty_{\theta'})$-module $F'_{C^\infty(\mathbb{T},A^\infty_{\theta'})}$, the Chern character $\mathrm{ch}F'$ is defined by

$$\begin{aligned}
(9) \qquad \mathrm{ch}F' &= \tau_{\theta'}(e^{\Omega'/2\pi i}) = \sum_{j=0} \frac{1}{j!}\tau_{\theta'}((\Omega')^j) \cdot \frac{1}{(2\pi i)^j} \\
&\in \Lambda^{\mathrm{even}}((L^{\mathbb{C}} \oplus \mathbb{C})^*) = \Lambda((L^{\mathbb{C}})^*),
\end{aligned}$$

where $\Omega' \in \mathrm{End}(F'_{C^\infty(\mathbb{T},A^\infty_{\theta'})}) \otimes \Lambda^2((L^{\mathbb{C}} \oplus \mathbb{C})^*)$ is the curvature of an arbitrary connection on F' (with respect to the action of $L^{\mathbb{C}} \oplus \mathbb{C}$ on $C^\infty(\mathbb{T}, A^\infty_{\theta'})$) and we have extended $\tau_{\theta'}$ to $\mathrm{End}(F'_{C^\infty(\mathbb{T},A^\infty_{\theta'})})$ as in the paragraph before Lemma 5.4. This determines the Chern character $\mathrm{ch} : \Lambda(\mathbb{Z}^n) = \mathrm{K}_0(A_{\theta'}) \oplus \mathrm{K}_1(A_{\theta'}) = \mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty_{\theta'})) \to \Lambda((L^{\mathbb{C}})^*)$ as a group homomorphism. The Chern character $\mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty_\theta)) \to \Lambda((L^{\mathbb{C}})^*)$ is defined similarly.

Now let us consider the finitely generated right $C^\infty(\mathbb{T}, A^\infty_\theta)$-module $F := F' \otimes_{C^\infty(\mathbb{T},A^\infty_{\theta'})} E$. To get a connection of $F_{C^\infty(\mathbb{T},A^\infty_\theta)}$ from that of $F'_{C^\infty(\mathbb{T},A^\infty_{\theta'})}$, let us extend φ to $L^{\mathbb{C}} \oplus \mathbb{C} \to L^{\mathbb{C}} \oplus \mathbb{C}$ as simply being the identity map on $\mathbb{C}$, and also extend ∇ to $L^{\mathbb{C}} \oplus \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(C^\infty(\mathbb{T}, E))$ in such a way that the action of $L^{\mathbb{C}}$ of $C^\infty(\mathbb{T}, E)$ is fibrewise the original ∇ and the action of the canonical unit vector of $\mathbb{C}$ on $C^\infty(\mathbb{T}, E)$ is the differentiation with respect to the anti-clockwise unit vector field on $\mathbb{T}$. Then ∇_X is a connection for both $(C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T},A^\infty_\theta)}, \delta_X)$ and $({}_{C^\infty(\mathbb{T},A^\infty_{\theta'})}C^\infty(\mathbb{T}, E), \delta_{\varphi(X)})$ for every $X \in L^{\mathbb{C}} \oplus \mathbb{C}$, and furthermore ∇ has constant curvature $\pi i \sum_{j,k} \Phi_{jk} a^j \wedge a^k$ in $\mathrm{End}(C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T},A^\infty_\theta)}) \otimes \Lambda^2((L^{\mathbb{C}} \oplus \mathbb{C})^*)$.

For any connection $\nabla'_{\varphi(X)}$ of $(F'_{C^\infty(\mathbb{T},A^\infty_{\theta'})}, \delta_{\varphi(X)})$, one checks readily that $\nabla'_{\varphi(X)} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X$ is a connection of $(F_{C^\infty(\mathbb{T},A^\infty_\theta)}, \delta_X)$. If we choose the connections of $(F_{C^\infty(\mathbb{T},A^\infty_\theta)}, \delta)$ in this way, then the curvature

Ω is calculated by

$$\Omega = \varphi^*(\Omega') + \pi i \sum_{j,k} \Phi_{jk} a^j \wedge a^k, \tag{10}$$

where φ^* denotes the linear isomorphism $\Lambda((L^{\mathbb{C}} \oplus \mathbb{C})^*) \to \Lambda((L^{\mathbb{C}} \oplus \mathbb{C})^*)$ induced by φ and we identify $\mathrm{End}(F'_{C^\infty(\mathbb{T}, A^\infty_{\theta'})})$ with a subalgebra of $\mathrm{End}(F_{C^\infty(\mathbb{T}, A^\infty_{\theta})})$ via identifying $T \in \mathrm{End}(F'_{C^\infty(\mathbb{T}, A^\infty_{\theta'})})$ with $T \otimes \mathrm{id}$. Let λ be the constant in Lemma 5.5. Since $\tau_{\theta'}$ (resp. τ_θ) was extended to $C^\infty(\mathbb{T}, A^\infty_{\theta'})$ (resp. $C^\infty(\mathbb{T}, A^\infty_{\theta})$) via tensoring with the Lebesgue integral on $\mathbb{T}$, the conclusion of Lemma 5.5 actually holds on $C^\infty(\mathbb{T}, A^\infty_{\theta'}) = \mathrm{End}(C^\infty(\mathbb{T}, E)_{C^\infty(\mathbb{T}, A^\infty_\theta)})$. Thus $\tau_\theta(a) = \lambda \tau_{\theta'}(a)$ for all $a \in \mathrm{End}(F'_{C^\infty(\mathbb{T}, A^\infty_{\theta'})})$. Therefore, we have

$$\begin{aligned} \mathrm{ch}F &= \tau(e^{\Omega/2\pi i}) = \tau(e^{\frac{1}{2}\sum_{j,k} \Phi_{jk} a^j \wedge a^k} \varphi(e^{\Omega'/2\pi i})) \\ &= \lambda e^{\frac{1}{2}\sum_{j,k} \Phi_{jk} a^j a^k} \varphi^*(\mathrm{ch}F'). \end{aligned} \tag{11}$$

Denote by $\mu(F')$ the equivalence class of F' in $\mathrm{K}_0(C^\infty(\mathbb{T}, A^\infty_{\theta'})) = \Lambda(\mathbb{Z}^n)$. By Theorem 4.2 of [16] (the sign there must be reversed; see page 137 of [15]), we have

$$\mathrm{ch}F' = e^{-\frac{1}{2}\sum_{j,k} \theta'_{jk} b_j b_k} \mu(F'). \tag{12}$$

Combining the equations (9), (11), and (12) together, we see that the map ψ is the restriction of the linear operator $V := V_1 V_2 V_3 V_4 \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}^*)$ on $\Lambda(\mathbb{Z}^n)$, where

$$\begin{aligned} V_1 f &= e^{\frac{1}{2}\sum_{j,k} \theta_{jk} b_j b_k} f, \\ V_2 f &= \lambda e^{\frac{1}{2}\sum_{j,k} \Phi_{jk} a^j a^k} f, \\ V_3 f &= \varphi^*(f), \\ V_4 f &= e^{-\frac{1}{2}\sum_{j,k} \theta'_{jk} b_j b_k} f, \end{aligned}$$

for $f \in \mathcal{F}^*$.

Each linear operator V_k is a *linear canonical transformation* in the sense that

$$V_k(\vec{b}, \vec{a}) V_k^{-1} = (\vec{b}, \vec{a}) g_k$$

for some $g_k \in M_{2n}(\mathbb{C})$. In fact, a simple calculation yields

$$g_1 = \begin{pmatrix} I & \theta \\ 0 & I \end{pmatrix}, \quad g_2 = \begin{pmatrix} I & 0 \\ \Phi & I \end{pmatrix}, \quad g_3 = \begin{pmatrix} \mathscr{A}^{-1} & 0 \\ 0 & \mathscr{A}^t \end{pmatrix}, \quad g_4 = \begin{pmatrix} I & -\theta' \\ 0 & I \end{pmatrix}.$$

Set $g = g_1 g_2 g_3 g_4$. Then the equations (6) and (7) hold.

Notice that each g_k belongs to $\mathrm{SO}(n, n|\,\mathbb{C})$, i.e. it satisfies the equations (1) and has determinant 1. Hence so also does g.

Since V extends the automorphism ψ of $\Lambda(\mathbb{Z}^n)$, g is easily seen to be in $M_{2n}(\mathbb{Z})$ by applying (7) to the canonical vectors 1 and a^j, $1 \le j \le n$, in the Fock space $\mathcal{F}^*$. Therefore, g belongs to $\mathrm{SO}(n,n|\,\mathbb{Z})$. $\square$

Remark 5.8. Let us indicate briefly how the proof of the part (2) of Theorem 1.1 leads to a new proof of the main result of [13], namely, if θ' and θ are in $\mathcal{T}_n' \cap \mathcal{T}_n^\flat$ and the algebras $A_{\theta'}^\infty$ and A_θ^∞ are isomorphic, then the bicharacters $\rho_{\theta'}$ and ρ_θ of $\mathbb{Z}^n$ are isomorphic. On using the given isomorphism $A_{\theta'}^\infty \to A_\theta^\infty$, the vector space $E = A_\theta^\infty$ becomes a Morita equivalence bimodule for $A_{\theta'}^\infty$ and A_θ^∞ in a natural way. The Chern character [8] of the free module $E_{A_\theta^\infty} (= {A_\theta^\infty}_{A_\theta^\infty})$ is 1. Therefore, the constant curvature connection of Theorem 5.3 has in fact curvature zero. It follows that we have $N = 0$ in (6). Since $g \in \mathrm{SO}(n,n|\,\mathbb{Z})$, the block entry S must belong to $\mathrm{GL}(n,\mathbb{Z})$. A simple calculation (which is trivial in the case also $R = 0$) shows that the bicharacters associated to θ' and $\theta = g\theta'$ are isomorphic (by means of the automorphism S of $\mathbb{Z}^n$).

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Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4
E-mail address: elliott@math.toronto.edu

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260, USA
E-mail address: hfli@math.buffalo.edu