Local entropy theory and applications

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Abstract

This paper is a survey about recent developments in the local entropy theory for topological dynamical systems and continuous group actions, with particular emphasis on the connections with other areas of dynamical systems and mathematics.

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1 Introduction

A measure-preserving transformation has completely positive entropy if every non-trivial factor has positive entropy. This is equivalent to saying that every non-trivial finite partition of the phase space has positive entropy. Bernoulli transformations are the prototypes for completely positive entropy, and it was an open question for some time whether these were the only examples. Nowadays there exist plenty of examples with completely positive entropy that are not Bernoulli.

Blanchard introduced two notions for topological dynamical systems that emulated the behaviour of completely positive entropy [7] in a topological context. A topological dynamical system has *completely positive entropy*, if every non-trivial factor has positive topological entropy, and has *uniform positive entropy* if the topological entropy of every non-dense finite open cover is positive. In contrast with measurable

systems, these properties turned out to be different. In order to comprehend them better, Blanchard introduced *entropy pairs* in [8]. Entropy pairs allow us to understand "where" the entropy lies in the system. The answer where, is not in the phase space, but in the product of the phase space. In vague terms, a pair of points is an entropy pair if every finite open cover that separates them has positive topological entropy.

It turns out that this approach of studying topological dynamics, nowadays known as local entropy theory, is not only useful for understanding positive entropy and completely positive entropy. With time it was discovered that this perspective is related to other dynamical objects like Li-Yorke chaos, Ellis semigroup, maximal equicontinuous factor, sequence entropy, homoclinic points, mean dimension, spectrum, nilsystems, and cardinality of ergodic measures. Furthermore, it has paved the way for novel interactions between dynamical systems and other areas of mathematics like operator algebras, combinatorics, group theory, descriptive set theory, and continuum theory.

This paper is not the first survey on local entropy theory (see [43]), nonetheless we consider that many important contributions have appeared since the publication of that survey, in particular many applications and connections with other fields. All the results presented here, except Theorem 4.33 and Theorem 4.35, appear in other papers or essentially follow from known results.

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2 IE-tuples for group actions

In this section we provide an introduction to the basic results of local entropy theory and combinatorial independence. We will provide the definition of entropy in different contexts but we will not go into details about the basic results of entropy theory. For a more complete exposition of this topic we refer the reader to [31, 84, 93] for entropy theory of \mathbb{Z} -actions, and to [65, 79] for entropy theory of actions of amenable and sofic groups.

Throughout this paper \mathbb{N} denotes the set of positive integers and G denotes a countably infinite group with identity e_G .

A (left) **action** of the group G on X is represented by $G \curvearrowright X$. In this paper we shall always assume that X is a compact metrizable space and that G acts by homeomorphisms. We denote by ρ a compatible metric on X.

Let $G \curvearrowright X$ and $G \curvearrowright Y$ be two actions. A function $\pi: X \to Y$ is *G*-equivariant if $g\pi(x) = \pi(gx)$ for every $g \in G$ and $x \in X$, in this situation we write $\pi: G \curvearrowright X \to G \curvearrowright Y$. If a *G*-equivariant function, $\pi: X \to Y$, is continuous and surjective, we say $G \curvearrowright Y$ is a factor of $G \curvearrowright X$ and π is a factor map. The following proposition is standard (e.g. see [28, Appendix E.11-3.]).

Proposition 2.1. If $\pi: G \curvearrowright X \to G \curvearrowright Y$ is a factor map then

$$R_{\pi} = \{(x, x') : \pi(x) = \pi(x')\} \subset X \times X$$

is a closed G-invariant equivalence relation. Conversely, if $Q \subset X \times X$ is a closed G-invariant equivalence relation then $\pi: G \curvearrowright X \to G \curvearrowright X/Q$ is a factor map where $G \curvearrowright X/Q$ is the induced action $gy = \pi(g\pi^{-1}(y))$.

2.1 Amenable groups

If K is a non-empty finite subset of G, we write $K \Subset G$.

Let $\delta > 0$ and $K \in G$. A set $F \in G$ is said to be (K, δ) -invariant if $|KF\Delta F| \leq \delta |F|$. A sequence of non-empty finite subsets, $\{F_n\}_{n\in\mathbb{N}}$, of G is said to be **Følner** if it is eventually (K, δ) -invariant for every $K \in G$ and $\delta > 0$. A countable group G is **amenable** if it admits a Følner sequence. Abelian groups and finitely generated groups of subexponential growth are always amenable. Given two finite open covers \mathcal{U}, \mathcal{V} of X, we define their **join** by

$$\mathcal{U} \lor \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \}.$$

Let $g \in G$. We define $g\mathcal{U} = \{g\mathcal{U} : U \in \mathcal{U}\}$ and $N(\mathcal{U})$ as the smallest cardinality of a subcover of \mathcal{U} . If $F \Subset G$, denote by \mathcal{U}^F the join

$$\mathcal{U}^F = \bigvee_{g \in F} g^{-1} \mathcal{U}.$$

Let G be an amenable group, $G \curvearrowright X$ an action, \mathcal{U} a finite open cover of X and $\{F_n\}_{n \in \mathbb{N}}$ a Følner sequence for G. We define the **topological entropy of** $G \curvearrowright X$ with respect to \mathcal{U} as

$$h_{\text{top}}(G \curvearrowright X, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}) \in [0, \infty).$$

The function $F \mapsto \log N(\mathcal{U}^F)$ is subadditive and thus the limit exists and does not depend on the choice of the Følner sequence (see for instance [65, page 220]). The **topological entropy** of $G \curvearrowright X$ is defined as

$$h_{top}(G \curvearrowright X) = \sup_{\mathcal{U}} h_{top}(G \curvearrowright X, \mathcal{U}) \in [0, \infty],$$

where \mathcal{U} ranges over finite open covers of X.

The concept of entropy pairs was introduced by Blanchard in [8]. Entropy tuples are a natural generalization that has been studied in several papers.

A tuple refers to an element of a product space Z^k for some $k \ge 1$. When k = 2, we simply call them pairs.

For each $k \in \mathbb{N}$, we define

$$\Delta_k(X) = \{ (x_1, \dots, x_k) \in X^k : x_1 = x_2 = \dots = x_k \}, \text{ and} \\ \Delta_k^2(X) = \{ (x_1, \dots, x_k) \in X^k : \text{there exist } n \neq m \in \{1, \dots, k\} \text{ such that } x_n = x_m \}$$

Definition 2.2. Let G be an amenable group, $G \cap X$ an action and $k \ge 2$. We call $\mathbf{x} = (x_1, ..., x_k) \in X^k \setminus \Delta_k(X)$ an entropy tuple if whenever $K_1, ..., K_l$ (with $l \ge 2$) are closed pairwise disjoint neighborhoods of some elements in $\{x_1, ..., x_k\}$, then

$$h_{\rm top}(G \curvearrowright X, \mathcal{U}) > 0,$$

where $\mathcal{U} = \{K_1^c, \ldots, K_l^c\}.$

Let $G \cap X$ be an action and $\mathbf{A} = (A_1, \ldots, A_k)$ a tuple of subsets of X. We say $J \subset G$ is an **independence set for A** if for every non-empty finite $I \subset J$ and every $\phi: I \to \{1, \ldots, k\}$ we have

$$\bigcap_{s\in I} s^{-1} A_{\phi(s)} \neq \emptyset$$

We define the **independence density of A** to be the largest $q \ge 0$ such that every set $F \Subset G$ has a subset of cardinality at least q |F| which is an independence set for **A**.

Definition 2.3. Let G be an amenable group, $G \cap X$ an action and $k \ge 1$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an independence entropy tuple (IE-tuple) if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} we have that (U_1, \ldots, U_k) has positive independence density. We denote the set of IE-tuples of length k by $IE_k(X, G)$.

Remark 2.4. The first definition of IE-tuples of an action of an amenable group (stated in [60]) is not exactly the previous definition but it is an equivalent notion [63, Theorem 4.8].

The following result is proved in [60, Theorem 3.16].

Theorem 2.5 (Kerr and Li [60]). Let G be an amenable group, $G \sim X$ an action, $k \ge 2$ and $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$. Then (x_1, \ldots, x_k) is an entropy tuple if and only if it is an IE-tuple.

IE-pairs are witnesses of topological entropy [60, Proposition 3.9] [63, Theorem 4.8, Theorem 8.1].

Proposition 2.6 (Blanchard [8], Kerr and Li [60, 63]). Let G be an amenable group and $G \curvearrowright X$ an action. The following statements are equivalent.

- 1. $h_{top}(G \curvearrowright X) > 0.$
- 2. $\operatorname{IE}_2(X,G) \setminus \Delta_2(X) \neq \emptyset$.
- 3. $\operatorname{IE}_k(X,G) \setminus \Delta_k^2(X) \neq \emptyset$ for every $k \geq 2$.

The first connection between combinatorial independence and entropy of actions of \mathbb{Z} was explored by Weiss in [94, Chapter 8]. Following this path, Huang and Ye [55] proved a result that is equivalent to Theorem 2.5 for actions of \mathbb{Z} . The proof comes from combinatorial arguments using the Sauer-Perles-Shelah Lemma and measure-theoretic tools. In order to generalize these results to actions of amenable groups a different approach was taken in [60].

Let Z be a non-empty finite set, $k \geq 2$, and \mathcal{U} the cover of

$$\{0, 1, \dots, k\}^Z = \prod_{z \in Z} \{0, 1, \dots, k\}$$

consisting of subsets of the form $\prod_{z \in Z} \{i_z\}^c$, where $1 \leq i_z \leq k$ for each $z \in Z$. For $S \subset \{0, 1, \ldots, k\}^Z$ we write $N_S(\mathcal{U})$ to denote the minimal number of sets in \mathcal{U} one needs to cover S.

The following result appeared in [60, Lemma 3.3] and is a key combinatorial result in the proof of Proposition 2.6 and Theorem 2.5.

Lemma 2.7 (Kerr and Li [60]). Let $k \ge 2$ and b > 0. There exists a constant c > 0, depending only on k and b, such that for every non-empty finite set Z and $S \subset \{0, 1, \ldots, k\}^Z$ with $N_S(\mathcal{U}) \ge k^{b|Z|}$, there exists $W \subset Z$ such that $|W| \ge c|Z|$ and $S|_W \supset \{1, \ldots, k\}^W$.

Here are some of the basic properties that IE-tuples satisfy (for the proofs see [60, Proposition 3.9 and Theorem 3.15]).

Theorem 2.8 (Blanchard [8], Glasner [39], Kerr and Li [60]). Let G be an amenable group, $G \cap X$, $G \cap Y$ actions, and $k \in \mathbb{N}$. We have that

- 1. $IE_k(X,G)$ is a closed subset of X^k that is invariant under the product action.
- 2. $\operatorname{IE}_k(X \times Y, G) = \operatorname{IE}_k(X, G) \times \operatorname{IE}_k(Y, G).$
- 3. If $\pi: G \curvearrowright X \to G \curvearrowright Y$ is a factor map, then

$$(\pi \times \cdots \times \pi)(\mathsf{IE}_k(X,G)) = \mathsf{IE}_k(Y,G).$$

Let $k \in \mathbb{N}$. Denote by $\mathcal{F}^k(X)$ the space of all pairwise disjoint tuples of closed subsets of X with length k.

Question 2.9. Let G be an amenable group and $G \sim X$ an action. Can $h_{top}(G \sim X)$ be computed with the use of combinatorial independence? In particular is it true that

$$h_{top}(G \cap X) = \sup_{k \in \mathbb{N}} \sup_{\mathbf{A} \in \mathcal{F}^k(X)} (independence \ density \ of \ \mathbf{A}) \log k?$$

2.2 Sofic groups

For $n \in \mathbb{N}$ we write $[n] = \{1, ..., n\}$, and $\operatorname{Sym}(n)$ for the group of permutations of [n]. A group G is **sofic** if there exist a sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers which goes to infinity and a sequence $\Sigma = \{\sigma_i \colon G \to \operatorname{Sym}(n_i)\}_{i\in\mathbb{N}}$ that satisfies

$$\lim_{i \to \infty} \frac{1}{n_i} \left| \{ v \in [n_i] : \sigma_i(st)v = \sigma_i(s)\sigma_i(t)v \} \right| = 1 \text{ for every } s, t \in G, \text{ and} \\ \lim_{i \to \infty} \frac{1}{n_i} \left| \{ v \in [n_i] : \sigma_i(s)v \neq \sigma_i(t)v \} \right| = 1 \text{ for every } s \neq t \in G.$$

In this case we say Σ is a **sofic approximation sequence** for *G*. Residually finite groups and amenable groups are all sofic. It is not known if there exists a group that is not sofic. We refer the reader to [18, 19, 83] for general information about sofic groups.

The following notion of topological entropy for sofic group actions was introduced by Kerr and Li [62] following the breakthrough of Bowen for measure-preserving actions [14].

Let $G \curvearrowright X$ be an action, $F \Subset G$, $\delta > 0$, $n \in \mathbb{N}$, and $\sigma \colon G \to \operatorname{Sym}(n)$.

Definition 2.10. We define $Map(\rho, F, \delta, \sigma)$ as the set of all maps $\varphi \colon [n] \to X$ such that

$$\left(\frac{1}{n}\sum_{v=1}^{n}\rho(\varphi(\sigma(s)v),s\varphi(v))^{2}\right)^{1/2} \leq \delta \text{ for every } s \in F$$

We write $N_{\varepsilon}(Y, \rho_{\infty})$ for the maximum cardinality of a subset Y' of $Y \subset X^{[n]}$ such that whenever φ_1, φ_2 are distinct in Y' then

$$\max_{v \in [n]} \rho(\varphi_1(v), \varphi_2(v)) \ge \varepsilon.$$

Let G be a sofic group and $\Sigma = \{\sigma_i : G \to \operatorname{Sym}(n_i)\}_{i \in \mathbb{N}}$ a sofic approximation sequence for G. The **topological sofic entropy of** $G \curvearrowright X$ (with respect to Σ) is defined by

$$h_{\text{top}}^{\Sigma}(G \cap X) = \sup_{\varepsilon > 0} \inf_{F \Subset G} \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{n_i} \log N_{\varepsilon}(\text{Map}(\rho, F, \delta, \sigma_i), \rho_{\infty}) \in \{-\infty\} \cup [0, \infty].$$

The value of $h_{\text{top}}^{\Sigma}(G \curvearrowright X)$ does not depend on the choice of ρ [65, Proposition 10.25], and hence it is invariant under conjugacy.

Remark 2.11. If G is an amenable group, then the topological entropy of any action $G \cap X$ coincides with the sofic entropy with respect to any sofic approximation sequence [64, Theorem 5.3] [65, Theorem 10.37].

Let G be a sofic group, $G \curvearrowright X$ an action and $\mathbf{A} = (A_1, \ldots, A_k)$ a tuple of subsets of X. Given $F \Subset G$, $\delta > 0$, $n \in \mathbb{N}$ and $\sigma: G \to \text{Sym}(n)$, we say $J \subset [n]$ is a $(\rho, F, \delta, \sigma)$ -independence set for A if for every $\omega: J \to [k]$ there exists $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ such that $\varphi(v) \in A_{\omega(v)}$ for every $v \in J$.

Definition 2.12. Let G be a sofic group, $G \curvearrowright X$ an action and $\Sigma = \{\sigma_i : G \to \text{Sym}(n_i)\}_{i \in \mathbb{N}}$ a sofic approximation sequence for G. We say that a tuple of subsets of X, $\mathbf{A} = (A_1, \ldots, A_k)$, has **positive upper** independence density over Σ , if there exists q > 0 such that for every $F \Subset G$ and $\delta > 0$, A has a $(\rho, F, \delta, \sigma_i)$ -independence set of cardinality at least qn_i for infinitely many i's.

We say $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is a sofic independence entropy tuple with respect to Σ (Σ -IEtuple) if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} the tuple $\mathbf{U} = (U_1, \ldots, U_k)$ has positive upper independence density over Σ . We denote the set of Σ -IE-tuples of length k by $\mathbf{IE}_k^{\Sigma}(X, G)$.

The previous property does not depend on the choice of the metric ρ , see [65, Lemma 10.24].

Remark 2.13. If G is an amenable group, the set of Σ -IE-tuples for any sofic approximation sequence Σ is the same as the set of IE-tuples [63, Theorem 4.8].

Remark 2.14. Zhang [96] defined entropy tuples with respect to a sofic approximation sequence Σ , using the topological sofic entropy with respect to a finite open cover, analogous to Definition 2.2. As it was noted in [63, Remark 4.4], a non-diagonal tuple is an entropy tuple with respect to Σ if and only if it is a Σ -IE-tuple.

For the proof of the next result see [63, Proposition 4.16 and Theorem 8.1].

Theorem 2.15 (Kerr and Li [63]). Let G be sofic, Σ a sofic approximation sequence for G, and $G \curvearrowright X$ an action. The following statements are equivalent.

- 1. $h_{top}^{\Sigma}(G \curvearrowright X) > 0.$
- 2. $\operatorname{IE}_2^{\Sigma}(X,G) \setminus \Delta_2(X) \neq \emptyset$.
- 3. $\operatorname{IE}_k^{\Sigma}(X,G) \setminus \Delta_k^2(X) \neq \emptyset$ for every $k \ge 2$.

The next result is obtained from [63, Proposition 4.16]. Note that in the second point we don't get equality as in Theorem 2.8. This comes from the fact that factors of actions with zero sofic entropy may have positive sofic entropy.

Theorem 2.16 (Kerr and Li [63]). Let G be a sofic group, Σ a sofic approximation sequence for G, $G \cap X$, $G \cap Y$ actions and $k \in \mathbb{N}$. We have that

- 1. $IE_k^{\Sigma}(X,G)$ is a closed subset of X^k that is invariant under the product action.
- 2. If $\pi: G \curvearrowright X \to G \curvearrowright Y$ is a factor map, then

$$(\pi \times \cdots \times \pi)(\operatorname{IE}_{k}^{\Sigma}(X,G)) \subset \operatorname{IE}_{k}^{\Sigma}(Y,G).$$

A product formula for Σ -IE-tuples is possible under extra assumptions on Σ [63, Theorem 5.2].

2.3 Countable groups

Let $G \curvearrowright X$ be an action and \mathcal{U} a finite open cover of X. We define the **naive topological entropy of** $G \curvearrowright X$ with respect to \mathcal{U} as

$$h_{\operatorname{top}}^{\operatorname{nv}}(G \frown X, \mathcal{U}) = \inf_{F \in G} \frac{1}{|F|} \log N(\mathcal{U}^F) \in [0, \infty).$$

The **naive topological entropy** of $G \curvearrowright X$ is defined as

$$h_{\rm top}^{\rm nv}(G \curvearrowright X) = \sup_{\mathcal{U}} h_{\rm top}^{\rm nv}(G \curvearrowright X, \mathcal{U}) \in [0, \infty],$$

where \mathcal{U} ranges over the set of finite open covers. The notion of naive topological entropy was introduced by Burton [17]. He showed that if G is not amenable then $h_{top}^{nv}(G \curvearrowright X)$ can only take the values in the set $\{0, \infty\}$.

Remark 2.17. If G is amenable, then the topological entropy of any action $G \curvearrowright X$ coincides with the naive topological entropy [32, Theorem 6.8].

The next definition uses notions from Section 2.1, and it is very similar to Definition 2.3, the only difference being that there is no condition on the group. For this reason we use the same notation.

Definition 2.18. Let $G \cap X$ be an action and $k \ge 1$ an integer. We say a tuple $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an **orbit independence entropy tuple (orbit IE-tuple)** if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , we have that (U_1, \ldots, U_k) has positive independence density. We denote the set of orbit IE-tuples of length k by $\mathbf{IE}_k(X, G)$.

Remark 2.19. If G is a sofic group and Σ a sofic approximation sequence for G, then every Σ -IE-tuple is an orbit IE-tuple [63, Proposition 4.6], but it is not known if there exist orbit IE-tuples that are not Σ -IE-tuples.

Positive naive entropy can be characterized with combinatorial independence but not at the level of orbit IE-pairs.

We now define a notion of multiple independence.

Definition 2.20. For any finite subset $\mathcal{A} \subset \mathcal{F}^2(X)$, we define the **independence density** of \mathcal{A} to be the largest $q \geq 0$ such that for every $F \in G$ there are some $J \subset F$ with $|J| \geq q|F|$ and some $\mathbf{A} \in \mathcal{A}$ so that J is an independence set for \mathbf{A} .

The proof of the following result appears in [69, Theorem 2.5].

Theorem 2.21 (Li and Rong [69]). Let $G \cap X$ be an action. We have that $h_{top}^{nv}(G \cap X) > 0$ if and only if there is a finite $\mathcal{A} \subset \mathcal{F}^2(X)$ with positive independence density. In particular, if $IE_2(X,G) \setminus \Delta_2(X) \neq \emptyset$, then $h_{top}^{nv}(G \cap X) > 0$.

There exists an action of the free group \mathbb{F}_2 with positive naive topological entropy and no non-trivial orbit IE-pairs [69, Proposition 5.2].

2.4 CPE and UPE

As mentioned in the introduction, one of the original motivations of studying entropy pairs was to obtain a better comprehension of uniform positive entropy and completely positive entropy.

We say a finite open cover is **non-dense** if every element of the cover is not dense.

Definition 2.22. Let $G \cap X$ be an action. We say that $G \cap X$ has naive uniform positive entropy (naive UPE) if for every non-dense open cover, $\mathcal{U} = \{U_1, U_2\}$, of X, we have

$$h_{\text{top}}^{\text{nv}}(G \curvearrowright X, \mathcal{U}) > 0.$$

If the group is amenable, we simply call this property UPE.

The following result was proved in [8] for actions of \mathbb{Z} , in [60] for actions of amenable groups (with the use of Lemma 2.7). The same proof works for actions of countable groups.

Proposition 2.23 (Blanchard [8], Kerr and Li [60]). Let $G \curvearrowright X$ be an action. We have $IE_2(X, G) = X^2$ if and only if $G \curvearrowright X$ has naive UPE.

Given a sofic approximation sequence for G, we say $G \curvearrowright X$ has Σ -**UPE** if $IE_2^{\Sigma}(X, G) = X^2$. While not used in the current literature, Σ -UPE is equivalent to having positive topological sofic entropy for every non-dense open cover consisting of two sets [63, Remark 4.4].

UPE behaves similarly when considering different forms of entropy, with CPE one has to be more careful because it deals with factors.

Definition 2.24. Let G be an amenable group and $G \curvearrowright X$ an action. We say that $G \curvearrowright X$ has completely positive entropy (CPE) if every non-trivial factor has positive topological entropy.

CPE can also be characterized using the **topological Pinsker factor** for actions of amenable groups, that is, the largest factor with zero topological entropy in the sense that every factor with zero topological entropy is also a factor of the topological Pinsker factor. An action of an amenable group has CPE if and only if the topological Pinsker factor is trivial. Using Proposition 2.6 and Theorem 2.8 (3) we obtain the following result.

Corollary 2.25 (Blanchard and Lacroix [12]). Let G be an amenable group and $G \cap X$ an action. The topological Pinsker factor of $G \cap X$ is obtained as the quotient of the smallest closed G-invariant equivalence relation on X containing the set of IE-pairs.

In particular, $G \curvearrowright X$ has CPE if and only if the smallest closed G-invariant equivalence relation on X containing the set of IE-pairs is X^2 .

As mentioned before, there exists an action with positive naive topological entropy and no non-trivial orbit IE-pairs [69, Proposition 5.2]. This implies that the previous corollary does not hold for actions of countable groups in the naive entropy context.

Given a sofic group G and Σ a sofic approximation sequence for G, we may also define the *topological* Σ -*Pinsker factor* for actions with $h_{top}^{\Sigma}(G \curvearrowright X) \geq 0$. The existence of these factors can be proved using [68, Lemma 2.13]. A topological Σ -Pinsker factor may still have factors with positive sofic topological entropy. If every non-trivial factor has positive sofic topological entropy with respect to Σ we say it has Σ -**CPE**. Using Theorem 2.15 and Theorem 2.16 (2), one can see that if the smallest closed G-invariant equivalence relation on X containing the set of Σ -IE-pairs is X^2 then $G \curvearrowright X$ has Σ -CPE. We do not know if the converse holds.

Question 2.26. Let G be a sofic group, Σ a sofic approximation sequence for G, and $G \curvearrowright X$ an action with Σ -CPE. Is X^2 the smallest closed G-invariant equivalence relation on X containing $IE_2^{\Sigma}(X,G)$?

3 Variations

Entropy pairs are incredibly flexible in the sense that several natural variations are indeed relevant and connect to other objects in dynamical systems like sequence entropy, Ellis semigroup, nilsystems, and mean dimension.

Some of these notions behave particularly well for minimal actions. We say an action $G \curvearrowright X$ is **minimal** if every *G*-orbit is dense.

3.1 Sequence entropy and IN-tuples

Let $G \cap X$ be an action. Following Goodman [44], for a sequence $S = \{s_n\}_{n \in \mathbb{N}}$ in G we define the topological sequence entropy of $G \cap X$ along S with respect to a finite open cover \mathcal{U} by

$$h_{\text{top}}(G \curvearrowright X, \mathcal{U}; S) = \limsup_{n \to \infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} s_{i}^{-1} \mathcal{U}\right) \in [0, \infty).$$

We define

$$h^*_{top}(G \curvearrowright X) = \sup_{\mathcal{U},S} h_{top}(G \curvearrowright X, \mathcal{U}; S),$$

where the supremum is taken over all finite open covers \mathcal{U} and all sequences S in G. Huang, Li, Shao, and Ye [52] and Huang, Maass and Ye [54] emulated the entropy pair/tuple definition for sequence entropy.

Definition 3.1. Let $G \curvearrowright X$ be an action and $k \ge 2$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$ is a sequence entropy tuple if whenever K_1, \ldots, K_l (with $l \ge 2$) are closed pairwise disjoint neighborhoods of some elements of $\{x_1, \ldots, x_k\}$, the open cover $\{K_1^c, \ldots, K_l^c\}$ has positive topological sequence entropy with respect to some sequence in G.

Definition 3.2. Let $G \curvearrowright X$ be an action and $k \ge 1$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an *IN-tuple* if for any product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , we have that (U_1, \ldots, U_k) has arbitrarily large finite independence sets. We denote the set of *IN*-tuples of length k by $\mathbf{IN}_k(X, G)$.

For the proof of the next result see [60, Theorem 5.9].

Theorem 3.3 (Kerr and Li [60]). Let $G \cap X$ be an action, $k \ge 2$ and $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$. Then (x_1, \ldots, x_k) is a sequence entropy tuple if and only if it is an IN-tuple.

We say that (X, G) is **null** if $h_{top}^*(G \cap X) = 0$. For a proof of the following result see [60, Proposition 5.4].

Proposition 3.4 (Huang, Li, Shao and Ye [52], Kerr and Li [60]). Let $G \curvearrowright X$ be an action. Then $G \curvearrowright X$ is null if and only if $IN_2(X,G) \subset \Delta_2(X)$.

Another application of combinatorial independence is a formula for topological sequence entropy using IN-tuples [56, Theorem 4.4].

Theorem 3.5 (Huang and Ye [56]). Let $G \curvearrowright X$ be an action and $k \ge 2$. We have that $IN_k(X, G) \subset \Delta_k^2(X)$ if and only if $h_{top}^*(G \curvearrowright X) < \log k$. Furthermore,

$$h^*_{top}(G \curvearrowright X) = \log n$$

for some $n \in \mathbb{N} \cup \{\infty\}$.

IN-tuples and IE-tuples have similar basic properties [60, Proposition 5.4].

Theorem 3.6 (Huang, Li, Shao and Ye [52], Kerr and Li [60]). Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions and $k \in \mathbb{N}$. We have that

- 1. $IN_k(X,G)$ is a closed subset of X^k that is invariant under the product action.
- 2. If $\pi: G \curvearrowright X \to G \curvearrowright Y$ is a factor map, then

$$(\pi \times \cdots \times \pi)(\operatorname{IN}_k(X,G)) = \operatorname{IN}_k(Y,G)$$

3.2 Tameness and IT-tuples

Given an action $G \curvearrowright X$, the **Ellis semigroup** $E(G \curvearrowright X)$ is defined as the closure of

$$\{x \mapsto gx : g \in G\} \subset X^X$$

in the product topology. Using the composition as the operation one can check that $E(G \curvearrowright X)$ is indeed a semigroup. We say that $G \curvearrowright X$ is **tame** if the cardinality of $E(G \curvearrowright X)$ is at most 2^{\aleph_0} (the cardinality of the real numbers). Tameness was originally defined by Köhler [66] by requiring that no continuous function $f: X \to \mathbb{R}$ has an infinite ℓ_1 -isomorphism set. We use the equivalent definition as stated by Glasner in [40].

In contrast to the density conditions in the context of entropy, here we are interested in the existence of infinite independence sets. This is related to Rosenthal's ℓ_1 Theorem [87] and the Ramsey method involved in its proof (see [45]).

Definition 3.7. Let $G \curvearrowright X$ be an action and $k \ge 1$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an IT-tuple if for any product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} the tuple (U_1, \ldots, U_k) has an infinite independence set. We denote the set of IT-tuples of length k by $\mathbf{IT}_k(X, G)$.

Every orbit IE-tuple is an IT-tuple [63, Corollary 7.2]. Every IT-tuple is an IN-tuple.

The following result was proved in [60, Proposition 6.4].

Theorem 3.8 (Kerr and Li [60]). Let $G \curvearrowright X$ be an action. Then $G \curvearrowright X$ is tame if and only if $IT_2(X, G) \subset \Delta_2(X)$.

Corollary 3.9. If an action $G \curvearrowright X$ is null then it must be tame.

There exist minimal actions of \mathbb{Z} that are tame but not null (see [60, Section 11] and [34]).

The following result was proved for minimal actions of \mathbb{Z} by Glasner [40], then Kerr and Li proved it for actions of \mathbb{Z} in [59] and for actions of amenable groups in [60]. The following generalization was proved by Li and Rong [69, Theorem 1.3].

Theorem 3.10 (Glasner [40], Kerr and Li [59, 60], Li and Rong [69]). If an action $G \sim X$ is tame then it has zero naive topological entropy.

For a proof of the following result see [60, Proposition 6.4].

Theorem 3.11 (Kerr and Li [60]). Let $G \curvearrowright X$, $G \curvearrowright Y$ be actions and $k \in \mathbb{N}$. We have that

1. $IT_k(X,G)$ is a closed subset of X^k that is invariant under the product action.

2. If $\pi: G \curvearrowright X \to G \curvearrowright Y$ is a factor map, then

$$(\pi \times \cdots \times \pi)(\mathrm{IT}_k(X,G)) = \mathrm{IT}_k(Y,G).$$

Null and tame actions are strongly related to their maximal equicontinuous factor.

We say $G \curvearrowright X$ is **equicontinuous** if for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $\rho(x, y) \le \delta$ then $\rho(gx, gy) \le \varepsilon$ for all $g \in G$, and $G \curvearrowright X$ is **uniquely ergodic** if there is only one *G*-invariant Borel probability measure. Minimal equicontinuous actions are uniquely ergodic. The **maximal equicontinuous** factor, $G \curvearrowright X_{eq}$, is the largest equicontinuous factor, i.e. every equicontinuous factor is also a factor of $G \curvearrowright X_{eq}$. The factor map to the maximal equicontinuous factor will be denoted by π_{eq} . Given a minimal action we denote the invariant Borel probability measure of the maximal equicontinuous factor by ν_{eq} . For an overview of minimal systems and maximal equicontinuous factors see [1].

A structure theorem for minimal systems involves the description of its natural factors or extensions. Let $N \in \mathbb{N}$ and $\pi : G \cap X \to G \cap Y$ a factor map, we say that π is **almost** *N*-to-1 if *N* is the smallest integer such that the G_{δ} set $\{x \in X : |\pi^{-1} \circ \pi(x)| \leq N\}$ is dense. If $G \cap X$ is minimal, we say that π is **almost finite-to-one** if there exists $y \in Y$ such that $|\pi^{-1}(y)| < \infty$; actually every almost finite-to-one factor map is almost *N*-to-1 for some $N \in \mathbb{N}$.

An action is **almost automorphic** if π_{eq} is almost 1-to-1. A minimal action is **regular** if $\nu_{eq}(\{y \in X_{eq} : |\pi_{eq}^{-1}(y)| = 1\}) = 1$. Every minimal regular action is uniquely ergodic and almost automorphic.

Almost automorphy and unique ergodicity were proven for minimal null actions of \mathbb{Z} by Huang, Li, Shao and Ye in [52], and for minimal tame actions of abelian groups by Huang [50] and Kerr and Li [60] independently. The following generalization was proved in [41, Corollary 5.4].

Theorem 3.12 (Glasner [41]). Every minimal tame action, which has an invariant Borel probability measure, is almost automorphic and uniquely ergodic.

Note that every action of an amenable group has an invariant Borel probability measure (e.g. see [65, Theorem 4.4]).

Regularity was proved for minimal null subshifts of amenable groups by García-Ramos [35] and for minimal null actions of \mathbb{Z} by García-Ramos, Jäger and Ye in [36]. The following generalization appeared in [33, Theorem 1.2].

Theorem 3.13 (Fuhrmann, Glasner, Jäger and Oertel [33]). Let $G \curvearrowright X$ be a minimal almost automorphic action. If $G \curvearrowright X$ is tame, then it is regular.

Combining Theorem 3.12 and Theorem 3.13 we obtain the following.

Corollary 3.14. Let G be an amenable group and $G \sim X$ a minimal tame action. Then $G \sim X$ is regular.

One may also wonder what happens when we restrict IT tuples of a certain size.

Conjecture 3.15 (Conjecture 1 in [53]). Let G be an amenable group, $G \curvearrowright X$ a minimal action and $k \ge 2$. If $IT_k(X,G) \subset \Delta_k^2(X)$, then π_{eq} is almost finite-to-one.

A result related to this conjecture was proved by Maass and Shao [76].

IT-tuples can also be used to bound ergodic measures [53, Theorem A].

Theorem 3.16 (Huang, Lian, Shao and Ye [53]). Let G be an amenable group and $G \sim X$ a minimal action. If π_{eq} is almost finite-to-one, and there is some integer $k \geq 2$ such that $IT_k(X,G) \subset \Delta_k^2(X)$, then $G \sim X$ has only finitely many ergodic probability measures.

3.3 Nilsystems and cubic independence tuples

Nilsystems are natural generalizations of equicontinuous systems. They provide connections with other areas of mathematics such as geometry and number theory. For a thorough exploration on these topics see [49]. In this subsection we define IN-tuples of order d and we will see their relation to nilsystems.

A topological dynamical system (TDS) is a pair (X,T), where X is a compact metrizable space and $T: X \to X$ is a homeomorphism.

Let H be a group. For $g, h \in H$ and $A, B \subset H$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of gand h and [A, B] for the subgroup generated by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups H_j , $j \geq 1$, are defined inductively by setting $H_1 = H$ and $H_{j+1} = [H_j, H]$. Let $d \geq 1$ be an integer. We say that H is d-step nilpotent if H_{d+1} is the trivial subgroup.

Let H be a d-step nilpotent Lie group and Γ a discrete cocompact subgroup of H. The compact manifold $X = H/\Gamma$ is called a d-step nilmanifold. The group H acts on X by left translations and we write this action as $(g, x) \mapsto gx$. For any $\tau \in H$, the transformation $x \mapsto \tau x$ of X is called a d-step nilsystem. A nilsystem is a d-step nilsystem for some $d \ge 1$. A 1-step nilsystem is simply a rotation on a compact abelian Lie group, and hence it must be equicontinuous.

We may also define nilsystems of order d using inverse limits. We say that a TDS (X, T) is an **inverse** limit of the TDSs $(X_i, T_i)_{i \in \mathbb{N}}$ if there exist factor maps $\pi_i : X_{i+1} \to X_i$, such that

- X is the subset of $\prod_{i \in \mathbb{N}} X_i$ given by $\{(x_i)_{i \in \mathbb{N}} : \pi_i(x_{i+1}) = x_i \ \forall i \in \mathbb{N}\},\$
- the topology on X is induced by the product topology on $\prod_{i \in \mathbb{N}} X_i$, and
- T acts as T_i on each coordinate.

A minimal system (X,T) is called a **nilsystem of order** d if it is an inverse limit of minimal d-step nilsystems. A minimal system (X,T) is called a **nilsystem of order** ∞ if it is an inverse limit of minimal nilsystems.

Let (X, T) be a minimal TDS and $d \in \mathbb{N} \cup \{\infty\}$. The **maximal nilfactor of order** d is the largest factor that is a nilsystem of order d. The factor map to the maximal nilfactor of order d will be denoted by $\pi_d : X \to X_d$. Using results from [67] one can show the existence of maximal nilfactors. Maximal nilfactors can also be constructed using regionally proximal pairs of finite order [90, Theorem 3.9] and infinite order [30, Remark 3.5]. The maximal nilfactor of order 1 is the maximal equicontinuous factor.

Definition 3.17. Let (X,T) be a TDS and $d, k \in \mathbb{N}$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an $IN^{[d]}$ -tuple if for all $\ell \in \mathbb{N}$ and product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , there exist vectors $\vec{p}_j \in \mathbb{N}^d$ for $j = 1, \ldots, \ell$ (with $\vec{p}_j \neq \vec{p}_{j'}$ for every $j \neq j' \in \{1, \ldots, \ell\}$), such that

$$\bigcup_{j=1}^{\ell} \{ \vec{p}_j \cdot \vec{\epsilon} : \vec{\epsilon} \in \{0,1\}^d \} \setminus \{0\}$$

(where \cdot is the usual dot product on \mathbb{R}^d) is an independence set for (U_1, \ldots, U_k) . We denote the set of $IN^{[d]}$ -tuples of length k by $IN^{[d]}_k(X,T)$.

Remark 3.18. Note that $IN_k^{[1]}(X,T)$ is defined similar to $IN_k(X,\mathbb{Z})$ but the independence sets are required to be in \mathbb{N} instead of \mathbb{Z} . Nonetheless one can check that $IN_k^{[1]}(X,T) = IN_k(X,\mathbb{Z})$.

A finite subset F of \mathbb{N} is called a **finite IP-set** if there exists a finite subset $\{p_1, p_2, \ldots, p_m\}$ of \mathbb{N} such that

$$F = FS(\{p_i\}_{i=1}^m) = \{p_{i_1} + \dots + p_{i_k} : 1 \le i_1 < \dots < i_k \le m \text{ and } k \in \mathbb{N}\}.$$

Definition 3.19. Let (X, T) be a TDS and $k \in \mathbb{N}$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an $IN^{[\infty]}$ -tuple if for any product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , there exist independence sets for $(U_1, ..., U_k)$ that contain arbitrarily long finite IP-sets. We denote the set of these tuples by $IN_k^{[\infty]}(X, T)$.

Every minimal nilsystem of order $d \in \mathbb{N} \cup \{\infty\}$ is uniquely ergodic [49, Section 11.2, Theorem 11]. We denote the unique invariant Borel probability measure on X_d by ν_d .

The following result was recently proved in [86] building on [30, Theorem 4.5].

Theorem 3.20 (Dong, Donoso, Maass, Shao and Ye [30], Qiu and Yu [86]). Let (X,T) be a minimal TDS. If $\mathbb{IN}_2^{[\infty]}(X,T) \subset \Delta_2(X)$, then (X,T) is uniquely ergodic and π_{∞} is regular, that is,

$$\nu_{\infty}(\{y \in X_{\infty} : |\pi_{\infty}^{-1}(y)| = 1\}) = 1.$$

Theorem 3.21 (Qiu and Yu [86]). Let (X,T) be a minimal TDS. If there is some integer $k \ge 2$ such that $IT_k^{\infty}(X,G) \subset \Delta_k^2(X)$, then (X,T) has only finitely many ergodic probability measures.

We also have the following finite order structure theorems. For the proof of the following result see [85, Theorem 5.7].

Theorem 3.22 (Qiu [85]). Let (X,T) be a minimal TDS and $d \in \mathbb{N}$. If $IN_2^{[d]}(X,T) \subset \Delta_2(X)$, then π_d is almost 1-to-1.

3.4 Mean dimension pairs

Mean dimension is a dynamical invariant introduced for actions of amenable groups by Gromov [47] and studied by Lindenstrauss and Weiss [74]. It is particularly useful for studying infinite entropy systems and deciding when a system can be embedded in another [73, 48]. Mean dimension was extended for actions of sofic groups by Li [68].

We say a finite open cover \mathcal{V} refines $\mathcal{U}, \mathcal{V} \succ \mathcal{U}$, if every member of \mathcal{V} is contained in a member of \mathcal{U} .

Definition 3.23. Let \mathcal{U} be a finite open cover of X. We denote

$$\operatorname{ord}(\mathcal{U}) = (\max_{x \in X} \sum_{U \in \mathcal{U}} 1_U(x)) - 1, \text{ and}$$

 $\mathcal{D}(\mathcal{U}) = \min_{\mathcal{V} \succ \mathcal{U}} \operatorname{ord}(\mathcal{V}).$

Definition 3.24. Let G be an amenable group, $G \cap X$ an action, $\{F_n\}_{n \in \mathbb{N}}$ a Følner sequence for G and \mathcal{U} a finite open cover of X. We define the **mean dimension of** $G \cap X$ with respect to \mathcal{U} as

$$\mathrm{mdim}(G \curvearrowright X, \mathcal{U}) = \lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U}^{F_n})}{|F_n|}.$$

The mean dimension of $G \cap X$ is the supremum of $\operatorname{mdim}(G \cap X, \mathcal{U})$ over all finite open covers and is denoted by $\operatorname{mdim}(G \cap X)$.

Mean dimension is invariant under conjugacy and does not depend on the choice of the Følner sequence [23, Theorem 9.4.1]. For a more thorough exposition on the theory of mean dimension for actions of amenable groups see [23, Chapter 10].

We will now define mean dimension for actions of sofic groups. Let \mathcal{U} be a finite open cover and $n \in \mathbb{N}$. We denote by $\mathcal{U}^{[n]}$ the finite open cover of $X^{[n]}$ consisting of $U_1 \times U_2 \times \cdots \times U_n$ for $U_1, \ldots, U_n \in \mathcal{U}$.

Let $F \in G$, $\delta > 0$, $n \in \mathbb{N}$ and $\sigma : G \to \text{Sym}(n)$. We have that $\text{Map}(\rho, F, \delta, \sigma)$ is a closed subset of $X^{[n]}$. Consider the restriction $\mathcal{U}^{[n]}|_{\text{Map}(\rho, F, \delta, \sigma)}$ of $\mathcal{U}^{[n]}$ to $\text{Map}(\rho, F, \delta, \sigma)$. Denote $\mathcal{D}(\mathcal{U}^{[n]}|_{\text{Map}(\rho, F, \delta, \sigma)})$ by $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)$.

Let $F \in G$ and $\delta > 0$. For a finite open cover \mathcal{U} of X and $\Sigma = \{\sigma_i : G \to \text{Sym}(n_i)\}_{i \in \mathbb{N}}$ a sofic approximation sequence, we define

$$\mathcal{D}_{\Sigma}(\mathcal{U}, \rho, F, \delta) = \overline{\lim_{i \to \infty}} \frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma_i)}{n_i},$$

 $\mathcal{D}_{\Sigma}(\mathcal{U}) = \inf_{F \Subset G} \inf_{\delta > 0} \mathcal{D}_{\Sigma}(\mathcal{U}, \rho, F, \delta).$

If $\operatorname{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large *i*, we set $\mathcal{D}_{\Sigma}(\mathcal{U}, \rho, F, \delta) = -\infty$. We define the **sofic** mean dimension (with respect to Σ) as

$$\mathrm{mdim}_{\Sigma}(G \curvearrowright X) = \sup_{\mathcal{U}} \mathcal{D}_{\Sigma}(\mathcal{U})$$

for \mathcal{U} ranging over finite open covers of X.

Remark 3.25. The quantities $\mathcal{D}_{\Sigma}(\mathcal{U})$ and $\operatorname{mdim}_{\Sigma}(G \curvearrowright X)$ do not depend on the choice of ρ [68]. In particular, $\operatorname{mdim}_{\Sigma}(G \curvearrowright X)$ is invariant under conjugcay. If G is an amenable group then $\operatorname{mdim}_{\Sigma}$ does not depend on the choice of a sofic approximation sequence, and the value coincides with the mean dimension as introduced by Gromov [68, Theorem 3.1].

Definition 3.26. Let G be a sofic group, Σ a sofic approximation sequence for G and $G \cap X$ an action. We say that $(x, y) \in X^2 \setminus \Delta_2(X)$ is a Σ -mean dimension pair if for any closed disjoint neighborhoods, K_x and K_y , of x and y respectively, we have that $\mathcal{D}_{\Sigma}(\{K_x^c, K_y^c\}) > 0$. We denote the set of Σ -mean dimension pairs by $D_m^{\Sigma}(X, G)$.

Theorem 3.27 (García-Ramos and Gutman [37]). Let G be a sofic group, Σ a sofic approximation sequence for G and $G \curvearrowright X$ an action. Then $\operatorname{mdim}_{\Sigma}(G \curvearrowright X) > 0$ if and only if $D_m^{\Sigma}(X,G) \neq \emptyset$.

Question 3.28. Let Σ be a sofic approximation sequence for G, and $G \curvearrowright X$ an action. Is every Σ -mean dimension pair a Σ -IE-pair?

We do not know the answer of the previous question even when $G = \mathbb{Z}$.

Let G be a sofic group and Σ a sofic approximation sequence for G. An action $G \curvearrowright X$ has **completely positive sofic mean dimension (with respect to** Σ) if every non-trivial factor has positive sofic mean dimension. Generalizing a result of Lindenstrauss and Weiss [74, Theorem 3.6], Li proved that the shift action $G \curvearrowright I^G$ of a sofic group G (where I is the unit interval) has completely positive sofic mean dimension [68, Theorem 7.5].

An action with zero mean dimension may still have a factor with positive mean dimension.

Theorem 3.29 (García-Ramos and Gutman [37]). Let Σ be a sofic approximation for G and $G \curvearrowright X$ an action. If the smallest closed G-invariant equivalence relation containing Σ -mean dimension pairs is X^2 , then $G \curvearrowright X$ has completely positive sofic mean dimension.

Question 3.30. Can positive mean dimension be characterized with some form of combinatorial independence?

4 Applications and connections

4.1 Homoclinic points of algebraic actions

In this subsection we are interested in how homoclinic points relate to entropy. This line of research began with the work of Lind and Schmidt [72] on algebraic actions of \mathbb{Z}^d .

We say an action $G \cap X$ is algebraic if X is a compact metrizable abelian group and G acts by automorphisms of X.

For algebraic actions it is more convenient to work with points rather than pairs or tuples because we have a reference fixed point, e_X , the identity of X.

Let $G \curvearrowright X$ be an algebraic action. We say $x \in X$ is an **IE-point** if $(x, e_X) \in IE_2(X, G)$. The set of IE-points is a *G*-invariant closed subgroup of X [63, Theorem 6.4], called the **IE-group** and will be denoted by IE(X, G). Furthermore, for each $k \in \mathbb{N}$, we have [63, Theorem 6.4]

 $IE_k(X,G) = \{(yx_1, \dots, yx_k) : x_1, \dots, x_k \in IE(X,G), y \in X\}.$

If G is sofic and Σ is a sofic approximation sequence for G, we say $x \in X$ is an \mathbf{IE}^{Σ} -point if $(x, e_X) \in \mathbf{IE}_2^{\Sigma}(X, G)$. The set of \mathbf{IE}^{Σ} -points is a G-invariant closed subset of X, and will be denoted by $\mathbf{IE}^{\Sigma}(X, G)$.

Given an algebraic action $G \curvearrowright X$, we say $x \in X$ is a **homoclinic point** [72] if $sx \to e_X$ as $G \ni s \to \infty$. The homoclinic points form a *G*-invariant subgroup of *X*, called the **homoclinic group** and will be denoted by A(X, G).

We say $G \curvearrowright X$ is **expansive** if there exists c > 0 such that whenever $x, y \in X$, if $x \neq y$ then there exists $g \in G$ such that $\rho(gx, gy) > c$.

The following result was proved for actions of amenable groups in [21]. The following generalization appeared in [3, Corollary 7.2].

Theorem 4.1 (Chung and Li [21], Barbieri, García-Ramos and Li [3]). Let $G \curvearrowright X$ be an expansive algebraic action. We have:

1. $A(X,G) \subset IE(X,G)$.

2. If G is sofic and Σ is any sofic approximation sequence for G, then $A(X,G) \subset IE^{\Sigma}(X,G)$.

The family of **elementary amenable groups** is the smallest class of groups containing all finite groups and all abelian groups and is closed under taking subgroups, quotient groups, extensions, and inductive limits [27, 20]. Elementary amenable groups are amenable, but there exist amenable groups that are not elementary amenable [46].

For each locally compact abelian group Y, we denote by \hat{Y} its **Pontryagin dual**. It consists of all continuous group homomorphisms $Y \to \mathbb{R}/\mathbb{Z}$, and becomes a locally compact abelian group under pointwise addition and the topology of uniform convergence on compact subsets. Then Y is compact metrizable if and only if \hat{Y} is countable discrete.

For a compact metrizable abelian group X, there is a natural one-to-one correspondence between algebraic actions of G on X and actions of G on \hat{X} by automorphisms. There is also a natural oneto-one correspondence between the latter and the left $\mathbb{Z}G$ -module structure on \hat{X} ($\mathbb{Z}G$ is the group ring which consists of all functions $f : G \to \mathbb{Z}$ of finite support, for more information see [81]). Thus, up to isomorphism, there is a natural one-to-one correspondence between algebraic actions of G and countable left $\mathbb{Z}G$ -modules.

We say an algebraic action $G \curvearrowright X$ is **finitely presented** if \hat{X} is a finitely presented left $\mathbb{Z}G$ -module. Expansive algebraic actions are always finitely generated (\hat{X} is finitely generated) [89, Proposition 2.2 and Corollary 2.16].

Theorem 4.2 (Chung and Li [21], Barbieri, García-Ramos and Li [3]). Let G be an elementary amenable group with an upper bound on the orders of finite subgroups of G, and $G \cap X$ a finitely presented expansive algebraic action. Then $\overline{A(X,G)} = IE(X,G)$.

The previous theorem was proved for polycyclic-by-finite groups by Chung and Li [21, Theorem 1.2], and generalized in [3, Corollary 7.5]. The statement of the result holds more generally, for amenable groups such that the strong Atiyah conjecture is satisfied, and there is an upper bound on the orders of finite subgroups (see [3] for definitions and details).

Question 4.3. Does the statement of Theorem 4.2 hold if we relax the hypothesis of elementary amenability for G and simply ask for amenability?

The statement of Theorem 4.2 is not satisfied for actions of the free group \mathbb{F}_2 in the context of naive topological entropy [3, Example 7.6].

Question 4.4. Is there a finitely presented expansive algebraic action of \mathbb{F}_2 with positive topological sofic entropy for some sofic approximation sequence and trivial homoclinic group?

As a corollary of Theorem 4.2 we obtain the following result.

Corollary 4.5. Let G be an elementary amenable group with an upper bound on the orders of finite subgroups of G, and $G \sim X$ an expansive finitely presented algebraic action. Then,

- 1. $G \curvearrowright X$ has positive topological entropy if and only if $A(X,G) \neq \{e_X\}$, and
- 2. $G \curvearrowright X$ has UPE if and only if $\overline{A(X,G)} = X$.

Remark 4.6. Meyerovitch proved that there exist an abelian group G and an expansive algebraic action $G \curvearrowright X$ with positive topological entropy and trivial homoclinic group [77, Theorem 1.2].

An algebraic action can be seen as a measure-preserving action with respect to the Haar measure of X. For algebraic actions of amenable groups, the Haar measure is a measure of maximal entropy, thus one has positive topological entropy if and only if one has positive measure-theoretical entropy [29, Theorem 2.2]. Furthermore, an algebraic action of an amenable group has UPE if and only if it has completely positive entropy in the measure-theoretic sense (every non-trivial measurable factor has positive measure-theoretic entropy) [21].

4.2 Continuum theory

In this subsection we will talk about a couple of interactions between topological dynamics and continuum theory. A topological space X is a **continuum** if it is a compact connected metrizable space. For an introduction to continuum theory see [80].

In the first topic that we will expose, the underlying theme is that complicated dynamics should imply existence of complicated continua. This line of investigation began with the work of Barge and Martin [4]; one of their results implies that if f is a continuous self-map of the interval I = [0, 1] with positive topological entropy, then the inverse limit space $\lim_{i \to \infty} (I, f)$ (the inverse limit by infinite copies of the same system) contains an indecomposable continuum. Indecomposable continua were introduced by Brouwer. A continuum is **indecomposable** if it contains more than one point and it cannot be expressed as the union of any two of its proper subcontinua. The dyadic solenoid and the pseudo-arc are indecomposable continua; the cube and the torus are not indecomposable.

Let Y be a compact metrizable space and $\varepsilon > 0$. A continuous mapping $g: X \to Y$ is an ε -mapping if for every $y \in Y$, the diameter of $g^{-1}(y)$ is less than ε . A continuum X is Y-like if for every $\varepsilon > 0$ there is an ε -mapping from X onto Y.

Using local entropy theory and introducing a new dynamical pair called zigzag pair the following result was proved in [26, Corollary 5.4 and Corollary 5.6].

Theorem 4.7 (Darji and Kato [26]). Let Y be a finite graph treated as a topological space, and X a Y-like continuum. If X admits a homeomorphism T such that (X,T) has positive topological entropy, then X contains an indecomposable continuum. Furthermore, if X admits a homeomorphism T such that (X,T) has UPE then X is an indecomposable continuum.

Now we will present a result where continuum theory and local entropy theory are applied to obtain a result regarding the possible values of sequence entropy.

Let $S_G(X)$ be the set of the values $h_{top}^*(G \curvearrowright X)$ for all possible actions $G \curvearrowright X$. Theorem 3.5 implies that $\{0\} \subset S_G(X) \subset \{0, \log 2, \log 3, ...\} \cup \{\infty\}$. The following result appeared in [92, Theorem 8.11].

Theorem 4.8 (Snoha, Ye and Zhang [92]). Let G be a finitely generated group such that there is a surjective homomorphism $G \to \mathbb{Z}$, and $\{0\} \subset A \subset \{0, \log 2, \log 3, ...\} \cup \{\infty\}$. There exists a one-dimensional continuum X such that $S_G(X) = A$.

Question 4.9. For which groups does the conclusion of Theorem 4.8 hold?

Question 4.10. Is it true that for every set $\{0\} \subset A \subset [0, \infty]$ such that $\mathbb{N}a \subset A$ for every $a \in A$, there exists a compact metrizable space X such that the possible values of topological entropy for all actions of \mathbb{Z} over X is A?

4.3 Descriptive complexity

Given a compact metrizable space, one may study the properties of the set of dynamical systems (or actions) with a certain property on that. One way to approach this is with the viewpoint of Baire category theory and descriptive set theory, which can give us information of how complicated a set is. This approach can allow us to discard possible characterizations of the studied properties. For example, Halmos and Rohlin proved that the properties of the sets of measure-preserving mixing and weak-mixing systems are different and thus concluded the existence of weak-mixing systems that are not mixing without explicitly constructing one.

While the existence of actions with CPE and not with UPE is known, in this subsection we will see a striking difference between these families.

This line of research began with the work of Pavlov when he found a sufficient condition for completely positive entropy on subshifts of finite type of \mathbb{Z}^d , and asked if the condition was also necessary [82]. This question was answered in the negative by Barbieri and García-Ramos by defining a rank (or hierarchy) for actions with completely positive entropy [2]. Westrick proved that this rank is actually a Π_1^1 -rank [95] (see also [25]), which implies that this rank can be used to prove that certain sets are complicated (not Borel) in the sense of descriptive set theory.

We will give a quick overview of classical descriptive complexity. For a more thorough exposition of these topics see [57, 78].

Given a compact metrizable space X, we define TDS(X) as the set of all continuous functions from X into X endowed with the uniform convergence topology. Note that TDS(X) is a **Polish space**, i.e., a separable space whose topology can be induced by a complete metric. In previous sections we defined UPE, CPE and expansiveness for group actions. The definition for semigroup actions and in particular continuous maps $T: X \to X$ is the natural adaptation. Let $T: X \to X$ be a continuous function. We say that T is **mixing** if for every pair of non-empty open sets U and V there exists $N \in \mathbb{N}$ such that $T^{-n}U \cap V \neq \emptyset$ for all $n \geq N$.

We also define the following subspaces

$$UPE(X) = \{T \in TDS(X) : T \text{ has UPE}\},\$$
$$CPE(X) = \{T \in TDS(X) : T \text{ has CPE}\},\$$
$$Mix(X) = \{T \in TDS(X) : T \text{ is mixing}\},\$$
and
$$Exp(X) = \{T \in TDS(X) : T \text{ is expansive}\}.$$

We say that a subset of a Polish space is **analytic** if it is the continuous image of a Polish space and **coanalytic** (or Π_1^1) if it is the complement of an analytic set. All Borel subsets of a Polish space are both analytic and coanalytic. Moreover, if a set is both analytic and coanalytic, then it must be Borel. However, in every uncountable Polish space there are analytic, and hence coanalytic, sets which are not Borel. Loosely speaking, if a set is analytic or coanalytic but not Borel, it means that the set cannot be described with countable information.

A standard way to prove that a coanalytic set is not Borel is to reduce it to a known combinatorial set which is not Borel. More specifically, if \mathcal{B} is a known non-Borel subset of some Polish space $\mathcal{Y}, \mathcal{A} \subset \mathcal{X}$ and $f: \mathcal{Y} \to \mathcal{X}$ is a Borel function such that $f^{-1}(\mathcal{A}) = \mathcal{B}$, then \mathcal{A} is not Borel. In this case, we say that \mathcal{B} is **Borel reducible** to \mathcal{A} . This inspires the following definition.

Definition 4.11. A coanalytic subset \mathcal{A} of a Polish space \mathcal{X} is complete coanalytic (or Π_1^1 -complete) if for every coanalytic subset \mathcal{B} of a Polish space \mathcal{Y} there exists a Borel function $f : \mathcal{Y} \to \mathcal{X}$ such that $f^{-1}(\mathcal{A}) = \mathcal{B}$.

Complete coanalytic sets are the most complicated coanalytic sets.

For a proof of the following result see [24, Corollary 3.2 and Proposition 2.12].

Theorem 4.12 (Darji and García-Ramos [24]). Let X be a compact metrizable space. Then UPE(X) is a Borel subset of TDS(X), and CPE(X) is a conalytic subset of TDS(X).

Whether CPE(X) is Borel or not will depend on X. There exist infinite compact metrizable spaces that do not admit dynamics with positive topological entropy, thus for these spaces CPE(X) is empty (and hence Borel) [22].

There are two techniques to prove that CPE(X) or subfamilies, are not Borel. One can use Borel reductions or the entropy rank introduced by Barbieri and García-Ramos.

The following result is a combination of [25, Theorem 3.7], [57, Theorem 35.23], and the proof of [2, Theorem 3.3],

Theorem 4.13 (Barbieri and García-Ramos [2]). Let X be a Cantor space. We have that CPE(X) is a coanalytic but not Borel subset of TDS(X).

For the proof of the following result see [24, Theorem 4.1].

Theorem 4.14 (Darji and García-Ramos [24]). For every $d \ge 1$ we have that $CPE(I^d)$ is a complete coanalytic (thus not Borel) subset of $TDS(I^d)$.

The previous result also holds for the circle, the torus and some other compact orientable manifolds.

Remark 4.15. If one restricts to the space of Lipschitz functions in TDS(I) one may still obtain a result similar to Theorem 4.14. We do not know what happens if we restrict to smooth functions.

One may wonder what happens if we consider dynamical systems with natural dynamical properties. The following result is a combination of [88, Theorem 3], [57, Theorem 35.23] and [25, Theorem 3.7].

Theorem 4.16 (Salo [88]). Let X be a Cantor space. We have that $Exp(X) \cap CPE(X)$ is a coanalytic but not Borel subset of TDS(X).

Remark 4.17. For any compact metrizable space X, the family of maps with shadowing and CPE is a Borel subset of TDS(X) [24, Proposition 3.5 and Corollary 3.4].

The following result can be obtained with techniques used by Blokh [13] (see [24, Corollary 3.9]).

Proposition 4.18. Let X be a finite graph treated as a topological space. Then $Mix(X) \cap CPE(X)$ is a Borel subset of TDS(X).

In particular the previous result applies to the interval and the circle.

In contrast to this, the following result was proved in [24, Theorem 5.25].

Theorem 4.19 (Darji and García-Ramos [24]). Let X be a Cantor space. Then $Mix(X) \cap CPE(X)$ is a coanalytic but not Borel subset of TDS(X).

Proving that CPE(X) and $Exp(X) \cap CPE(X)$ are complete coanalytic for a Cantor space X might be accessible, but some extra arguments are missing. The following question seems harder to approach.

Question 4.20. Let X be a Cantor space. Is $Mix(X) \cap CPE(X)$ a complete coanalytic subset of TDS(X)?

Question 4.21. Let X be a torus of dimension at least 2. Is $Mix(X) \cap CPE(X)$ a Borel subset of TDS(X)?

A *G*-subshift over a finite set *A* is a closed subset of A^G that is invariant under the shift action of *G*. Thus, for every *G*-subshift *X* one can naturally assign an action $G \sim X$. A subshift is of finite type if it can be determined using a finite number of forbidden patterns. It is not difficult to see that for every group *G* and every finite alphabet *A* the family of subshifts of finite type in A^G is countable and hence any subset of this family will be a Borel (F_σ) set. Nonetheless, the complexity of these families can be studied using another branch of descriptive set theory that uses computable functions instead of Borel functions. We will not give the precise definitions of effective Borel and coanalytic sets, since they are more involved than their topological counterparts. The interested reader can check [78]. For the following theorem see [95, Theorem 1].

Theorem 4.22 (Westrick [95]). The property of completely positive entropy is effective complete coanalytic in the set of \mathbb{Z}^2 -subshifts of finite type in $A^{\mathbb{Z}^2}$ for a finite set A.

Completely positive mean dimension can also be studied from this point of view. In this situation a different model is more natural.

We now recall the basics of the Hausdorff metric. Given a metric space (Y, ρ) we denote with K(Y) the space of all non-empty compact subsets of Y. We equip K(Y) with the Vietoris topology, which is generated by the **Hausdorff metric** defined as

$$\rho_H(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\},\$$

where $C_{\delta} = \{x \in Y : \rho(x, C) < \delta\}.$

If Y is compact, then so is K(Y). For more information see [57, Section 4.F].

Let A be a compact metric space. We equip $A^{\mathbb{Z}}$ with the product topology, and we denote the shift action by $\sigma: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$. We define

$$\mathcal{S}(A) = \{ X \subset A^{\mathbb{Z}} : X \text{ is closed, non-empty and } \sigma(X) = X \}.$$

Since $\mathcal{S}(A)$ is a closed subset of $K(A^{\mathbb{Z}})$, we also equip it with the induced topology. Note that $\mathcal{S}(A)$ is also a compact metrizable space, and hence it is Polish. We also define the following subspace

 $S_{c+}(A) = \{X \in S(A) : (X, \sigma) \text{ has completely positive mean dimension}\}.$

Theorem 4.23 (García-Ramos and Gutman [37]). We have that $S_{c+}(I)$ is a complete coanalytic subset of S(I).

4.4 Li-Yorke chaos

In the seminal work of Li and Yorke it was proved that the existence of a periodic point of order 3 of an interval map implies some form of disorder [71]. This notion is now known as Li-Yorke chaos. An active line of research has been to understand under which conditions positive topological entropy implies Li-Yorke chaos.

Let S be an infinite subset of G. A pair $(x, y) \in X \times X$ is called S-scrambled if

 $\limsup_{S \ni s \to \infty} \rho(sx,sy) > 0 \quad \text{ and } \quad \liminf_{S \ni s \to \infty} \rho(sx,sy) = 0.$

A subset $K \subset X$ with at least two points is called *S*-scrambled if any two distinct points $x, y \in K$ form an *S*-scrambled pair.

Definition 4.24. Let S be an infinite subset of G. We say that $G \cap X$ is Li-Yorke chaotic along S if there exists an uncountable S-scrambled subset of X. We say that $G \cap X$ is Li-Yorke chaotic if it is Li-Yorke chaotic along G.

Blanchard, Glasner, Kolyada, and Maass [10] proved, using local entropy theory and measure-theoretic methods, that any TDS with positive topological entropy must be Li-Yorke chaotic. This result was then proved for actions of amenable groups by Kerr and Li [60] using local entropy theory and purely topological arguments. The following generalization appears in [51, Theorem 1.1].

Theorem 4.25 (Huang, Li and Ye [51]). Let G be an amenable group and $G \curvearrowright X$ an action. If $G \curvearrowright X$ has positive topological entropy, then for any infinite subset $S \subset G$, $G \curvearrowright X$ is Li-Yorke chaotic along S.

The following result was obtained using combinatorial independence in the sense of Theorem 2.21 in [69, Theorem 1.1].

Theorem 4.26 (Blanchard et al [10], Kerr and Li [60, 63], Li and Rong [69]). Every action $G \curvearrowright X$ with positive naive topological entropy is Li-Yorke chaotic.

An action $G \curvearrowright X$ is **distal** if for every pair of distinct points $x, y \in X$ one has $\inf_{g \in G} \rho(gx, gy) > 0$. Every nilsystem (See Section 3.3) is distal. The following result is a direct corollary of Theorem 4.26 and answers a question of Bowen [15, Question 8].

Corollary 4.27 (Li and Rong [69]). Every distal action $G \sim X$ has zero topological naive entropy.

Question 4.28. Is every action $G \cap X$ with positive naive topological entropy Li-Yorke chaotic along every infinite $S \subset G$?

4.5 Induced actions

Bauer and Sigmund initiated the study of induced dynamical systems on the space of Borel probability measures. In some sense, this is like studying dynamics with random initial conditions. Local entropy theory has been very useful for proving several results that compare the dynamics of an action with the induced dynamics.

Let X be a compact metrizable space, \mathcal{B}_X the collection of Borel subsets of X, and C(X) the space of continuous maps from X to \mathbb{R} endowed with the uniform convergence topology generated by the sup metric, and $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with the **weak*-topology**, which is the smallest topology making the map $\mu \mapsto \int_X g d\mu$ continuous for every $g \in C(X)$. The space $\mathcal{M}(X)$ is compact and metrizable.

Furthermore, given an action $G \curvearrowright X$ we naturally induce an action $G \curvearrowright \mathcal{M}(X)$, with the pushforwards. For a TDS (X,T) the induced map on $\mathcal{M}(X)$ will be denoted by T_* .

For the proof of the following result see [5, Proposition 6] and [42, Theorem A].

Theorem 4.29 (Bauer and Sigmund [5], Glasner and Weiss [42]). Let (X,T) be a TDS. Then (X,T) has positive topological entropy if and only if $(\mathcal{M}(X), T_*)$ has positive topological entropy if and only if $(\mathcal{M}(X), T_*)$ has infinite topological entropy.

While the proof of the previous result does not use local entropy theory, the proofs of the following results use it in some form. The following result was proved for TDSs in [6, Theorem 4] and for actions of amenable groups in [75, Theorem 1.1].

Theorem 4.30 (Bernardes, Darji and Vermersch [6], Liu and Wei [75]). Let G be an amenable group and $G \curvearrowright X$ an action. Then $G \curvearrowright X$ has UPE if and only if $G \curvearrowright \mathcal{M}(X)$ has UPE.

Remark 4.31. The previous result was also proved for relativized versions of UPE in [75].

The following result regarding sequence entropy was proved for actions of \mathbb{Z} in [59, Theorem 5.10]; it follows from [60, Proposition 5.8].

Theorem 4.32 (Kerr and Li [60]). Let $G \curvearrowright X$ be an action. Then $h^*_{top}(G \curvearrowright X) > 0$ if and only if $h^*_{top}(G \curvearrowright \mathcal{M}(X)) > 0$.

We also have the following result.

Theorem 4.33. Let $G \curvearrowright X$ be an action. Then $h_{top}^*(G \curvearrowright X) > 0$ if and only if $h_{top}^*(G \curvearrowright \mathcal{M}(X)) > 0$ if and only if $h_{top}^*(G \curvearrowright \mathcal{M}(X)) = \infty$.

Proof. The first equivalence is Theorem 4.32.

Assume that $h_{top}^*(G \curvearrowright X) > 0$. There exists $(x, y) \in IN_2(X, G) \setminus \Delta_2(X)$ (Proposition 3.4). Let $k \in \mathbb{N}$. We define $\mu_i = \frac{i\delta_x + (k-i)\delta_y}{k}$ for every $0 \le i \le k$ (where δ_x is the Dirac measure supported on $\{x\}$).

Let $U_0 \times \cdots \times U_k$ an open product neighbourhood of $(\mu_0, \mu_1, \dots, \mu_k)$. Pick U_x, U_y disjoint open neighbourhoods of x, y sufficiently small so the following condition is satisfied for every $0 \le i \le k$: if $p_1, \dots, p_i \in U_x$

and $p_{i+1}, ..., p_k \in U_y$ then the measure $\frac{\sum_{j=1}^k \delta_{p_j}}{k} \in U_i$. Since $(x, y) \in IN_2(X, G) \setminus \Delta_2(X)$, there exists an independence set, J, for (U_x, U_y) of size n. Let $\phi: J \to \{0, \ldots, k\}$. There exist a set of functions $\phi_i: J \to \{x, y\}$ for $1 \le i \le k$ such that

$$|\{1 \le i \le k : \phi_i(g) = x\}| = \phi(g)$$

for every $g \in J$. Since J is an independence set for (U_x, U_y) , for every $1 \leq i \leq k$ there exists $z_i \in X$ such that $gz_i \in U_{\phi_i(g)}$ for every $g \in J$.

Let $\mathbf{z} = \{z_1, \ldots, z_k\}$. We have that

$$\begin{aligned} &|\{i \in \{1, ..., k\} : gz_i \in U_x\}| = \phi(g), \text{ and} \\ &|\{i \in \{1, ..., k\} : gz_i \in U_y\}| = k - \phi(g), \end{aligned}$$

for every $g \in J$.

Let $\mu_{\mathbf{z}} = \frac{\sum_{i=1}^{k} \delta_{z_i}}{k}$. Using the condition for U_x and U_y we have that $g\mu_{\mathbf{z}} \in U_{\phi(g)}$ for every $g \in J$. Thus J is an independence set for (U_0, \ldots, U_k) and $(\mu_0, \mu_1, \ldots, \mu_k) \in IN_{k+1}(\mathcal{M}(X), G)$.

Using Theorem 3.5 we conclude the result.

The next result follows from [60, Proposition 6.6].

Theorem 4.34 (Kerr and Li [60]). Let $G \cap X$ be an action. Then $G \cap X$ is not tame if and only if $G \cap \mathcal{M}(X)$ is not tame.

Theorem 4.35. Let $G \curvearrowright X$ be an action. Then $G \curvearrowright X$ is not tame if and only if $G \curvearrowright \mathcal{M}(X)$ is not tame if and only if $\mathrm{IT}_k(\mathcal{M}(X), G) \setminus \Delta_k^2(\mathcal{M}(X)) \neq \emptyset$ for every $k \geq 2$.

Proof. The proof follows the same lines as the proof of Theorem 4.33.

The following result was proved for $G = \mathbb{Z}$ in [16, Main Theorem] and for amenable groups in [91, Main Theorem].

Theorem 4.36 (Burguet and Shi [16], Shi and Zhang [91]). Let G be an amenable group and $G \cap X$ an action. Then $G \cap X$ has positive topological entropy if and only if $G \cap \mathcal{M}(X)$ has positive mean dimension if and only if $G \cap \mathcal{M}(X)$ has infinite mean dimension.

5 Measure tuples

Let $G \cap X$ be an action. A Borel probability measure μ on X is G-invariant if for every Borel subset $A \subset X$ and $g \in G$, we have $\mu(A) = \mu(g^{-1}A)$. In this case we say that $G \cap (X, \mu)$ is a **probability measure-preserving (p.m.p.) action**. We denote the space of all G-invariant Borel probability measures by $\mathcal{M}(G \cap X)$.

Let $G \sim (X, \mu)$ be a p.m.p. action. For a finite partition \mathcal{P} of X consisting of Borel sets the **Shannon** entropy of \mathcal{P} with respect to μ is given by

$$H_{\mu}(\mathcal{P}) = \sum_{A \in \mathcal{P}} -\mu(A) \log \mu(A).$$

For an amenable group G, the **measure-theoretical entropy of** $G \curvearrowright (X, \mu)$ with respect to \mathcal{P} is given by

$$h_{\mu}(G \curvearrowright X, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{|F_n|} \log H_{\mu}(\bigvee_{g \in F_n} g^{-1} \mathcal{P}),$$

where $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence. The **measure-theoretical entropy** of $G \curvearrowright (X, \mu)$ is the supremum of $h_{\mu}(G \curvearrowright X, \mathcal{P})$ taken over all finite Borel partitions:

$$h_{\mu}(G \curvearrowright X) = \sup_{\mathcal{P} \text{ finite}} h_{\mu}(G \curvearrowright X, \mathcal{P}).$$

The variational principle relates the topological entropy with the measure-theoretical entropy through the following formula [79, Theorem 5.2.7].

Theorem 5.1. Let G be an amenable group and $G \curvearrowright X$ an action. We have

$$h_{top}(G \curvearrowright X) = \sup_{\mu \in \mathcal{M}(G \curvearrowright X)} h_{\mu}(G \curvearrowright X).$$

Remark 5.2. The theory of measure-theoretical entropy for actions of sofic groups is well developed but we will not use it in this subsection. See [14, 62, 58] for more information.

Let $G \curvearrowright X$ be an action, μ a *G*-invariant Borel probability measure, $S = \{s_i\}_{i \in \mathbb{N}}$ a sequence in *G* and \mathcal{P} a finite Borel partition of *X*. The **sequence entropy of** $G \curvearrowright (X, \mu)$ **along** *S* **with respect to** \mathcal{P} is defined by

$$h^{S}_{\mu}(G \curvearrowright X, \mathcal{P}) = \limsup \frac{1}{n} H_{\mu} \left(\bigvee_{i=0}^{n-1} T^{-s_{i}} \mathcal{P} \right).$$

The sequence entropy of $G \curvearrowright (X, \mu)$ along S is

$$h^{S}_{\mu}(G \curvearrowright X) = \sup_{\mathcal{P}} h^{S}_{\mu}(G \curvearrowright X, \mathcal{P}),$$

where the supremum is taken over all finite Borel partitions \mathcal{P} .

5.1 Measure entropy tuples

Measure entropy pairs were introduced by Blanchard, Host, Maass, Martinez and Rudolph [11]. Measure entropy tuples are a natural generalization.

Definition 5.3. Let G be an amenable group, $G \cap X$ an action, $\mu \in \mathcal{M}(G \cap X)$, $k \ge 2$ and $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$. We say (x_1, \ldots, x_k) is a μ -entropy tuple if for every finite Borel partition \mathcal{P} such that $\{x_1, \ldots, x_k\} \notin \overline{P}$ for every $P \in \mathcal{P}$, we have that

$$h_{\mu}(G \curvearrowright X, \mathcal{P}) > 0.$$

Combinatorial independence can also be used to study μ -entropy pairs.

We denote the family of Borel subsets of X by $\mathcal{B}(X)$. For $\delta > 0$ denote by $\mathcal{B}(\mu, \delta)$ the collection of all Borel subsets D of X such that $\mu(D) \ge 1 - \delta$, and by $\mathcal{B}'(\mu, \delta)$ the collection of all maps $D : G \to \mathcal{B}(X)$ such that $\inf_{s \in G} \mu(D_s) \ge 1 - \delta$.

Definition 5.4. Let $G \sim X$ be an action, $\mathbf{A} = (A_1, \ldots, A_k)$ a tuple of subsets of X, and D be a map from G to the power set 2^X of X, with the image of $s \in G$ written as D_s . We say that a set $J \subset G$ is an independence set for A relative to D if for every $F \Subset J$ and map $\phi : F \to \{1, \ldots, k\}$ we have $\bigcap_{s \in F} (D_s \cap s^{-1}A_{\phi(s)}) \neq \emptyset$. For a subset $D \subset X$, we say that J is an independence set for A relative to D if for every $F \Subset J$ and map $\phi : F \to \{1, \ldots, k\}$ we have $D \cap \bigcap_{s \in F} s^{-1}A_{\sigma(s)} \neq \emptyset$, i.e., if J is an independence set for A relative to the map $G \to 2^X$ with constant value D.

Let $A = (A_1, \ldots, A_k)$ be a tuple of subsets of X and $\delta > 0$. For every $F \subseteq G$ we define

$$\varphi_{\boldsymbol{A},\delta}(F) = \min_{D \in \mathcal{B}(\mu,\delta)} \max\left\{ |F \cap J| : J \text{ is an independence set for } \boldsymbol{A} \text{ relative to } D \right\}$$

The limit of $\frac{1}{|F|}\varphi_{\mathbf{A},\delta}(F)$ as F becomes more and more invariant might not exist. We define $\overline{I}_{\mu}(\mathbf{A},\delta)$ to be the limit supremum of $\frac{1}{|F|}\varphi_{\mathbf{A},\delta}(F)$ as F becomes more and more invariant. We refer to the quantity $\sup_{\delta>0} \overline{I}_{\mu}(\mathbf{A},\delta)$ as the **upper** μ -independence density of \mathbf{A} .

Definition 5.5. Let $G \cap X$ be an action, $\mu \in \mathcal{M}(G \cap X)$ and $k \ge 1$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is a μ -**IE-tuple** if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , we have that (U_1, \ldots, U_k) has positive upper μ -independence density. We denote the set of μ -IE-tuples of length k by $\mathbf{IE}_k^{\mu}(X, G)$.

The following result appeared in [61, Theorem 2.27].

Theorem 5.6 (Kerr and Li [61]). Let G be an amenable group, $G \cap X$ an action, $\mu \in \mathcal{M}(G \cap X)$, $k \ge 2$ and $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$. Then (x_1, \ldots, x_k) is a μ -entropy tuple if and only if it is a μ -IE-tuple.

For a proof of the following result see [61, Theorem 2.23].

Theorem 5.7 (Blanchard et al [11], Blanchard et al [9], Kerr and Li [61]). Let G be an amenable group, $G \curvearrowright X$ an action and $k \ge 2$. We have that

- 1. $\operatorname{IE}_{k}^{\mu}(X,G) \subset \operatorname{IE}_{k}(X,G)$ for every $\mu \in \mathcal{M}(G \curvearrowright X)$ and
- 2. there exists $\mu \in \mathcal{M}(G \curvearrowright X)$ such that $IE_k^{\mu}(X,G) = IE_k(X,G)$.

5.2 Measure sequence entropy tuples

Measure sequence entropy pairs were introduced by Huang, Maass and Ye [54].

Definition 5.8. Let $G \curvearrowright X$ be an action, $\mu \in \mathcal{M}(G \curvearrowright X)$ and $k \ge 2$. We say that $(x_1, ..., x_k) \in X^k \setminus \Delta_k(X)$ is a μ -sequence entropy pair if for any finite Borel partition \mathcal{P} such that $\{x_1, ..., x_k\} \not\subset \overline{\mathcal{P}}$ for every $P \in \mathcal{P}$, there exists a sequence S in G with $h^{\mathcal{L}}_{\mu}(G \curvearrowright X, \mathcal{P}) > 0$.

Definition 5.9. Let $G \curvearrowright X$ be an action and $\mu \in \mathcal{M}(G \curvearrowright X)$. For $A_1, \ldots, A_k \subset X$ and $\varepsilon > 0$ we say that (A_1, \ldots, A_k) has (ε, μ) -independence over arbitrarily large finite sets if there exists c > 0 such that for every N > 0 there is a finite set $F \subset G$ with |F| > N such that for every $E \in \mathcal{B}'(\mu, \varepsilon)$ there is an independence set $I \subset F$ for (A_1, \ldots, A_k) relative to E with $|I| \ge c|F|$.

Definition 5.10. Let $G \cap X$ be an action, $\mu \in \mathcal{M}(G \cap X)$ and $k \ge 1$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is a μ -**IN-tuple** if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} there exists $\varepsilon > 0$ such that (U_1, \ldots, U_k) has (ε, μ) -independence over arbitrarily large finite sets.

For a proof of the following theorem see [61, Theorem 4.9].

Theorem 5.11 (Kerr and Li [61]). Let $G \curvearrowright X$ be an action, $\mu \in \mathcal{M}(G \curvearrowright X)$ and $k \ge 2$. A tuple $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$ is a μ -IN-tuple if and only if it is a μ -sequence entropy tuple.

Let G be an amenable group, $\{F_n\}_{n\in\mathbb{N}}$ a Følner sequence for G and $S \subset G$. We define the upper density of S as

$$\overline{D}(S) = \limsup_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}.$$

Measure theoretic mean sensitivity is a strong form of sensitivity of initial conditions introduced in [35]. It is invariant under (measure) isomorphisms and related to discrete spectrum. This notion can be localized as follows. **Definition 5.12.** Let G be an amenable group, $\{F_n\}$ a Følner sequence for G, $G \cap X$ an action and $\mu \in \mathcal{M}(G \cap X)$. We say that $\mathbf{x} = (x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$ is a μ -mean sensitivity tuple if for every product neighborhood $U_1 \times \cdots \times U_k$ of \mathbf{x} , there exists $\varepsilon > 0$ such that for every Borel subset A of positive measure, there exist $p_1, \ldots, p_k \in A$ such that

$$\overline{D}(\{s \in G : sp_i \in U_i \ \forall i \in \{1, ..., k\}\}) > \epsilon.$$

Let $\{F_n\}$ be a Følner sequence of G. We say that $\{F_n\}_{n\in\mathbb{N}}$ is **tempered** if there exists C > 0 such that

$$\left|\bigcup_{i=1}^{n-1} F_i^{-1} F_n\right| \le C|F_n|, \text{ for all } n > 1.$$

Given a group action, we say a measure μ is **ergodic** if $\mu \in \mathcal{M}(G \cap X)$, and every G-invariant Borel set has full or null measure.

The following result appears in [38, Theorem 3.6].

Theorem 5.13 (García-Ramos and Muñoz-López [38]). Let G be an abelian group, $G \cap X$ an action, μ an ergodic measure and $(x, y) \in X^2 \setminus \Delta_2(X)$. The following are equivalent:

- 1. (x, y) is a μ -sequence entropy pair.
- 2. (x, y) is a μ -mean sensitivity pair with respect to a tempered Følner sequence.
- 3. (x, y) is a μ -mean sensitivity pair with respect to every tempered Følner sequence.

Remark 5.14. A consequence of the proof of the previous result is that for abelian groups and ergodic measures, one can define μ -IN-pairs using $\mathcal{B}(\mu, \delta)$ instead of $\mathcal{B}'(\mu, \delta)$ [38, Theorem 3.8].

The following related result [70, Theorem 1.3] was obtained independently using different techniques.

Theorem 5.15 (Li, Liu, Tu and Yu [70]). Let (X,T) be a TDS, μ an ergodic measure and $k \geq 2$. We have that $(x_1, \ldots, x_k) \in X^k \setminus \Delta_k(X)$ is a μ -sequence entropy tuple if and only if (x_1, \ldots, x_k) is a μ -mean sensitivity tuple (with respect to $F_n = \{1, \ldots, n\}$).

The previous result fails if the measure is not ergodic [70, Theorem 1.6].

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