INDEPENDENCE IN TOPOLOGICAL AND C^*-DYNAMICS

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Abstract. We develop a systematic approach to the study of independence in topological dynamics with an emphasis on combinatorial methods. One of our principal aims is to combinatorialize the local analysis of topological entropy and related mixing properties. We also reframe our theory of dynamical independence in terms of tensor products and thereby expand its scope to C^*-dynamics.

1. Introduction

The probabilistic notion of independence underlies several key concepts in ergodic theory as means for expressing randomness or indeterminism. Strong mixing, weak mixing, and ergodicity all capture the idea of asymptotic independence, the first in a strict sense and the second two in a mean sense. Moreover, positive entropy in $\mathbb{Z}$-systems is reflected via the Shannon-McMillan-Breiman theorem in independent behaviour along positive density subsets of iterates (see for example Section 3 of [36]).

One can also speak of independence in topological dynamics, in which case the issue is not the size of certain intersections as in the probabilistic context but rather the simple combinatorial fact of their nonemptiness. Although the topological analogues of strong mixing, weak mixing, and ergodicity and their relatives form the subject of an extensive body of research (stemming in large part from Furstenberg's seminal work on disjointness [28, 32]), a systematic approach to independence as a unifying concept for expressing and analyzing recurrence and mixing properties seems to be absent in the topological dynamics literature. The present paper aims to establish such an approach, with a particular emphasis on combinatorial arguments. At the same time we propose a recasting of the theory in terms of tensor products. This alternative formulation has the advantage of being applicable to general C^*-dynamical systems, in line with the principle in operator space theory that it is the tensor product viewpoint which typically enables the quantization of concepts from Banach space theory [24, 66].

In fact it is from within the theory of Banach spaces that the inspiration for our combinatorial approach to dynamical independence originates. In the Banach space context, independence at the dual level is associated with ℓ_1 structure, as was strikingly demonstrated by Rosenthal in the proof of his characterization of Banach spaces containing ℓ_1 isomorphically [70]. Rosenthal's groundbreaking ℓ_1 theorem initiated a line of research based in Ramsey methods that led to the work of Bourgain, Fremlin, and Talagrand on pointwise compact sets of Baire class one functions [13] (see [38, 75] for general references). The transfer of these ideas to the dynamical realm was initiated by Köhler, who applied

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the Bourgain-Fremlin-Talagrand dichotomy for spaces of Baire class one functions to obtain a corresponding statement for enveloping semigroups of continuous interval maps [54]. This dynamical Bourgain-Fremlin-Talagrand dichotomy was recently extended by Glasner and Megrelishvili to general metrizable systems [34] (see also [33]). The dichotomy hinges on the notion of tameness, which was introduced by Köhler (under the term regularity) and refers to the absence of orbits of continuous functions which contain an infinite subset equivalent to the standard basis of ℓ_1.

The link between topological entropy and ℓ_1 structure via coordinate density was discovered by Glasner and Weiss, who proved using techniques from the local theory of Banach spaces that if a homeomorphism of a compact metric space X has zero entropy then so does the induced weak* homeomorphism of the space of probability measures on X [36]. This connection to Banach space geometry was pursued in [48] with applications to Voiculescu-Brown entropy in C^*-dynamics and then in [49] within a general Banach space framework. It was shown in [49] that functions in the topological Pinsker algebra can be described by the property that their orbits do not admit a positive density subset equivalent to the standard basis of ℓ_1. While tameness is concerned with infinite subsets of orbits with no extra structure and thereby calls for the application of Ramsey methods, the positive density condition in the case of entropy reflects a tie to quantitative results in the local theory of Banach spaces involving the Sauer-Perles-Shelah lemma and Hilbertian geometry. What is common to both cases is the dynamical appearance of ℓ_1 as a manifestation of combinatorial independence. How this link between linear geometry and combinatorial structure plays out in the study of local dynamical behaviour is in general not so straightforward however, and a major goal of this paper is to understand the local situation from a combinatorial standpoint.

Over the last ten years a substantial local theory of topological entropy for $\mathbb{Z}$-systems has unfolded around the concept of entropy pair introduced by Blanchard in [9]. Remarkably, every significant result to date involving entropy pairs has been obtained using measure-dynamical techniques by way of a variational principle. This has raised the question of whether more direct topological-combinatorial arguments can be found (see for example [30]). Applying a local variational principle, Huang and Ye have recently obtained a characterization of entropy pairs, and more generally of entropy tuples, in terms of an independence property [43]. We will give a combinatorial proof of this result in Section 3 with a key coordinate density lemma inspired by work of Mendelson and Vershynin [59]. Our argument has the major advantage of portability and provides the basis for a versatile combinatorial approach to the local analysis of entropy. It works equally well for noninvertible surjective continuous maps and actions of discrete amenable groups, applies to sequence entropy where no variational principle exists (Section 5), and is potentially of use in the study of Banach space geometry. The tuples of points enjoying the independence property relevant to entropy we call IE-tuples, and in analogy we also define IN-tuples and IT-tuples as tools for the local study of sequence entropy and tameness, respectively. While positive entropy is keyed to independent behaviour along positive density subsets of iterates, what matters for positive sequence entropy and untameness are independence along arbitrarily large finite subsets and independence along infinite subsets, respectively (see Sections 5 and 6). We investigate how these local independence properties are interrelated at the global level (Section 8), as well as how their quantizations are connected to various types of asymptotic Abelianness in C^*-dynamical systems (Section 9).

We begin the main body of the paper in Section 2 by laying down the general notation and definitions that will form the groundwork for subsequent discussions. Our setting will be that of topological semigroups with identity acting continuously on compact Hausdorff spaces by surjective continuous maps, or strongly continuously on unital C^*-algebras by injective unital *-endomorphisms, with notable specializations to singly generated systems in Section 3 (except for the last part on actions of discrete amenable groups) and Section 4 and to actions of groups in Section 7 (where the main results are in fact for the Abelian case), the second half of Section 9, and Section 10. In Section 3 we introduce the notion of IE-tuple and establish several basic properties in parallel with those of entropy tuples, including behaviour under taking products, which we prove by measure and density arguments in a 0–1 product space. We then argue that entropy tuples are IE-tuples by applying the key coordinate density result which appears as Lemma 3.3. In particular, we recover the result of Glasner on entropy pairs in products [30] without having invoked a variational principle. Using IE-tuples we give an alternative proof of the fact due to Blanchard, Glasner, Kolyada, and Maass [11] that positive entropy implies Li-Yorke chaos. Our arguments show moreover that the set of entropy pairs contains a dense subset consisting of Li-Yorke pairs. To conclude Section 3 we discuss how the theory readily extends to actions of discrete amenable groups. Note in contrast that the measure-dynamical approach as it has been developed for $\mathbb{Z}$-systems does not directly extend to the general amenable case, as one needs for example to find a substitute for the procedure of taking powers of a single automorphism (see [43] and Section 19.3 of [31]).

We continue our discussion on entropy in Section 4 by shifting to the tensor product perspective and determining how concepts like uniformly positive entropy translate, with a view towards formulating the notion of a C^*-algebraic K-system.

In Section 5 we define IN-tuples as the analogue of IE-tuples for topological sequence entropy. We give a local description of nullness (i.e., the vanishing of sequence entropy over all sequences) at the Banach space level in terms of IN-pairs and show that nondiagonal IN-pairs are the same as sequence entropy pairs as defined in [41]. Here our combinatorial approach is essential, as there is no variational principle for sequence entropy.

Section 6 concerns independence in the context of tameness. We define IT-tuples and establish several properties in relation to untameness. While nullness implies tameness, we illustrate with a WAP subshift example that the converse is false. Section 7 investigates tame extensions of minimal systems. Under the hypothesis of an Abelian acting group, we establish in Theorem 7.15 a proximal-equicontinuous decomposition that answers a question of Glasner from [33] and show in Theorem 7.19 that tame minimal systems are uniquely ergodic. These theorems generalize results from [41] which cover the metrizable null case.

In Section 8 we define I-independence as a tensor product property that may be thought of as a C^*-dynamical analogue of measure-theoretic weak mixing. Theorem 8.6 asserts that, for systems on compact Hausdorff spaces, I-independence is equivalent to uniform untameness (resp. uniform nonnullness) of all orders and the weak mixing (resp. transitivity) of all of the n-fold product systems, and, in the case of an Abelian acting group, to untameness, nonnullness, and weak mixing. We also demonstrate that, for general C^*-dynamical systems, I-independence implies complete untameness.

Section 9 focuses on independence in the noncommutative context. For dynamics on simple unital nuclear C^*-algebras, we show that independence essentially amounts to Abelianness. It is also observed that in certain situations the existence of a faithful weakly mixing state implies independence along a thickly syndetic set. In the opposite direction, we prove in Section 10 that I-independence in the setting of a UHF algebra implies weak mixing for the unique tracial state. Moreover, for Bogoliubov actions on the even CAR algebra, I-independence is actually equivalent to weak mixing for the unique tracial state. Continuing with the theme of UHF algebras, we round out Section 10 by showing that, in the type d^∞ case, I-independence for *-automorphisms is point-norm generic.

In the final two sections we construct an example of a tame nonnull Toeplitz subshift (Section 11) and prove that the action of a convergence group is null (Section 12).

After this paper was finished Wen Huang informed us that he has shown that every tame minimal action of an Abelian group on a compact metrizable space is a highly proximal extension of an equicontinuous system and is uniquely ergodic [40]. Corollary 7.16 and Theorem 7.19 in our paper strengthen these results from the perspective of our geometric formulation of tameness. We also remark that our paper answers the last three questions in [40], the first negatively and the second two positively.

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2. General notation and basic notions

By a *dynamical system* we mean a pair (X, G) where X is a compact Hausdorff space and G is a topological semigroup with identity with a continuous action $(s, x) \mapsto sx$ on X by surjective continuous maps. By a *C^*-dynamical system* we mean a triple (A, G, α) where A is a unital C^*-algebra, G is a topological semigroup with identity, and α is an action $(s, a) \mapsto \alpha_s(a)$ of G on A by injective unital *-endomorphisms. The identity of G will be written e. We denote by G_0 the set $G \setminus \{e\}$.

A dynamical system (X, G) gives rise to an action α of the opposite semigroup $G^{\rm op}$ on $C(X)$ defined by $\alpha_s(f)(x) = (sf)(x) = f(sx)$ for all $s \in G^{\rm op}$, $f \in C(X)$, and $x \in X$, using the same notation for corresponding elements of G and $G^{\rm op}$. Whenever we define a property for C^*-dynamical systems we will also speak of the property for dynamical systems and surjective continuous maps by applying the definition to this associated C^*-dynamical system. If a C^*-dynamical system is defined by a single *-endomorphism then we will talk about properties of the system as properties of the *-endomorphism whenever convenient, with a similar comment applying to singly generated dynamical systems.

For a semigroup with identity, we write $\mathcal{N}$ for the collection of nonempty subsets of G_0, $\mathcal{I}$ for the collection of infinite subsets of G_0, and $\mathcal{TS}$ for the collection of thickly syndetic subsets of G_0. Recall that a subset K of G is *syndetic* if there is a finite subset F of G such that $FK = G$ and *thickly syndetic* if for every finite subset F of G the set $\bigcap_{s \in F} sK$

is syndetic. When $G = \mathbb{Z}$ we say that a subset $I \subseteq G$ has *positive density* if the limit

$$\lim_{n\to\infty} \frac{|I \cap \{-n, -n+1, \dots, n\}|}{2n+1}$$

exists and is nonzero. When $G = \mathbb{Z}_{\geq 0}$ we say that a subset $I \subseteq G$ has *positive density* if the limit

$$\lim_{n\to\infty} \frac{|I \cap \{0, 1, \dots, n\}|}{n+1}$$

exists and is nonzero.

Given a C^*-dynamical system (A, G, α) and an element $a \in A$, a subset $I \subseteq G$ is said to be an *ℓ_1-isomorphism set* for a if the set $\{\alpha_s(a)\}_{s\in I}$ is equivalent to the standard basis of ℓ_1^I. If this a λ-equivalence for some $\lambda \geq 1$ then we also refer to I as an *ℓ_1-λ-isomorphism set.* These definitions also make sense more generally for actions on Banach spaces by isometric endomorphisms. For a dynamical system (X, G) we will speak of ℓ_1-isomorphism sets for elements of $C(X)$ in reference to the induced action of G^{op} described above.

For a dynamical system (X, G) we denote by $M(X)$ the weak* compact convex set of Borel probability measures on X and by $M(X, G)$ the weak* closed convex subcollection of G-invariant Borel probability measures.

For subsets Y and Z of a metric space (X, d) and $\varepsilon > 0$ we write $Y \subseteq_\varepsilon Z$ and say that Z approximately includes Y to within ε if for every $y \in Y$ there is a $z \in Z$ with $d(z, y) < \varepsilon$. For a set X and an $m \in \mathbb{N}$ we write $\Delta_m(X) = \{(x, \dots, x) \in X^m : x \in X\}$. An element of X^m which is not contained in $\Delta_m(X)$ is said to be *nondiagonal.* The linear span of a set Ω of elements in a linear space will often be written $[\Omega]$.

A partition of unity $\{f_1, \dots, f_n\}$ of a compact Hausdorff space X is said to be *effective* if $\max_{x\in X} f_i(x) = 1$ for each $i = 1, \dots, n$, which is equivalent to the existence of $x_1, \dots, x_n \in X$ such that $f_i(x_j) = \delta_{ij}$. In this case the linear map $\operatorname{span}\{f_1, \dots, f_n\} \to \mathbb{C}^n$ given by evaluation at the points $x_1, \dots, x_n$ is an isometric order isomorphism. See Section 8 of [7] for more information on order structure and finite-dimensional approximation in commutative C^*-algebras.

An *operator space* is a closed subspace of a C^*-algebra, or equivalently of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The distinguishing characteristic of an operator space is its collection of matrix norms, in terms of which one can formulate an abstract definition, the equivalence of which with the concrete definition is a theorem of Ruan. A linear map $\varphi : V \to W$ between operator spaces is said to be *completely bounded* if $\sup_{n\in\mathbb{N}} \|\mathrm{id}_{M_n} \otimes \varphi\| < \infty$, in which case we refer to this supremum as the c.b. (completely bounded) norm, written $\|\varphi\|_{\mathrm{cb}}$. We say that a map $\varphi : V \to W$ between operator spaces is a *λ-c.b.-isomorphism* if it is invertible and the c.b. norms of φ and φ^{-1} satisfy $\|\varphi\|_{\mathrm{cb}}\|\varphi^{-1}\|_{\mathrm{cb}} \leq \lambda$. The minimal tensor product of operator spaces $V \subseteq \mathcal{B}(\mathcal{H})$ and $W \subseteq \mathcal{B}(\mathcal{K})$, written $V \otimes W$, is the closure of the algebraic tensor product of V and W under its canonical embedding into $\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$. When applied to closed subspaces of commutative C^*-algebras, the minimal operator space tensor product is the same as the Banach space injective tensor product (ignoring the matricial data).

An *operator system* is a closed unital self-adjoint subspace of a unital C^*-algebra. Let V be an operator system and I, I' nonempty finite sets with $I \subseteq I'$. We regard $V^{\otimes I}$ as an operator subsystem of $V^{\otimes I'}$ under the complete order embedding given by $v \mapsto v \otimes 1 \in$

$V^{\otimes I} \otimes V^{\otimes I' \setminus I} = V^{\otimes I'}$. With respect to such inclusions we define $V^{\otimes J}$ for any index set J as a direct limit over the finite subsets of J. For general references on operator spaces and operator systems see [24, 66].

A collection $\{(A_{i,0}, A_{i,1})\}_{i \in I}$ of pairs of disjoint subsets of a set X is said to be *independent* if for every finite set $F \subseteq I$ and $\sigma \in \{0,1\}^F$ we have $\bigcap_{i \in F} A_{i,\sigma(i)} \neq \emptyset$.

Our basic concept of dynamical independence and its quantization are given by the following definitions.

Definition 2.1. Let (X, G) be a dynamical system. For a tuple $\boldsymbol{A} = (A_1, A_2, \dots, A_k)$ of subsets of X, we say that a set $J \subseteq G$ is an *independence set* for $\boldsymbol{A}$ if for every nonempty finite subset $I \subseteq J$ and function $\sigma : I \to \{1, 2, \dots, k\}$ we have $\bigcap_{s \in J} s^{-1} A_{\sigma(s)} \neq \emptyset$, where $s^{-1}A$ for a set $A \subseteq X$ refers to the inverse image $\{x \in X : sx \in A\}$.

Definition 2.2. Let (A, G, α) be a C^*-dynamical system, and let V be a finite-dimensional operator subsystem of A. Associated to every tuple $(s_1, \dots, s_k)$ of elements of G is the linear dynamical multiplication map $V^{\otimes[1,k]} \to A$ determined on elementary tensors by $a_1 \otimes \cdots \otimes a_k \mapsto \alpha_{s_1}(a_1) \cdots \alpha_{s_k}(a_k)$. For $\lambda \geq 1$, we say that a tuple of elements of G is a λ-*contraction tuple* for V if the associated multiplication map has c.b. norm at most λ, a λ-*expansion tuple* for V if the multiplication map has an inverse with c.b. norm at most λ, and a λ-*independence tuple* for V if the multiplication map is a λ-c.b.-isomorphism onto its image. A subset $J \subseteq G$ is said to be a λ-*contraction set*, λ-*expansion set*, or λ-*independence set* if every tuple of distinct elements in J is of the corresponding type.

Remark 2.3. If A is a commutative C^*-algebra, then for each of the linear maps in Definition 2.2 the norm and c.b. norm coincide. Moreover the linear map $V^{\otimes[1,k]} \to A$ is contractive and thus φ is a λ-isomorphism onto its image if and only if it has a bounded inverse of norm at most λ. Thus in this case a λ-expansion tuple (resp. set) is the same as a λ-independence tuple (resp. set).

Notice that if a tuple $(s_1, \dots, s_k)$ of elements of G is a λ-expansion tuple for a finite-dimensional operator subsystem $V \subseteq A$ with λ close to one, then the associated multiplication map φ gives rise to an operator space matrix norm on $V^{\otimes[1,k]}$ which is close to being a cross norm in the sense that for all $a_1, \dots, a_k \in V$ we have

$$\begin{aligned} \lambda^{-1}\|a_1\| \cdots \|a_k\| = \lambda^{-1}\|a_1 \otimes \cdots \otimes a_k\| &\leq \|\varphi(a_1 \otimes \cdots \otimes a_k)\| \\ &= \|\alpha_{s_1}(a_1) \cdots \alpha_{s_k}(a_k)\| \leq \|a_1\| \cdots \|a_k\|. \end{aligned}$$

Definition 2.4. Let (A, G, α) be a C^*-dynamical system, and let V be a finite-dimensional operator subsystem of A. For $\varepsilon \geq 0$, we define $\mathrm{Con}(\alpha, V, \varepsilon)$ to be the set of all $s \in G_0$ such that (e, s) is a $(1+\varepsilon)$-contraction tuple, $\mathrm{Exp}(\alpha, V, \varepsilon)$ to be the set of all $s \in G_0$ such that (e, s) is a $(1+\varepsilon)$-expansion tuple, and $\mathrm{Ind}(\alpha, V, \varepsilon)$ to be the set of all $s \in G_0$ such that (e, s) is a $(1+\varepsilon)$-independence tuple. For a collection $\mathcal{C}$ of subsets of G_0 which is closed under taking supersets, we say that the system (A, G, α) or the action α is $\mathcal{C}$-*contractive* if for every finite-dimensional operator subsystem $V \subseteq A$ and $\varepsilon > 0$ the set $\mathrm{Con}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$, $\mathcal{C}$-*expansive* if the same criterion holds with respect to the set $\mathrm{Exp}(\alpha, V, \varepsilon)$, and $\mathcal{C}$-*independent* if the criterion holds with respect to the set $\mathrm{Ind}(\alpha, V, \varepsilon)$.

We will mainly be interested in applying Definition 2.4 to the collections $\mathcal{N}$, $\mathcal{I}$, and $\mathcal{TS}$ as defined above. Note that, by Remark 2.3, for dynamical systems $\mathcal{C}$-contractivity is automatic and $\mathcal{C}$-expansivity and $\mathcal{C}$-independence amount to the same thing.

To verify $\mathcal{C}$-contractivity, $\mathcal{C}$-expansivity, or $\mathcal{C}$-independence it suffices to check the condition in question over a collection of operator subsystems with dense union. This fact is recorded in Proposition 2.6 and rests on the following perturbation lemma, which is a slight variation on Lemma 2.13.2 of [66] with essentially the same proof.

Lemma 2.5. *Let V be a finite-dimensional operator space with Auerbach system $\{(v_i, f_i)\}_{i=1}^n$ and let $\varepsilon > 0$ be such that $\varepsilon(1+\varepsilon) < 1$. Let W be an operator space, $\rho : V \to W$ a linear map which is an isomorphism onto its image with $\max(\|\rho\|_{\mathrm{cb}}, \|\rho^{-1}\|_{\mathrm{cb}}) < 1+\varepsilon$, and $w_1, \dots, w_n$ elements of W such that $\|\rho(v_i) - w_i\| < \dim(V)^{-1}\varepsilon$ for each $i = 1, \dots, n$. Then the linear map $\varphi : V \to W$ determined by $\varphi(v_i) = w_i$ for $i = 1, \dots, n$ is an isomorphism onto its image with $\|\varphi\|_{\mathrm{cb}} < 1 + 2\varepsilon$ and $\|\varphi^{-1}\|_{\mathrm{cb}} < (1 + \varepsilon)(1 - \varepsilon(1 + \varepsilon))^{-1}$.*

Proof. Define the linear map $\delta : V \to W$ by $\delta(v) = \sum_{i=1}^n f_i(v)(w_i - \rho(v_i))$ for all $v \in V$. Since the norm and c.b. norm of a rank-one linear map coincide, we have $\|\delta\|_{\mathrm{cb}} \leq \sum_{i=1}^n \|f_i\|\|w_i - \rho(v_i)\| < \varepsilon$, and thus since $\varphi = \rho + \delta$ we see that $\|\varphi\|_{\mathrm{cb}} < 1 + 2\varepsilon$. The bound $\|\varphi^{-1}\|_{\mathrm{cb}} < (1 + \varepsilon)(1 - \varepsilon(1 + \varepsilon))^{-1}$ follows by applying Lemma 2.13.1 of [66]. □

Proposition 2.6. *Let (A, G, α) be a C^*-dynamical system. Let $\mathfrak{S}$ be a collection of finite-dimensional operator subsystems of A with the property that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is a $V \in \mathfrak{S}$ such that $\Omega \subseteq_\varepsilon V$. Let $\mathcal{C}$ be a collection of subsets of G_0 which is closed under taking supersets. Then α is $\mathcal{C}$-contractive if and only if for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set $\mathrm{Con}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$, $\mathcal{C}$-expansive if and only if for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set $\mathrm{Exp}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$, and $\mathcal{C}$-independent if and only if for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set $\mathrm{Ind}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$.*

Proof. We will show the third equivalence, the first two involving similar perturbation arguments. For the nontrivial direction, suppose that for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set $\mathrm{Ind}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$. Let V be a finite-dimensional operator subsystem of A and let $\varepsilon > 0$. Let δ be a positive real number to be further specified below. By assumption we can find a $W \in \mathfrak{S}$ such that the set $\mathrm{Ind}(\alpha, W, \delta)$ is a member of $\mathcal{C}$ and the unit ball of V is approximately included to within δ in W. By Lemma 2.5, if δ is sufficiently small then, taking an Auerbach basis $\mathcal{S} = \{v_i\}_{i=1}^r$ for V and choosing $w_1, \dots, w_r \in W$ with $\|w_i - v_i\| < \delta$ for each $i = 1, \dots r$, the linear map $\rho : V \to W$ determined on $\mathcal{S}$ by $\rho(v_i) = w_i$ is an isomorphism onto its image with $\|\rho\|_{\mathrm{cb}}\|\rho^{-1}\|_{\mathrm{cb}} < \sqrt[4]{1+\varepsilon}$. Now let $s \in \mathrm{Ind}(\alpha, W, \delta)$. Let $\varphi : W \otimes W \to A$ be the dynamical multiplication map determined on elementary tensors by $w_1 \otimes w_2 \mapsto w_1\alpha_s(w_2)$. Consider the linear map $\theta : [\rho(V)\alpha_s(\rho(V))] \to [V\alpha_s(V)]$ determined by $\rho(v_1)\alpha_s(\rho(v_2)) \mapsto v_1\alpha_s(v_2)$ for $v_1, v_2 \in V$, which is well defined by our choice of ρ and s. Another application of Lemma 2.5 shows that if δ is small enough then the composition $\theta \circ \varphi$ is an isomorphism onto its image satisfying $\|\theta \circ \varphi\|_{\mathrm{cb}}\|(\theta \circ \varphi)^{-1}\|_{\mathrm{cb}} < \sqrt{1+\varepsilon}$. Notice now that the dynamical multiplication map $\psi : V \otimes V \to A$ determined on elementary tensors by $v_1 \otimes v_2 \mapsto v_1\alpha_s(v_2)$ factors as $\theta \circ \varphi \circ (\rho \otimes \rho)$, and hence

$$\|\psi\|_{\mathrm{cb}}\|\psi^{-1}\|_{\mathrm{cb}} \leq \|\theta \circ \varphi\|_{\mathrm{cb}}\|(\theta \circ \varphi)^{-1}\|_{\mathrm{cb}}\|\rho\|_{\mathrm{cb}}^2\|\rho^{-1}\|_{\mathrm{cb}}^2 \leq 1 + \varepsilon.$$

Thus $s \in \mathrm{Ind}(\alpha, V, \varepsilon)$, and we conclude that α is $\mathcal{C}$-independent. □

3. Topological entropy and combinatorial independence

The local theory of topological entropy based on entropy pairs is developed in the literature for $\mathbb{Z}$-systems, but here will we consider general continuous surjective maps. In fact one of the novel features of our combinatorial approach is that it applies not only to singly generated systems but also to actions of any discrete amenable group, as we will indicate in the last part of the section. Thus, with the exception of the last part of the section, G will be one of the additive semigroups $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ and we will denote the generating surjective endomorphism of X by T.

Recall that the topological entropy $\mathrm{h}_{\mathrm{top}}(T, \mathcal{U})$ of an open cover $\mathcal{U}$ of X with respect to T is defined as $\lim_{n\to\infty} \frac{1}{n} \ln N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U})$, where $N(\cdot)$ denotes the minimal cardinality of a subcover. The topological entropy $\mathrm{h}_{\mathrm{top}}(T)$ of T is the supremum of $\mathrm{h}_{\mathrm{top}}(T, \mathcal{U})$ over all open covers $\mathcal{U}$ of X. A pair $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ is said to be an *entropy pair* if whenever U_1 and U_2 are closed disjoint subsets of X with $x_1 \in \mathrm{int}(U_1)$ and $x_2 \in \mathrm{int}(U_2)$, the open cover $\{U_1^{\mathrm{c}}, U_2^{\mathrm{c}}\}$ has positive topological entropy. More generally, following [35] we call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k \setminus \Delta_k(X)$ an *entropy tuple* if whenever $U_1, \dots, U_l$ are closed pairwise disjoint neighbourhoods of the distinct points in the list $x_1, \dots, x_k$, the open cover $\{U_1^{\mathrm{c}}, \dots, U_l^{\mathrm{c}}\}$ has positive topological entropy.

Definition 3.1. We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ an *IE-tuple* (or an *IE-pair* in the case $k = 2$) if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the tuple $(U_1, \dots, U_k)$ has an independence set of positive density. We denote the set of IE-tuples of length k by $\mathrm{IE}_k(X, G)$.

The argument in the second paragraph of the proof of Theorem 3.2 in [36] shows the following lemma, which will be repeatedly useful for converting finitary density statements to infinitary ones.

Lemma 3.2. *A tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of X has an independence set of positive density if and only if there exists a $d > 0$ such that for any $M > 0$ we can find an interval I in G with $|I| \geq M$ and an independence set J for $\boldsymbol{A}$ contained in I for which $|J| \geq d|I|$.*

On page 684 of [36] a pair $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ is defined to be an *E-pair* if for any neighbourhoods U_1 and U_2 of x_1 and x_2, respectively, there exists a $\delta > 0$ and a k_0 such that for every $k \geq k_0$ there exists a sequence $0 \leq n_1 < n_2 < \cdots < n_k < k/\delta$ such that $\bigcap_{j=1}^k T^{-n_j}(U_{\sigma(j)}) \neq \emptyset$ for every $\sigma \in \{1, 2\}^k$. From Lemma 3.2 we see that E-pairs are the same as nondiagonal IE-pairs.

We now proceed to establish some facts concerning the set of IE-pairs as captured in Propositions 3.9, 3.10, and Theorem 3.15. For $\mathbb{Z}$-systems these can be proved via a local variational principle route by combining the known analogues for entropy pairs (see [9, 31, 49]) with Huang and Ye's characterization (in different terminology) of entropy pairs as IE-pairs [43], which itself will be reproved and extended to cover noninvertible surjective continuous maps in Theorem 3.16 (see also the end of the section for actions of discrete amenable groups).

Let $k \geq 2$ and let Z be a nonempty finite set. Let $\mathcal{U}$ be the cover of $\{0, 1, \dots, k\}^Z = \prod_{z\in Z}\{0, 1, \dots, k\}$ consisting of subsets of the form $\prod_{z\in Z}\{i_z\}^{\mathrm{c}}$, where $1 \leq i_z \leq k$ for each $z \in Z$. For $S \subseteq \{0, 1, \dots, k\}^Z$ we write F_S to denote the minimal number of sets in $\mathcal{U}$ one needs to cover S.

The following result plays a key role in our combinatorial approach to the study of IE-tuples (and IN-tuples in Section 5). The idea of considering the property (i) in its proof comes from the proof of Theorem 4 in [59].

Lemma 3.3. *Let $k \geq 2$ and let $b > 0$ be a constant. There exists a constant $c > 0$ depending only on k and b such that for every finite set Z and $S \subseteq \{0, 1, \dots, k\}^Z$ with $F_S \geq k^{b|Z|}$ there exists a $W \subseteq Z$ with $|W| \geq c|Z|$ and $S|_W \supseteq \{1, \dots, k\}^W$.*

Proof. Pick a constant $0 < \lambda < \frac{1}{3}$ such that $b_1 := b + \log_k(1-\lambda) > 0$. Set $b_2 := \log_k\left(\frac{1-\lambda}{\lambda}\right) > 0$ and $t = (2b_2)^{-1} b_1 \log_2\left(\frac{k+1}{k}\right)$.

Denote by H_S the number of non-empty subsets W of Z such that $S|_W \supseteq \{1, \dots, k\}^W$. By Stirling's formula there is a constant $c > 0$ depending only on t (and hence depending only on k and b) such that $\sum_{1 \leq j \leq cn} \binom{n}{j} < 2^{tn}$ for all n large enough. If $H_S \geq 2^{t|Z|}$ and $|Z|$ is large enough, then there exists a $W \subseteq Z$ for which $|W| \geq c|Z|$ and $S|_W \supseteq \{1, \dots, k\}^W$. It thus suffices to show that $H_S \geq 2^{t|Z|}$.

Set $S_0 = S$ and $Z_0 = Z$. We shall construct $Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_m$ for $m := \lceil t|Z| / \log_2 \frac{k+1}{k} \rceil$ and $S_j \subseteq \{0, 1, \dots, k\}^{Z_j}$ for all $1 \leq j \leq m$ with the following properties:

(i) $H_{S_{j-1}} \geq \frac{k+1}{k} H_{S_j}$ for all $1 \leq j \leq m$,
(ii) $F_{S_j} \geq k^{b|Z|}(1-\lambda)^{|Z \setminus Z_j| - j} \lambda^j$ for all $0 \leq j \leq m$.

Suppose that we have constructed $Z_0, \dots, Z_j$ and $S_0, \dots, S_j$ with the above properties for some $0 \leq j < m$. If we have a $Q \subseteq Z_j$ and a $\sigma \in \{1, \dots, k\}^{Z_j \setminus Q}$ such that $F_{S_{j,\sigma}} \geq (1-\lambda)^{|Z_j \setminus Q|} F_{S_j}$, where $S_{j,\sigma}$ is the restriction of $\{f \in S_j : f(x) \neq \sigma(x) \text{ for all } x \in Z_j \setminus Q\}$ on Q, then

$$\begin{aligned}
|Q| &\geq \log_k(F_{S_{j,\sigma}}) \\
&\geq (|Z_j \setminus Q|) \log_k(1-\lambda) + \log_k(F_{S_j}) \\
&\geq (|Z_j \setminus Q| + |Z \setminus Z_j| - j) \log_k(1-\lambda) + b|Z| + j \log_k \lambda \\
&= (|Z| - |Q| - j)(b + \log_k(1-\lambda)) + b(|Q| + j) + j \log_k \lambda \\
&= (|Z| - |Q|) b_1 + b|Q| - j b_2 \\
&\geq (|Z| - |Q|) b_1 + b|Q| - b_2 \Big(1 + t|Z| / \log_2 \Big(\frac{k+1}{k}\Big)\Big) \\
&= (|Z| - |Q|) b_1 + b|Q| - b_2 - \frac{b_1}{2}|Z|
\end{aligned}$$

and hence $|Q| \geq \frac{|Z| b_1/2 - b_2}{1 + b_1 - b} \geq 2$ when $|Z|$ is large enough. Take Q and σ as above such that $|Q|$ is minimal. Then $|Q| \geq 2$. Pick a $z \in Q$, and set $S_{j,i}$ to be the restriction of $\{f \in S_{j,\sigma} : f(z) = i\}$ to $Z_{j+1} := Q \setminus \{z\}$ for $i = 1, \dots, k$. Then

$$F_{S_{j,i}} \geq \lambda(1-\lambda)^{|Z_j \setminus Q|} F_{S_j} \geq k^{b|Z|}(1-\lambda)^{|Z \setminus Z_{j+1}| - (j+1)} \lambda^{j+1}$$

for $i = 1, \dots, k$ (here one needs the fact that $|Q| \geq 2$). Now take S_{j+1} to be one of the sets among $S_{j,1}, \dots, S_{j,k}$ with minimal H-value, say $S_{j,l}$. For each $1 \leq i \leq k$ denote by B_i the set of nonempty subsets $W \subseteq Z_{j+1}$ such that $S_{j,i}|_W \supseteq \{1, \dots, k\}^W$. Note that

$H_{S_j} \geq |\bigcup_{i=1}^k B_i| + |\bigcap_{i=1}^k B_i|$. If $|\bigcup_{i=1}^k B_i| \geq \frac{k+1}{k}|B_l|$, then $H_{S_j} \geq \frac{k+1}{k}|B_l| = \frac{k+1}{k}H_{S_{j+1}}$. Suppose that $|\bigcup_{i=1}^k B_i| < \frac{k+1}{k}|B_l|$. Note that

$$\left|\bigcap_{i=1}^k B_i\right| \cdot k + \left(\left|\bigcup_{i=1}^k B_i\right| - \left|\bigcap_{i=1}^k B_i\right|\right)(k-1) \geq \sum_{i=1}^k |B_i| \geq k|B_l|.$$

Thus

$$\left|\bigcap_{i=1}^k B_i\right| \geq k|B_l| - (k-1)\left|\bigcup_{i=1}^k B_i\right| \geq k|B_l| - (k-1)\cdot\frac{k+1}{k}|B_l| = \frac{1}{k}|B_l|.$$

Therefore $H_{S_j} \geq |\bigcup_{i=1}^k B_i| + |\bigcap_{i=1}^k B_i| \geq |B_l| + \frac{1}{k}|B_l| = \frac{k+1}{k}H_{S_{j+1}}$. Hence the properties (i) and (ii) are also satisfied for $j+1$.

A simple calculation shows that $k^{b|Z|}(1-\lambda)^{|Z\setminus Z_m|-m}\lambda^m \geq k^{b|Z|}(1-\lambda)^{|Z|-m}\lambda^m > 1$ when $|Z|$ is large enough. Thus $F_{S_m} > 1$ according to property (ii) and hence $H_{S_m} \geq 1$. By property (i) we have $H_S \geq (\frac{k+1}{k})^m H_{S_m} \geq (\frac{k+1}{k})^m \geq 2^{t|Z|}$. This completes the proof of the proposition. $\square$

For a cover $\mathcal{U}$ of X we denote by $\mathrm{h_c}(T,\mathcal{U})$ the combinatorial entropy of $\mathcal{U}$ with respect to T, which is defined using the same formula as for the topological entropy of open covers.

Lemma 3.4. *Let $k \geq 2$. Let $U_1, \dots, U_k$ be pairwise disjoint subsets of X and set $\mathcal{U} = \{U_1^{\mathrm{c}}, \dots, U_k^{\mathrm{c}}\}$. Then $\boldsymbol{U} := (U_1, \dots, U_k)$ has an independence set of positive density if and only if $\mathrm{h_c}(T,\mathcal{U}) > 0$.*

Proof. The "only if" part is trivial. For the "if" part, set $b := \mathrm{h_c}(X,\mathcal{U})$ and consider the map $\varphi_n : X \to \{0,1,\dots,k\}^{\{1,\dots,n\}}$ defined by

$$(\varphi_n(x))(j) = \begin{cases} i, & \text{if } T^j(x) \in U_i \text{ for some } 1 \leq i \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $N\big(\bigvee_{i=1}^n T^{-i}\mathcal{U}\big) = F_{\varphi_n(X)}$, and so $F_{\varphi_n(X)} > e^{\frac{b}{2}n}$ for all large enough n. By Lemma 3.3 there exists a constant $c > 0$ depending only on k and b such that $\varphi_n(X)|_W \supseteq \{1,\dots,k\}^W$ for some $W \subseteq \{1,\dots,n\}$ with $|W| \geq cn$ when n is large enough. Then W is an independence set for the tuple $\boldsymbol{U}$. Thus by Lemma 3.2 $\boldsymbol{U}$ has an independence set of positive density. $\square$

We will need the following consequence of Karpovsky and Milman's generalization of the Sauer-Perles-Shelah lemma [71, 72, 45]. It also follows directly from Lemma 3.3.

Lemma 3.5 ([45])**.** *Given $k \geq 2$ and $\lambda > 1$ there is a constant $c > 0$ such that, for all $n \in \mathbb{N}$, if $S \subseteq \{1,2,\dots,k\}^{\{1,2,\dots,n\}}$ satisfies $|S| \geq ((k-1)\lambda)^n$ then there is an $I \subseteq \{1,2,\dots,n\}$ with $|I| \geq cn$ and $S|_I = \{1,2,\dots,k\}^I$.*

The case $|Z| = 1$ of the following lemma appeared in [63].

Lemma 3.6. *Let Z be a finite set such that $Z \cap \{1,2,3\} = \emptyset$. There exists a constant $c > 0$ depending only on $|Z|$ such that, for all $n \in \mathbb{N}$, if $S \subseteq (Z \cup \{1,2\})^{\{1,2,\dots,n\}}$ is such that $\Gamma_n|_S : S \to (Z \cup \{3\})^{\{1,2,\dots,n\}}$ is bijective, where $\Gamma_n : (Z \cup \{1,2\})^{\{1,2,\dots,n\}} \to (Z \cup \{3\})^{\{1,2,\dots,n\}}$ converts the coordinate values 1 and 2 to 3, then there is some $I \subseteq \{1,2,\dots,n\}$ with $|I| \geq cn$ and either $S|_I \supseteq (Z \cup \{1\})^I$ or $S|_I \supseteq (Z \cup \{2\})^I$.*

Proof. The case $Z = \emptyset$ is trivial. So we assume that Z is nonempty. Fix a (small) constant $0 < t < \frac{1}{8}$ which we shall determine later. Denote by S' the elements of S taking value in Z on at least $(1-4t)n$ many coordinates in $\{1, 2, \dots, n\}$. Then $|S'| \geq \binom{n}{3tn}|Z|^{(1-4t)n}$ when n is large enough. Note that each $\sigma \in S'$ takes values 1 or 2 on at most $4tn$ many coordinates in $\{1, 2, \dots, n\}$. For $i = 1, 2$, set S'_i to be the elements in S' taking value i on at most $2tn$ many coordinates in $\{1, 2, \dots, n\}$. Then $\max(|S'_1|, |S'_2|) \geq \frac{1}{2}|S'| \geq \frac{1}{2}\binom{n}{3tn}|Z|^{(1-4t)n}$ when n is large enough. Without loss of generality, we may assume that $|S'_1| \geq \frac{1}{2}\binom{n}{3tn}|Z|^{(1-4t)n}$. For each $\beta \subseteq \{1, 2, \dots, n\}$ with $|\beta| \leq 2tn$ denote by S^β the set of elements in S'_1 taking value 1 exactly on β. The number of different β is

$$\sum_{0 \leq m \leq 2tn} \binom{n}{m} \leq (2tn+1)\binom{n}{2tn}.$$

By Stirling's formula we can find $M_1, M_2 > 0$ such that for all $n \in \mathbb{N}$ we have

$$\binom{n}{2tn} \leq \frac{M_1}{\sqrt{tn}}\left(\frac{1}{(1-2t)^{1-2t}(2t)^{2t}}\right)^n$$

and

$$\binom{n}{3tn} \geq \frac{M_2}{\sqrt{tn}}\left(\frac{1}{(1-3t)^{1-3t}(3t)^{3t}}\right)^n.$$

Therefore, when n is large enough we can find some β such that

$$|S^\beta| \geq \frac{\frac{1}{2}\binom{n}{3tn}|Z|^{(1-4t)n}}{(2tn+1)\binom{n}{2tn}} \geq M|Z|^{(1-4t)n}(f(t))^n\frac{1}{2tn+1},$$

where $M := \frac{M_2}{2M_1} > 0$ and $f(t) := \frac{(1-2t)^{1-2t}(2t)^{2t}}{(1-3t)^{1-3t}(3t)^{3t}}$. Note that $\lim_{t\to 0^+} t^{-1}\ln f(t) = \infty$. Fix t such that $f(t) \geq (2|Z|)^{4t}$. Then there is some $n_0 > 0$ such that

$$|S^\beta| \geq M(|Z|2^{4t})^n\frac{1}{2tn+1} \geq (|Z|2^t)^n$$

for all $n \geq n_0$. By Lemma 3.5 there exists a constant $c > 0$ depending only on $|Z|$ such that for all $n \geq n_0$ we can find an $I \subseteq \{1, 2, \dots, n\} \setminus \beta$ for which $|I| \geq c|\{1, 2, \dots, n\} \setminus \beta| \geq c(1-2t)n$ and $S^\beta|_I = (Z \cup \{2\})^I$. Now we may reset c to be $\min(c(1-2t), 1/n_0)$. ☐

As an immediate consequence of Lemma 3.6 we have:

Lemma 3.7. *Let c be as in Lemma 3.6 for $|Z| = k-1$. Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a k-tuple of subsets of X and suppose $A_1 = A_{1,1} \cup A_{1,2}$. If H is a finite independence set for $\boldsymbol{A}$, then there exists some $I \subseteq H$ such that $|I| \geq c|H|$ and I is an independence set for $(A_{1,1}, \dots, A_k)$ or $(A_{1,2}, \dots, A_k)$.*

The next lemma follows directly from Lemmas 3.2 and 3.7.

Lemma 3.8. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a k-tuple of subsets of X which has an independence set of positive density. Suppose that $A_1 = A_{1,1} \cup A_{1,2}$. Then at least one of the tuples $(A_{1,1}, \dots, A_k)$ and $(A_{1,2}, \dots, A_k)$ has an independence set of positive density.*

Proposition 3.9. *The following are true:*

(1) *Let $(A_1, \dots, A_k)$ be a tuple of closed subsets of X which has an independence set of positive density. Then there exists an IE-tuple $(x_1, \dots, x_k)$ with $x_j \in A_j$ for all $1 \le j \le k$.*
(2) $\mathrm{IE}_2(X,T) \setminus \Delta_2(X)$ *is nonempty if and only if* $\mathrm{h_{top}}(T) > 0$.
(3) $\mathrm{IE}_k(X,T)$ *is a closed* $T \times \cdots \times T$*-invariant subset of* X^k.
(4) *Let* $\pi : (X,T) \to (Y,S)$ *be a factor map. Then* $(\pi\times\cdots\times\pi)(\mathrm{IE}_k(X,T)) = \mathrm{IE}_k(Y,S)$.
(5) *Suppose that* Z *is a closed* T*-invariant subset of* X. *Then* $\mathrm{IE}_k(Z,T|_Z) \subseteq \mathrm{IE}_k(X,T)$.

Proof. Assertion (1) follows from Lemma 3.8 and a simple compactness argument. One can easily show that $\mathrm{h_{top}}(T) > 0$ if and only if there is some two-element open cover $\mathcal{U} = \{U_1, U_2\}$ of X with positive topological entropy (see for instance the proof of Proposition 1 in [9]). Then assertion (2) follows directly from assertion (1) and Lemma 3.4. Assertions (3)–(5) either are trivial or follow directly from assertion (1). $\square$

We remark that (2) and (4) of Proposition 3.9 show that the topological Pinsker factor (i.e., the largest zero-entropy factor) of (X,T) is obtained from the closed invariant equivalence relation on X generated by the set of IE-pairs (cf. [12]).

In [49] we introduced the notion of CA entropy for isometric automorphisms of a Banach space. One can easily extend the definition to isometric endomorphisms of Banach spaces and check that Theorem 3.5 in [49] holds in this general setting. In particular, $\mathrm{h_{top}}(T) > 0$ if and only if there exists an $f \in C(X)$ with an ℓ_1-isomorphism set (with respect to the induced *-endomorphism $f \mapsto f \circ T$) of positive density. Thus if f is a function in $C(X)$ with an ℓ_1-isomorphism set of positive density then the dynamical factor of (X,T) spectrally generated by f has positive entropy and hence by Proposition 3.9(2) has a nondiagonal IE-pair, from which we infer using Proposition 3.9(4) that (X,T) has an IE-pair (x,y) such that $f(x) \neq f(y)$. This yields one direction of the following proposition. The other direction follows by a standard argument which appears in the proof of the Rosenthal-Dor ℓ_1 theorem [23].

Proposition 3.10. *Let $f \in C(X)$. Then f has an ℓ_1-isomorphism set of positive density if and only if there is an IE-pair (x,y) with $f(x) \neq f(y)$.*

We next describe in Proposition 3.12 the set $\mathrm{IE}_1(X,T)$, which can also be identified with $\mathrm{IE}_k(X,T) \cap \Delta_k(X)$ for each k. The following lemma is mentioned on page 35 of [10]. For completeness we provide a proof here.

Lemma 3.11. *Let A be a closed subset of X. Then A has an independence set of positive density if and only if there exists a $\mu \in M(X,T)$ with $\mu(A) > 0$.*

Proof. Suppose $\mu(A) > 0$ for some $\mu \in M(X,T)$. Let H be a finite subset of G. Denote by M the maximum over all $x \in X$ of the cardinality of the set of $s \in H$ such that $x \in s^{-1}A$. Then $\sum_{s\in H} 1_{s^{-1}A} \le M$. Thus

$$M \ge \int \sum_{s\in H} 1_{s^{-1}A}\, d\mu = |H|\mu(A).$$

Therefore we can find a subset $I \subseteq H$ such that $|I| \ge |H|\mu(A)$ and I is an independence set for A. By Lemma 3.2 A has an independence set of positive density. This proves the "if" part.

Conversely, suppose that A has an independence set H with density $d > 0$. We consider the case $G = \mathbb{Z}$. The case $G = \mathbb{Z}_{\geq 0}$ is dealt with similarly. Let $y \in \bigcap_{s\in H} s^{-1}A$. Then

$$\liminf_{n\to\infty} \left(\frac{1}{2n+1} \sum_{j=-n}^{n} \delta_{T^j(y)} \right)(A) \geq d.$$

Take an accumulation point μ of the sequence $\{(2n+1)^{-1} \sum_{j=-n}^{n} \delta_{T^j(y)})\}_{n\in\mathbb{N}}$ in $M(X)$. Then $\mu \in M(X,T)$ and $\mu(A) \geq d$. This proves the "only if" part. ☐

As a consequence of Lemma 3.11 we have:

Proposition 3.12. *The set* $\mathrm{IE}_1(X,T)$ *is the closure of the union of* $\mathrm{supp}(\mu)$ *over all* $\mu \in M(X,T)$.

We describe next in Theorem 3.15 the IE-tuples of a product system. The corresponding statement for entropy pairs was proved by Glasner in [30, Theorem 3.(7)(9)] (see also [31, Theorem 19.24]) for a product of metrizable systems using a local variational principle.

For any tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of X, denote by $\mathcal{P}_{\boldsymbol{A}}$ the set of all independence sets for $\boldsymbol{A}$. Identifying subsets of G with elements of $\Omega_2 := \{0,1\}^G$ by taking indicator functions, we may think of $\mathcal{P}_{\boldsymbol{A}}$ as a subset of Ω_2. Endow Ω_2 with the shift induced from addition by 1 on G. Clearly $\mathcal{P}_{\boldsymbol{A}}$ is closed and shift-invariant (i.e., the image of $\mathcal{P}_{\boldsymbol{A}}$ under the shift coincides with $\mathcal{P}_{\boldsymbol{A}}$). We say a closed shift-invariant subset $\mathcal{P} \subseteq \Omega_2$ has *positive density* if it has an element with positive density. Then by definition $\boldsymbol{A}$ has an independence set of positive density exactly when $\mathcal{P}_{\boldsymbol{A}}$ has positive density. We also say $\mathcal{P}$ is *hereditary* if any subset of any element in $\mathcal{P}$ is an element in $\mathcal{P}$. Note that $\mathcal{P}_{\boldsymbol{A}}$ is hereditary.

For $s \in G$ we denote by $[s]$ the set of subsets of G containing s.

Lemma 3.13. *Let* $\mathcal{P}$ *be a closed shift-invariant subset of* Ω_2. *Then* $\mathcal{P}$ *has positive density if and only if* $\mu(\mathcal{P} \cap [0]) > 0$ *for some shift-invariant Borel probability measure* μ *on* $\mathcal{P}$.

Proof. Notice that $\mathcal{P}$ has positive density if and only if $\mathcal{P} \cap [0]$ has an independence set of positive density. The lemma then follows from Lemma 3.11. ☐

Lemma 3.14. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be hereditary closed shift-invariant subsets of* Ω_2 *with positive density. Then* $\mathcal{P} \cap \mathcal{Q}$ *is also a hereditary closed shift-invariant subset of* Ω_2 *with positive density.*

Proof. Clearly $\mathcal{P} \cap \mathcal{Q}$ is a hereditary closed shift-invariant subset of Ω_2. By Lemma 3.13 there is a shift-invariant Borel probability measure μ (resp. ν) on $\mathcal{P}$ (resp. $\mathcal{Q}$) such that $\mu(\mathcal{P} \cap [0]) > 0$ (resp. $\nu(\mathcal{Q} \cap [0]) > 0$). Then $(\mu \times \nu)(U) > 0$ for $U := (\mathcal{P} \cap [0]) \times (\mathcal{Q} \cap [0])$. By Lemma 3.11, U has an independence set H of positive density. Take a pair

$$(x', y') \in \bigcap\nolimits_{s\in H} s^{-1}U = (\mathcal{P} \times \mathcal{Q}) \cap \left(\left(\bigcap\nolimits_{s\in H} [s] \right) \times \left(\bigcap\nolimits_{s\in H} [s] \right) \right).$$

Then $H \subseteq x', y'$. Since both $\mathcal{P}$ and $\mathcal{Q}$ are hereditary, $H \in \mathcal{P} \cap \mathcal{Q}$. This finishes the proof. ☐

Theorem 3.15. *Let* (X,T) *and* (Y,S) *be dynamical systems. Then*

$$\mathrm{IE}_k(X \times Y, T \times S) = \mathrm{IE}_k(X,T) \times \mathrm{IE}_k(Y,S).$$

Proof. By Proposition 3.9.(4) the left hand side is included in the right hand side. The other direction follows from Lemma 3.14. □

We next record the fact that, as an immediate consequence of Lemma 3.4, nondiagonal IE-tuples are the same as entropy tuples. This was established for $\mathbb{Z}$-systems in [43] using a local variational principle.

Theorem 3.16. *Let $(x_1, \dots, x_k)$ be a tuple in $X^k \setminus \Delta_k(X)$ with $k \geq 2$. Then $(x_1, \dots, x_k)$ is an entropy tuple if and only if it is an IE-tuple.*

In [11, Corollary 2.4(2)] Blanchard, Glasner, Kolyada, and Maass showed using the variational principle that positive entropy implies Li-Yorke chaos. We will next give an alternative proof of this fact using IE-tuples. The notion of Li-Yorke chaos was introduced in [11] and is based on ideas from [55]. In the case that X is a metric space with metric ρ, a pair (x_1, x_2) of points in X is said to be a *Li-Yorke pair* (with modulus δ) if

$$\limsup_{n\to\infty} \rho(T^n x_1, T^n x_2) = \delta > 0 \quad \text{and} \quad \liminf_{n\to\infty} \rho(T^n x_1, T^n x_2) = 0.$$

A set $Z \subseteq X$ is said to be *scrambled* if all nondiagonal pairs of points in Z are Li-Yorke. The system (X, T) is said to be *Li-Yorke chaotic* if X contains an uncountable scrambled set.

We begin by establishing the following lemma, in which we use the notation established just before Lemma 3.13. For any subset $\mathcal{P}$ of Ω_2, we say that a finite subset $J \subset G$ has *positive density with respect to* $\mathcal{P}$ if there exists a $K \subseteq G$ with positive density such that $(K - K) \cap (J - J) = \{0\}$ and $K + J \in \mathcal{P}$. We say that a subset $J \subseteq G$ has *positive density with respect to* $\mathcal{P}$ if every finite subset of J has positive density with respect to $\mathcal{P}$.

Lemma 3.17. *Let $\mathcal{P}$ be a hereditary closed shift-invariant subset of Ω_2 with positive density. Then there exists a $J \subseteq \mathbb{Z}_{\geq 0}$ with positive density which also has positive density with respect to $\mathcal{P}$.*

Proof. Denote by $\mathcal{Q}$ the set of subsets of $\mathbb{Z}_{\geq 0}$ which have positive density with respect to $\mathcal{P}$. Then $\mathcal{Q}$ is a hereditary closed shift-invariant subset of $\{0,1\}^{\mathbb{Z}_{\geq 0}}$. Take $H \in \mathcal{P}$ with density $d > 0$. Fix $0 < d' < \frac{d}{3}$. We claim that if n is large enough so that $0 < \frac{d+d'}{2d'n} - \frac{2}{n+2} =: b$ and $d'n \geq 2$, then there exists an $S \in \mathcal{Q}$ such that $|S| = c_n := \lfloor d'n \rfloor$ and $S \subseteq [0, n]$. When m is large enough, we have $|[-m, m] \cap H| > \frac{d+d'}{2} m$. Arrange the elements of $[-m, m] \cap H$ in increasing order as $a_1 < a_2 < \cdots < a_k$, where $k > \frac{d+d'}{2} m$. Consider the numbers $a_{jc_n} - a_{(j-1)c_n+1}$ for $1 \leq j \leq d_{n,m} := \left\lfloor \frac{\frac{d+d'}{2} m}{c_n} \right\rfloor$. Set $M_{n,m} = |\{1 \leq j \leq d_{n,m} : a_{jc_n} - a_{(j-1)c_n+1} \leq n\}|$. Then

$$2m + 1 \geq a_k - (a_1 - 1) \geq (d_{n,m} - M_{n,m})(n + 2).$$

Thus

$$M_{n,m} \geq d_{n,m} - \frac{2m+1}{n+2} \geq \frac{\frac{d+d'}{2} m}{d'n} - 1 - \frac{2m+1}{n+2} > \frac{b}{2} m$$

when m is large enough. Note that if $a_{jc_n} - a_{(j-1)c_n+1} \leq n$, then $0 < a_{(j-1)c_n+2} - a_{(j-1)c_n+1} < \cdots < a_{jc_n} - a_{(j-1)c_n+1}$ are contained in $[0, n]$. Consequently, when m is large enough, there exist an $S_m \subseteq [0, n]$ with $|S_m| = c_n$ and a $W_m \subseteq [-m, m] \cap H$ with

$|W_m| \geq \frac{b}{2\binom{n}{cn-1}} m$ such that $(W_m - W_m) \cap (S_m - S_m) = \{0\}$ and $W_m + S_m \subseteq H$. Since $\mathcal{P}$ is hereditary, $W_m + S_m \in \mathcal{P}$. Then there exists an $S \subseteq [0, n]$ which coincides with infinitely many of the S_m. Note that the collection $\mathcal{W}_S$ consisting of the sets $W \subseteq G$ such that $(W - W) \cap (S - S) = \{0\}$ and $W + S \in \mathcal{P}$ is a closed shift-invariant subset of Ω_2. By Lemma 3.2 $\mathcal{W}_S \cap [0]$ has an independence set of positive density. Thus $S \in \mathcal{Q}$. By Lemma 3.2 again we see that $\mathcal{Q} \cap [0]$ has an independence set of positive density. In other words, $\mathcal{Q}$ has positive density. This finishes the proof. $\square$

Theorem 3.18. *Suppose that X is metrizable with a metric ρ. Suppose that $k \geq 2$ and $\boldsymbol{x} = (x_1, \dots, x_k)$ is an* IE*-tuple in $X^k \setminus \Delta_k(X)$. For each $1 \leq j \leq k$, let A_j be a neighbourhood of x_j. Then there exist a $\delta > 0$ and a Cantor set $Z_j \subseteq A_j$ for each $j = 1, \dots, k$ such that the following hold:*

(1) *every nonempty finite tuple of points in $Z := \bigcup_j Z_j$ is an* IE*-tuple;*
(2) *for all $m \in \mathbb{N}$, distinct $y_1, \dots, y_m \in Z$, and $y_1', \dots, y_m' \in Z$ one has*

$$\liminf_{n\to\infty} \max_{1\leq i\leq m} \rho(T^n y_i, y_i') = 0.$$

Proof. We may assume that the A_j are closed and pairwise disjoint. We shall construct, via induction on m, closed nonempty subsets A_σ for $\sigma \in \Sigma_m := \{1, 2, \dots, k\}^{\{1,2,\dots,m\}}$ with the following properties:

(a) when $m = 1$, $A_\sigma = A_{\sigma(1)}$ for all $\sigma \in \Sigma_m$,
(b) when $m \geq 2$, $A_\sigma \subseteq A_{\sigma|_{\{1,2,\dots,m-1\}}}$ for all $\sigma \in \Sigma_m$,
(c) when $m \geq 2$, for every map $\gamma : \Sigma_m \to \Sigma_{m-1}$ there exists an $h_\gamma \in G$ with $h_\gamma \geq m$ such that $h_\gamma(A_\sigma) \subseteq A_{\gamma(\sigma)}$ for all $\sigma \in \Sigma_m$,
(d) when $m \geq 2$, $\mathrm{diam}(A_\sigma) \leq 2^{-m}$ for all $\sigma \in \Sigma_m$,
(e) for every m, the collection $\{A_\sigma : \sigma \in \Sigma_m\}$, ordered into a tuple, has an independence set of positive density.

Suppose that we have constructed such A_σ over all m. Then the A_σ for all σ in a given Σ_m are pairwise disjoint because of property (c). Set $\Sigma = \{1, 2, \dots, k\}^{\mathbb{N}}$. Properties (b) and (d) imply that for each $\sigma \in \Sigma$ we have $\bigcap_m A_{\sigma|_{\{1,2,\dots,m\}}} = \{z_\sigma\}$ for some $z_\sigma \in X$ and that $Z_j = \{z_\sigma : \sigma \in \Sigma \text{ and } \sigma(1) = j\}$ is a Cantor set for each $j = 1, \dots, k$. Property (a) implies that $Z_j \subseteq A_j$. Condition (1) follows from properties (d) and (e). Condition (2) follows from properties (c) and (d).

We now construct the A_σ. Define A_σ for $\sigma \in \Sigma_1$ according to property (a). By assumption property (e) is satisfied for $m = 1$. Assume that we have constructed A_σ for all $\sigma \in \Sigma_j$ and $j = 1, \dots, m$ with the above properties. Set $\boldsymbol{A}_m$ to be $\{A_\sigma : \sigma \in \Sigma_m\}$ ordered into a tuple. By Lemma 3.17 there exist an $H \subseteq \mathbb{Z}_{\geq 0}$ with positive density which also has positive density with respect to $\mathcal{P}_{\boldsymbol{A}_m}$. Then for any nonempty finite subset J of H, the sets $A_{J,\omega} := \bigcap_{h\in J} h^{-1} A_{\omega(h)}$ for all $\omega \in (\Sigma_m)^J$, taken together as a tuple, have an independence set of positive density. Replacing H by $H - h$ for the smallest $h \in H$ we may assume that $0 \in H$ and hence may require that $0 \in J$. For each $\gamma \in (\Sigma_m)^{\Sigma_{m+1}}$ take an $h_\gamma \in J$ with $h_\gamma \geq m + 1$. As we can take $|J|$ to be arbitrarily large, we may assume that $h_\gamma \neq h_{\gamma'}$ for $\gamma \neq \gamma'$ in $(\Sigma_m)^{\Sigma_{m+1}}$. Take a map $f : \Sigma_{m+1} \to (\Sigma_m)^J$ such that $(f(\sigma))(0) = \sigma|_{\{1,\dots,m\}}$ and $(f(\sigma))(h_\gamma) = \gamma(\sigma)$ for all $\sigma \in \Sigma_{m+1}$ and $\gamma \in (\Sigma_m)^{\Sigma_{m+1}}$. Set

$A_\sigma = A_{J,f(\sigma)}$ for all $\sigma \in \Sigma_{m+1}$. Then properties (b), (c), and (e) hold for $m+1$. For each $\sigma \in \Sigma_{m+1}$ write A_σ as the union of finitely many closed subsets each with diameter no bigger than $2^{-(m+1)}$. Using Lemma 3.8 we may replace A_σ by one of these subsets. Consequently, property (d) is also satisfied for $m+1$. This completes the induction procedure and hence the proof of the theorem. □

The set Z in Theorem 3.18 is clearly scrambled. As a consequence of Proposition 3.9(2) and Theorem 3.18 we obtain the following corollary.

Corollary 3.19. *Suppose that X is metrizable. If $\mathrm{h}_{\mathrm{top}}(T) > 0$, then (X,T) is Li-Yorke chaotic.*

As mentioned above, Corollary 3.19 was proved in [11, Corollary 2.4.(2)] using measure-dynamical techniques.

Denote by $\mathrm{LY}(X,T)$ the set of Li-Yorke pairs in $X \times X$. Employing a local variational principle, Glasner showed in [30, Theorem 4.(3)] (see also [31, Theorem 19.27]) that for $\mathbb{Z}$-systems the set of proximal entropy pairs is dense in the set of entropy pairs. As a consequence of Theorem 3.18 we have the following improvement:

Corollary 3.20. *Suppose that X is metrizable. Then $\mathrm{LY}(X,T) \cap \mathrm{IE}_2(X,T)$ is dense in $\mathrm{IE}_2(X,T) \setminus \Delta_2(X)$.*

The next corollary is a direct consequence of Theorem 3.18 in the case X is metrizable and follows from the proof of Theorem 3.18 in the general case.

Corollary 3.21. *Let W be a neighbourhood of an IE-pair (x_1, x_2) in $X^2 \setminus \Delta_2(X)$. Then $W \cap \mathrm{IE}_2(X,T)$ is not of the form $\{x_1\} \times Y$ for any subset Y of X. In particular, $\mathrm{IE}_2(X,T)$ does not have isolated points.*

Corollary 3.21, as stated for entropy pairs, was proved by Blanchard, Glasner, and Host in [10, Theorem 6].

We round out this section by explaining how the facts about IE-tuples captured in Propositions 3.9, 3.10, and 3.12 and Theorems 3.15 and 3.16 apply to actions of any discrete amenable group. So for the remainder of the section G will be a discrete amenable group. For a finite $K \subseteq G$ and $\delta > 0$ we denote by $M(K,\delta)$ the set of all nonempty finite subsets F of G which are (K,δ)-invariant in the sense that

$$|\{s \in F : Ks \subseteq F\}| \geq (1-\delta)|F|.$$

According to the Følner characterization of amenability, $M(K,\delta)$ is nonempty for every finite set $K \subseteq G$ and $\delta > 0$. This is equivalent to the existence of a *Følner net*, i.e., a net $\{F_\gamma\}_\gamma$ of nonempty finite subsets of G such that $\lim_\gamma |sF_\gamma \Delta F_\gamma|/|F_\gamma| = 0$ for all $s \in G$. For countable G we may take this net to be a sequence, in which case we speak of a *Følner sequence*.

For countable G, a sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty finite subsets of G is said to be *tempered* if for some $c > 0$ we have $|\bigcup_{k=1}^{n-1} F_k^{-1} F_n| \leq c|F_n|$ for all $n \in \mathbb{N}$. By [56, Prop. 1.4], every Følner sequence has a tempered subsequence. Below we will make use of the pointwise ergodic theorem of Lindenstrauss [56], which applies to tempered Følner sequences.

The following subadditivity result was established by Lindenstrauss and Weiss for countable G [57, Theorem 6.1]. We will reduce the general case to their result. Note that there is also a version for left invariance.

Proposition 3.22. *If φ is a real-valued function which is defined on the set of finite subsets of G and satisfies*

(1) $0 \leq \varphi(A) < +\infty$ *and* $\varphi(\emptyset) = 0$,
(2) $\varphi(A) \leq \varphi(B)$ *for all* $A \subseteq B$,
(3) $\varphi(As) = \varphi(A)$ *for all finite* $A \subseteq G$ *and* $s \in G$,
(4) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ *if* $A \cap B = \emptyset$,

then $\frac{1}{|F|}\varphi(F)$ *converges to some limit b as the set F becomes more and more invariant in the sense that for every $\varepsilon > 0$ there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that* $\left|\frac{1}{|F|}\varphi(F) - b\right| < \varepsilon$ *for all (K,δ)-invariant sets $F \subseteq G$.*

Proof. For a finite $K \subseteq G$ and $\delta > 0$ we denote by $Z(K,\delta)$ the closure of $\left\{\frac{1}{|F|}\varphi(F) : F \in M(K,\delta)\right\}$. The collection of sets $Z(K,\delta)$ has the finite intersection property and clearly the conclusion of the proposition is equivalent to the set $Z := \bigcap_{(K,\delta)} Z(K,\delta)$ containing only one point. Suppose that x and y are distinct points of Z. Then one can find a sequence $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ of pairs of finite subsets of G such that $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are both Følner sequences for the subgroup H of G generated by $\bigcup_{n\in\mathbb{N}}(A_n \cup B_n)$ and $\max\left(\left|x - \frac{1}{|A_n|}\varphi(A_n)\right|, \left|y - \frac{1}{|B_n|}\varphi(B_n)\right|\right) < 1/n$ for each $n \in \mathbb{N}$. Since subgroups of discrete amenable groups are amenable [64, Proposition 0.16], H is amenable. This contradicts the result of Lindenstrauss and Weiss. Thus Z contains only one point. □

Let (X,G) be a dynamical system. We define the topological entropy of (X,G) by first defining the topological entropy of a finite open cover of X using Proposition 3.22 and then taking a supremum over all finite open covers (this was originally introduced in [60] without the subadditivity result). For a finite tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of X, Proposition 3.22 also applies to the function $\varphi_{\boldsymbol{A}}$ given by

$$\varphi_{\boldsymbol{A}}(F) = \max\{|F \cap J| : J \text{ is an independence set for } \boldsymbol{A}\}.$$

This permits us to define the *independence density* $I(\boldsymbol{A})$ of $\boldsymbol{A}$ as the limit of $\frac{1}{|F|}\varphi_{\boldsymbol{A}}(F)$ as F becomes more and more invariant, providing a numerical measure of the dynamical independence of $\boldsymbol{A}$.

Proposition 3.23. *Let (X,G) be a dynamical system. Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X. Let $c > 0$. Then the following are equivalent:*

(1) $I(\boldsymbol{A}) \geq c$,
(2) *for every $\varepsilon > 0$ there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that for every $F \in M(K,\delta)$ there is an independence set J for $\boldsymbol{A}$ with $|J \cap F| \geq (c - \varepsilon)|F|$.*
(3) *for every finite set $K \subseteq G$ and $\varepsilon > 0$ there exist an $F \in M(K,\varepsilon)$ and an independence set J for $\boldsymbol{A}$ such that $|J \cap F| \geq (c - \varepsilon)|F|$.*

When G is countable, these conditions are also equivalent to:

4. *for every tempered Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G there is an independence set J for $\boldsymbol{A}$ such that* $\lim_{n\to\infty} \frac{|F_n \cap J|}{|F_n|} \geq c$.

5. *there are a tempered Følner sequence* $\{F_n\}_{n\in\mathbb{N}}$ *of* G *and an independence set* J *for* $\boldsymbol{A}$ *such that* $\lim_{n\to\infty} \frac{|F_n \cap J|}{|F_n|} \geq c$.

Proof. The equivalences (1)⇔(2)⇔(3) follow from Proposition 3.22. Assume now that G is countable. Then the implications (4)⇒(5)⇒(3) are trivial. Suppose that (3) holds and let us show (4). Then one can easily show that there is a G-invariant Borel probability measure μ on $\mathcal{P}_{\boldsymbol{A}} \subseteq \{0,1\}^G$ with $\mu([e]\cap\mathcal{P}_{\boldsymbol{A}}) \geq c$, as in the proof of Lemma 3.11. Here $\mathcal{P}_{\boldsymbol{A}}$ is defined as before Lemma 3.13 and $\{0,1\}^G$ is equipped with the shift given by $sx(t) = x(ts)$ for all $x \in \{0,1\}^G$ and $s,t \in G$. Replacing μ by a suitable ergodic G-invariant Borel probability measure in the ergodic decomposition of μ, we may assume that μ is ergodic. Let $\{F_n\}_{n\in\mathbb{N}}$ be a tempered Følner sequence for G. The pointwise ergodic theorem [56, Theorem 1.2] asserts that $\lim_{n\to\infty} \frac{1}{|F_n|}\sum_{s\in F_n} f(sx) = \int f\,d\mu$ μ-a.e. for every $f \in L^1(\mu)$. Setting f to be the characteristic function of $[e]\cap\mathcal{P}_{\boldsymbol{A}}$ and taking J to be some x satisfying the above equation, we get (4). □

Effectively extending Definition 3.1, we call a tuple $\boldsymbol{x} = (x_1,\dots,x_k) \in X^k$ an *IE-tuple* (or an *IE-pair* in the case $k = 2$) if for every tuple $\boldsymbol{U} = (U_1,\dots,U_k)$ associated to a product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the independence density $I(\boldsymbol{U})$ is nonzero. Denoting the set of IE-tuples of length k by $\mathrm{IE}_k(X,G)$ and replacing everywhere the existence of positive density independence sets for a tuple $\boldsymbol{A}$ by the nonvanishing of the independence density $I(\boldsymbol{A})$ in our earlier discussion for singly generated systems, we see that Propositions 3.9 and 3.12 and Theorem 3.15 continue to hold in our current setting. In particular, the topological Pinsker factor (i.e., the largest zero-entropy factor) of (X,G) arises from the closed invariant equivalence relation on X generated by the set of IE-pairs.

Given an action α of G by isometric automorphisms of a Banach space V and an element $v \in V$, we can apply the left invariance version of Proposition 3.22 to the function $\varphi_{v,\lambda}$ given by

$$\varphi_{v,\lambda}(F) = \max\{|F\cap J| : J \text{ is an } \ell_1\text{-}\lambda\text{-isomorphism set for } v\}$$

and define the ℓ_1-λ-*isomorphism density* $I(v,\lambda)$ of v as the limit of $\frac{1}{|F|}\varphi_{v,\lambda}(F)$ as F becomes more and more invariant. Defining the CA entropy of α by taking a limit supremum of averages as in Section 2 of [49] but this time along a Følner net, one can check that the analogue of Theorem 3.5 of [49] holds. In particular, the topological entropy of (X,G) is nonzero if and only if there exists an $f \in C(X)$ with nonvanishing ℓ_1-λ-isomorphism density for some $\lambda \geq 1$. Then we obtain the analogue of Proposition 3.10, i.e., a function $f \in C(X)$ has nonvanishing ℓ_1-λ-isomorphism density for some $\lambda \geq 1$ if and only if there is an IE-pair (x,y) in $X \times X$ with $f(x) \neq f(y)$.

Finally, if we define entropy tuples in the same way as for singly generated systems, then Theorem 3.16 still holds.

4. Topological entropy and tensor product independence

In this section we will see how combinatorial independence in the context of entropy translates into the language of tensor products, with a hint at how the theory might thereby be extended to noncommutative C^*-dynamical systems. As in the previous section, our dynamical systems here will have acting semigroup $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$ with generating endomorphism T.

To start with, we remark that, given a dynamical system (X, T) and denoting by α_T the induced *-endomorphism $f \mapsto f \circ T$ of $C(X)$, the following conditions are equivalent:

(1) $\mathrm{h}_{\mathrm{top}}(T) > 0$,
(2) $\mathrm{IE}_2(X, T) \setminus \Delta_2(X) \neq \emptyset$,
(3) there is an $f \in C(X)$ with an ℓ_1-isomorphism set of positive density,
(4) there is a 2-dimensional operator subsystem $V \subseteq C(X)$ and a $\lambda \geq 1$ such that V has a λ-independence set of positive density for α_T.

The equivalence (1)⇔(3) was established in [49] for $\mathbb{Z}$-systems and is similarly seen to be valid for endomorphisms, and (1)⇔(2) is Proposition 3.9(2) of Section 3. To show (2)⇒(4) we simply need to take a pair (A, B) of disjoint closed subsets of X with an independence set of positive density and consider the operator system generated by a norm-one self-adjoint function in $C(X)$ taking the constant values 1 and -1 on A and B, respectively. Finally, the implication (4)⇒(3) is a consequence of the following lemma, which expresses in a form suited to our context a well-known phenomenon observed by Rosenthal in the proof of his ℓ_1 theorem [70, 69], namely that ℓ_1 geometry ensues in a natural way from independence.

Lemma 4.1. *Let V be an operator system and let v be a nonscalar element of V. For each $j \in \mathbb{N}$ set $v_j = 1 \otimes v \otimes 1 \in V^{\otimes[1,j-1]} \otimes V \otimes V^{\otimes[j+1,\infty)} = V^{\otimes\mathbb{N}}$. Set $Z = \{\sigma(v) : \sigma \in S(V)\}$ where $S(V)$ is the state space of V. Let $0 < \eta < \frac{1}{2}\mathrm{diam}(Z)$. Then for all $n \in \mathbb{N}$ and complex scalars $c_1, \dots, c_n$ we have*

$$\frac{\eta}{4} \sum_{j=1}^{n} |c_j| \leq \left\| \sum_{j=1}^{n} c_j v_j \right\|.$$

Proof. Choose points $b_1, b_2 \in Z$ such that $|b_1 - b_2| = \mathrm{diam}(Z)$, and take disks D_1 and D_2 in the complex plane centred at b_1 and b_2, respectively, with common diameter d and at distance greater than $\max(2\eta, 2d)$ from each other. For each $j \in \mathbb{N}$ define the subsets

$$\begin{aligned} U_j &= \{\sigma \in S(V^{\otimes\mathbb{N}}) : \sigma(v_j) \in D_1\}, \\ V_j &= \{\sigma \in S(V^{\otimes\mathbb{N}}) : \sigma(v_j) \in D_2\} \end{aligned}$$

of the state space $S(V^{\otimes\mathbb{N}})$. Then the collection of pairs (U_j, V_j) for $j \in \mathbb{N}$ is independent, and so we obtain the result by the proof in [23]. □

What can be said about the link between ℓ_1 structure and independence in connection with the global picture of entropy production? This question is tied to the divergence in topological dynamics between the notions of completely positive entropy and uniformly positive entropy.

Recall that the dynamical system (X, T) is said to have *completely positive entropy* or *c.p.e.* if each of its nontrivial factors has positive topological entropy, i.e., if its Pinsker factor is trivial [8]. The functions in $C(X)$ which lie in the topological Pinsker algebra (the C^*-algebraic manifestation of the Pinsker factor) are characterized by the fact that they lack an ℓ_1-isomorphism set of positive density for the induced *-endomorphism of $C(X)$ [49]. Thus (X, T) has c.p.e. precisely when every nonscalar function in $C(X)$ has an ℓ_1-isomorphism set of positive density.

The system (X, T) is said to have *uniformly positive entropy* or *u.p.e.* if every nondiagonal pair in $X \times X$ is an entropy pair [8]. More generally, following [35] we say that (X, T) has *u.p.e. of order* n if every tuple in $X^n \setminus \Delta_n(X)$ is an entropy tuple (see the beginning of Section 3). By Theorem 3.16, this is equivalent to every n-tuple of nonempty open subsets of X having an independence set of positive density. Finally, we say that (X, T) has *u.p.e. of all orders* if it has u.p.e. of order n for each $n \geq 2$. U.p.e. implies c.p.e., but the converse is false. Also, in [43] it is shown that, for every $n \geq 2$, u.p.e. of order n does not imply u.p.e. of order $n + 1$.

The following propositions supply functional-analytic characterizations of u.p.e. and u.p.e. of all orders, complementing the characterization of c.p.e. in terms of ℓ_1-isomorphism sets.

Proposition 4.2. *Let X be a compact Hausdorff space and $T : X \to X$ a surjective continuous map. Then the following are equivalent:*

(1) *(X, T) has u.p.e. of all orders,*
(2) *for every finite set $\Omega \subseteq C(X)$ and $\delta > 0$ there is a finite-dimensional operator subsystem $V \subseteq C(X)$ which approximately includes Ω to within δ and has a 1-independence set of positive density,*
(3) *for every finite set $\Omega \subseteq C(X)$ and $\delta > 0$ there is a finite-dimensional operator subsystem $V \subseteq C(X)$ which approximately includes Ω to within δ and has a λ-independence set of positive density for some $\lambda \geq 1$.*

Proof. (1)⇒(2). Let Ω be a finite subset of A and let $\delta > 0$. Then we can construct a partition of unity $\{g_1, \dots, g_k\}$ in $C(X)$ such that the operator system it spans approximately includes Ω to within δ and for each $i = 1, \dots, k$ we have $g_i(x) = 1$ for all x in some nonempty open set U_i. By (1) and Theorem 3.16, the tuple $(U_1, \dots, U_k)$ has an independence set I of positive density. Let $(s_1, \dots, s_n)$ be a tuple of distinct elements of I and let $\varphi : V^{\otimes[1,n]} \to C(X)$ be the contractive linear map determined on elementary tensors by $f_1 \otimes \cdots \otimes f_n \mapsto (f_1 \circ T^{s_1}) \cdots (f_n \circ T^{s_n})$. Then the collection $\{(g_{\sigma(1)} \circ T^{s_1}) \cdots (g_{\sigma(n)} \circ T^{s_n}) : \sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}\}$ is an effective k^n-element partition of unity of X and hence is isometrically equivalent to the standard basis of $\ell_\infty^{k^n}$. Since the subset $\{g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)} : \sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}\}$ of $V^{\otimes[1,n]}$ is also isometrically equivalent to the standard basis of $\ell_\infty^{k^n}$, we conclude that I is a 1-independence set for V, yielding (2).

(2)⇒(3). Trivial.

(3)⇒(1). Let $k \geq 2$ and let $(U_1, \dots, U_k)$ be a k-tuple of nonempty open subsets of X with pairwise disjoint closures. To obtain (1) it suffices to show that $(U_1, \dots, U_k)$ has an independence set of positive density. Since the sets U_i have pairwise disjoint closures we can construct positive norm-one functions $g_1, \dots, g_k \in C(X)$ such that, for each i, the set on which g_i takes the value 1 is a closed subset of U_i. By (3), given a $\delta > 0$ we can find an operator subsystem $V \subseteq C(X)$ such that V has a λ-independence set J of positive density for some $\lambda \geq 1$ and there exist $f_1, \dots, f_k \in V$ for which $\|f_i - g_i\| < \delta$ for each $i = 1, \dots, k$. Since for each i continuity implies that $g_i^{-1}((\rho, 1]) \subseteq U_i$ for some $\rho \in (0, 1)$, by taking δ small enough we may ensure that the norm-one self-adjoint elements

$h_i = \|(f_i + f_i^*)/2\|^{-1}(f_i + f_i^*)/2 \in V$ for $i = 1, \dots, k$ are defined and for some $\theta \in (0,1)$ satisfy $h_i^{-1}((\theta, 1]) \subseteq U_i$ for each i.

Choose an $r \in \mathbb{N}$ large enough so that $\theta^r < \lambda^{-1}$ and a $b > 0$ small enough so that $\frac{k}{k-1}2^{-2b} > 1$. By Stirling's formula there is a $c \in (0, 1/2)$ such that $\binom{n}{cn} \leq 2^{bn}$ for all $n \in \mathbb{N}$. By Lemma 3.5 there is a $d > 0$ such that, for all $n \in \mathbb{N}$, if $\Gamma \subseteq \{1, \dots, k\}^{\{1,\dots,n\}}$ and $|\Gamma| \geq (k2^{-2b})^n = \big((k-1)\big(\frac{k}{k-1}2^{-2b}\big)\big)^n$ then there exists a set $I \subseteq \{1, \dots, n\}$ such that $|I| \geq dn$ and $\Gamma|_I = \{1, \dots, k\}^I$.

By shifting J we may assume that it contains 0, and so by positive density there is an $a > 0$ such that for each $m \in \mathbb{Z}$ the set $J_m := J \cap \{-m, -m+1, \dots, m\}$ has cardinality at least am. Now suppose we are given an $m \in \mathbb{N}$ with $m \geq \max(r/ab, r/ac)$. Enumerate the elements of J_m as $j_1, \dots, j_n$. For each $\sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}$ we define a set $K_\sigma \subseteq I$ as follows. Pick an $x \in X$ such that $h_{\sigma(1)}(T^{j_1}x) \cdots h_{\sigma(n)}(T^{j_n}x) = \|(h_{\sigma(1)} \circ T^{j_1}) \cdots (h_{\sigma(n)} \circ T^{j_n})\|$. Since $\|(h_{\sigma(1)} \circ T^{j_1}) \cdots (h_{\sigma(n)} \circ T^{j_n})\| \geq \lambda^{-1} > \theta^r$, there exists a $K_\sigma \subseteq \{1, \dots, n\}$ with $|K_\sigma| = n - r$ such that $h_{\sigma(i)}(T^{j_i}x) > \theta$ for each $i \in K_\sigma$. Hence $x \in \bigcap_{i \in K_\sigma} T^{-j_i}U_{\sigma(i)}$, so that $\bigcap_{i \in K_\sigma} T^{-j_i}U_{\sigma(i)}$ is nonempty.

Now since $r \leq cam \leq cn$, we have

$$\big|\{K_\sigma : \sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}\}\big| \leq \binom{n}{r} \leq 2^{bn}.$$

We can thus find a $K \in \{K_\sigma : \sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}\}$ such that the set $\mathcal{R}$ of all $\sigma \in \{1, \dots, k\}^{\{1,\dots,n\}}$ for which $K_\sigma = K$ has cardinality at least $k^n/2^{bn}$. Then the set of all restrictions of elements of $\mathcal{R}$ to K has cardinality at least $|\mathcal{R}|/2^{n-|K|} \geq 2^{-r}(k2^{-b})^n \geq (k2^{-2b})^n$. It follows that there is a set $I_m \subseteq J_m$ with $|I_m| \geq d|J_m| \geq dam$ such that the set of all restrictions of elements of $\mathcal{R}$ to I_m is $\{1, \dots, k\}^{I_m}$. Since $\bigcap_{i \in I_m} T^{-j_i}U_{\sigma(i)} \neq \emptyset$ for every $\sigma \in \{1, \dots, k\}^{I_m}$, I_m is an independence set for $(U_1, \dots, U_k)$. By Lemma 3.2 we conclude that $(U_1, \dots, U_k)$ has an independence set of positive density, finishing the proof. □

We denote by $\mathcal{S}_2(X)$ the collection of 2-dimensional operator subsystems of $C(X)$ equipped with the metric given by the Hausdorff distance between unit balls.

Proposition 4.3. *For a $\mathbb{Z}$-dynamical system (X, T) the following are equivalent:*

1. *(X, T) has u.p.e.,*
2. *the collection of 2-dimensional operator subsystems of $C(X)$ which have a 1-independence set of positive density is dense in $\mathcal{S}_2(X)$,*
3. *the collection of 2-dimensional operator subsystems of $C(X)$ which have a λ-independence set of positive density for some $\lambda \geq 1$ is dense in $\mathcal{S}_2(X)$.*

Proof. (1)⇒(2). Let V be a 2-dimensional operator subsystem of $C(X)$. Then V has a linear basis of the form $\{1, g\}$ for some nonscalar $g \in C(X)$, which we may assume to be self-adjoint by replacing it with $g + g^*$ if necessary. By scaling and scalar-translating g, we may furthermore assume that the spectrum of g is a subset of $[0, 1]$ containing 0 and 1. Given $\delta > 0$, by a simple perturbation argument we can construct a positive norm-one $h \in C(X)$ such that $\|h - g\| < \delta$, $h(x) = 0$ for all x in some nonempty open set U_0, and $h(x) = 1$ for all x in some nonempty open set U_1. By taking δ small enough we can make the unit ball of the operator system $W = \operatorname{span}\{1, h\}$ as close as we wish to the unit ball of V, and so we will obtain (2) once we show that W has a 1-independence set of positive density, and this can be done as in the proof of the corresponding implication in Proposition 4.2 using the partition of unity $\{h, 1-h\}$.

(2)⇒(3). Trivial.

(3)⇒(1). Argue as in the proof of the corresponding implication in Proposition 4.2. □

In [43] Huang and Ye proposed u.p.e. of all orders as the most suitable topological analogue of a K-system. In view of Proposition 4.2 we might then define a unital *-endomorphism α of a unital C^*-algebra A to be a C^*-algebraic K-system if for every finite set $\Omega \subseteq A$ and $\delta > 0$ there is a finite-dimensional operator subsystem $V \subseteq A$ which approximately includes Ω to within δ and has a $(1+\varepsilon)$-independence set of positive density for every $\varepsilon > 0$. This property holds prototypically for the shift on the infinite minimal tensor product $A^{\otimes\mathbb{Z}}$ for any unital C^*-algebra A, and it implies completely positive Voiculescu-Brown entropy [50], as can be see from Remark 3.10 of [48] and Lemma 4.1. In fact to deduce completely positive Voiculescu-Brown entropy all we need is that the collection of 2-dimensional operator subsystems of A which have a λ-independence set of positive density for some $\lambda \geq 1$ is dense in the collection of all 2-dimensional operator subsystems of A with respect to the metric given by Hausdorff distance between unit balls.

Remark 4.4. Following up on the discussion in the last part of Section 3, we point out that the results of this section hold more generally for any dynamical system (X, G) with G discrete and amenable if the relevant terms are interpreted or reformulated as follows. U.p.e., u.p.e of order n, and u.p.e. of all orders are defined in the same way as for singly generated systems. For a finite-dimensional operator subsystem $V \subseteq C(X)$ and a $\lambda \geq 1$, the left invariance version of Proposition 3.22 applies to the function $\varphi_{V,\lambda}$ given by $\varphi_{V,\lambda}(F) = \max\{|F \cap J| : J \text{ is a } \lambda\text{-independence set for } V\}$, so that we may define the *λ-independence density* $I(V, \lambda)$ of V as the limit of $\frac{1}{|F|}\varphi_{V,\lambda}(F)$ as F becomes more and more invariant. We can then replace "λ-independence set of positive density" everywhere above by "nonvanishing λ-independence density".

5. Topological sequence entropy, nullness, and independence

In this section we will examine the local theory of topological sequence entropy and nullness from the viewpoint of independence. Sequence entropy is developed in the literature for single continuous maps, but here we will work in the framework of a general dynamical system (X, G).

Following Goodman [37], for a sequence $\mathfrak{s} = \{s_n\}_{n\in\mathbb{N}}$ in G we define the topological sequence entropy of (X, G) with respect to $\mathfrak{s}$ and a finite open cover $\mathcal{U}$ of X by

$$\mathrm{h}_{\mathrm{top}}(X, \mathcal{U}; \mathfrak{s}) = \limsup_{n\to\infty} \frac{1}{n} \log N\bigg(\bigvee_{i=1}^{n} s_i^{-1}\mathcal{U}\bigg)$$

where $N(\cdot)$ denotes the minimal cardinality of a subcover. Following Huang, Li, Shao, and Ye [41], we call a nondiagonal pair $(x, y) \in X \times X$ a *sequence entropy pair* if for any disjoint closed neighbourhoods U and V of x and y, respectively, there exists a sequence $\mathfrak{s}$ in G such that $\mathrm{h}_{\mathrm{top}}(X, \{U^{\mathrm{c}}, V^{\mathrm{c}}\}; \mathfrak{s}) > 0$. More generally, following Huang, Maass, and Ye

[42] we call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k \setminus \Delta_k(X)$ a *sequence entropy tuple* if whenever $U_1, \dots, U_l$ are closed pairwise disjoint neighbourhoods of the distinct points in the list $x_1, \dots, x_k$, the open cover $\{U_1^{\rm c}, \dots, U_l^{\rm c}\}$ has positive topological sequence entropy with respect to some sequence in G. We say that (X, G) is *null* if $\mathrm{h_{top}}(X, \mathcal{U}; \mathfrak{s}) = 0$ for all open covers $\mathcal{U}$ of X and all sequences $\mathfrak{s}$ in G. We say that (X, G) is *nonnull* otherwise. Then the basic facts recorded in [41, Proposition 2.1] and [42, Proposition 3.2] also hold in our general setting.

Definition 5.1. We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ an *IN-tuple* (or an *IN-pair* in the case $k = 2$) if for any product neighbourhood $U_1 \times \dots \times U_k$ of $\boldsymbol{x}$ the tuple $(U_1, \dots, U_k)$ has arbitrarily large finite independence sets. We denote the set of IN-tuples of length k by $\mathrm{IN}_k(X, G)$.

We will show in Proposition 5.9 that sequence entropy tuples are exactly nondiagonal IN-tuples.

First we record some basic facts pertaining to IN-tuples. For a cover $\mathcal{U}$ of X and a sequence $\mathfrak{s}$ in G we denote by $\mathrm{h_c}(T, \mathcal{U}; \mathfrak{s})$ the combinatorial sequence entropy of $\mathcal{U}$ with respect to $\mathfrak{s}$, which is defined using the same formula as for the topological sequence entropy of open covers. From Lemma 3.3 we infer the following analogue of Lemma 3.4.

Lemma 5.2. *Let $k \geq 2$. Let $U_1, \dots, U_k$ be disjoint subsets of X and set $\mathcal{U} = \{U_1^{\rm c}, \dots, U_k^{\rm c}\}$. Then $\boldsymbol{U} := (U_1, \dots, U_k)$ has arbitrarily large finite independence sets if and only if $\mathrm{h_c}(T, \mathcal{U}; \mathfrak{s}) > 0$ for some sequence $\mathfrak{s}$ in G.*

From Lemma 3.7 we infer the following analogue of Lemma 3.8.

Lemma 5.3. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a k-tuple of subsets of X with arbitrarily large finite independence sets. Suppose that $A_1 = A_{1,1} \cup A_{1,2}$. Then at least one of the tuples $(A_{1,1}, \dots, A_k)$ and $(A_{1,2}, \dots, A_k)$ has arbitrarily large finite independence sets.*

Using Lemmas 5.2 and 5.3 we obtain the following analogue of Proposition 3.9.

Proposition 5.4. *The following are true:*

1. *Let $(A_1, \dots, A_k)$ be a tuple of closed subsets of X which has arbitrarily large finite independence sets. Then there exists an IN-tuple $(x_1, \dots, x_k)$ with $x_j \in A_j$ for all $1 \leq j \leq k$.*
2. $\mathrm{IN}_2(X, G) \setminus \Delta_2(X)$ *is nonempty if and only if (X, G) is nonnull.*
3. $\mathrm{IN}_k(X, G)$ *is a closed G-invariant subset of X^k.*
4. *Let $\pi : (X, G) \to (Y, G)$ be a factor map. Then $(\pi \times \dots \times \pi)(\mathrm{IN}_k(X, G)) = \mathrm{IN}_k(Y, G)$.*
5. *Suppose that Z is a closed G-invariant subset of X. Then $\mathrm{IN}_k(Z, G) \subseteq \mathrm{IN}_k(X, G)$.*

We remark that (2) and (4) of Proposition 5.4 show that the largest null factor of (X, G) is obtained from the closed invariant equivalence relation on X generated by the set of IN-pairs.

Corollary 5.5. *For a fixed G, the class of null G-systems is preserved under taking factors, subsystems and products.*

Corollary 5.6. *Suppose that (X,G) is nonnull. Then there exist an open cover $\mathcal{U} = \{U,V\}$ of X and a sequence $\mathfrak{s}$ in G such that $\mathrm{h}_{\mathrm{top}}(X,\mathcal{U};\mathfrak{s}) = \log 2$. In the case $G = \mathbb{Z}$ the sequence can be taken in $\mathbb{N}$.*

Definition 5.7. We say that a function $f \in C(X)$ is *null* if there does not exist a $\lambda \geq 1$ such that f has arbitrarily large finite ℓ_1-λ-isomorphism sets. Otherwise f is said to be *nonnull*.

The proof of the following proposition is similar to that of Proposition 3.10, where this time we use Theorem 5.8 in [49] as formulated for the more general context of isometric endomorphisms of Banach spaces.

Proposition 5.8. *Let $f \in C(X)$. Then f is nonnull if and only if there is an IN-pair (x,y) with $f(x) \neq f(y)$.*

In analogy with the situation for entropy, nondiagonal IN-tuples turn out to be the same as sequence entropy tuples, as follows from Lemma 5.2.

Theorem 5.9. *Let $(x_1,\dots,x_k)$ be a tuple in $X^k \setminus \Delta_k(X)$ with $k \geq 2$. Then $(x_1,\dots,x_k)$ is a sequence entropy tuple if and only if it is an IN-tuple.*

In parallel with Blanchard's definition of uniform positive entropy [8], we say that (X,G) is *uniformly nonnull* if $\mathrm{IN}_2(X,G) = X \times X$, or, equivalently, if any pair of nonempty open subsets of X has arbitrarily large finite independence sets. By Theorem 5.9 this is the same as s.u.p.e. as defined on page 1507 of [41]. We say that (X,G) is *completely nonnull* if the maximal null factor of (X,G) is trivial. Uniform nonnullness implies complete nonnullness, but the converse is false. Blanchard showed in [8, Example 8] that the shift action on $X = \{a,b\}^{\mathbb{Z}} \cup \{a,c\}^{\mathbb{Z}}$ has completely positive entropy but is not transitive. Hence this action is completely nonnull but by Theorem 8.6 fails to be uniformly nonnull.

We now briefly examine the tensor product viewpoint. In analogy with the case of entropy (see the beginning of Section 4), it can be shown that the system (X,G) is nonnull if and only if there is a 2-dimensional operator subsystem $V \subseteq C(X)$ and a $\lambda \geq 1$ such that V has arbitrarily large finite λ-independence sets for the induced C^*-dynamical system. Using arguments similar to those in the proof of Proposition 4.3 (with $\mathcal{S}_2(X)$ defined as in the discussion there) one can show:

Proposition 5.10. *For a dynamical system (X,G) the following are equivalent:*

1. *(X,G) is uniformly nonnull,*
2. *the collection of 2-dimensional operator subsystems of $C(X)$ which have arbitrarily large finite 1-independence sets is dense in $\mathcal{S}_2(X)$,*
3. *for every 2-dimensional operator subsystem $V \subseteq C(X)$ there is a $\lambda \geq 1$ such that V has arbitrarily large finite λ-independence sets.*

For $n \geq 2$ we say that (X,G) is *uniformly nonnull of order n* if $\mathrm{IN}_n(X,G) = X^n$, or, equivalently, if every n-tuple of nonempty open subsets of X has arbitrarily large finite independence sets. We say that (X,G) is *uniformly nonnull of all orders* if it is uniformly nonnull of order n for every $n \geq 2$. The analogue of Proposition 5.10 for uniform nonnullness of all orders is also valid and will be subsumed as part of Proposition 8.4 and Theorem 8.6 in connection with I-independence, to which uniform nonnullness of all orders is equivalent.

6. Tameness and independence

Let (X, G) be a dynamical system. We say that (X, G) is *tame* if no element $f \in C(X)$ has an infinite ℓ_1-isomorphism set and *untame* otherwise. The concept of tameness was introduced by Köhler in [54] under the term regularity. Here we are following the terminology of [33]. Actually in [33] the system (X, G) is defined to be tame if its enveloping semigroup is separable and Fréchet, which is equivalent to our geometric definition when X is metrizable.

Definition 6.1. We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ an *IT-tuple* (or an *IT-pair* in the case $k = 2$) if for any product neighbourhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ the tuple $(U_1, \dots, U_k)$ has an infinite independence set. We denote the set of IT-tuples of length k by $\mathrm{IT}_k(X, G)$.

In contrast to the density conditions in the context of entropy and sequence entropy, we are interested here in the existence of infinite sets along which independence (or, in the definition of tameness, equivalence to the standard ℓ_1 basis) occurs. This places us in the realm of Rosenthal's ℓ_1 theorem [70] and the Ramsey methods involved in its proof (see [38]). Indeed we begin by observing the following immediate consequence of [70, Theorem 2.2]. Note that though Theorem 2.2 of [70] is stated for sequences of pairs of disjoint subsets, the proof there works for sequences of tuples of (not necessarily disjoint) subsets.

Lemma 6.2. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a tuple of closed subsets of X. Then the pair $\boldsymbol{A}$ has an infinite independence set if and only if there is an infinite set $H \subseteq G$ such that for every infinite set $W \subseteq H$ there exists an $x \in X$ for which the sets $(x, A_j, W)^\perp := \{s \in W : sx \in A_j\}$ for $j = 1, \dots, k$ are all infinite.*

Lemma 6.3. *Let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a k-tuple of closed subsets of X with an infinite independence set. Suppose that $A_1 = A_{1,1} \cup A_{1,2}$ and that both $A_{1,1}$ and $A_{1,2}$ are closed. Then at least one of the tuples $(A_{1,1}, \dots, A_k)$ and $(A_{1,2}, \dots, A_k)$ has an infinite independence set.*

Proof. Let $H \subseteq G$ be as in Lemma 6.2 for $\boldsymbol{A}$. Suppose that neither of $(A_{1,1}, \dots, A_k)$ and $(A_{1,2}, \dots, A_k)$ has an infinite independence set. Then by Lemma 6.2 we can find infinite subsets $W_1 \supseteq W_2$ of H such that for every $x \in X$ and $i = 1, 2$ at least one of the sets $(x, A_{1,i}, W_i)^\perp$ and $(x, A_j, W_i)^\perp$ for $2 \le j \le k$ is finite. But there exists an $x \in X$ such that $(x, A_j, W_2)^\perp$ is infinite for all $1 \le j \le k$. Thus $(x, A_{1,1}, W_1)^\perp$ and $(x, A_{1,2}, W_2)^\perp$ are finite. Since $(x, A_1, W_2)^\perp \subseteq (x, A_{1,1}, W_2)^\perp \cup (x, A_{1,2}, W_2)^\perp$, we obtain a contradiction. Therefore at least one of the tuples $(A_{1,1}, \dots, A_k)$ and $(A_{1,2}, \dots, A_k)$ has an infinite independence set. $\square$

Proposition 6.4. *The following are true:*

1. *Let $(A_1, \dots, A_k)$ be a tuple of closed subsets of X which has an infinite independence set. Then there exists an IT-tuple $(x_1, \dots, x_k)$ with $x_j \in A_j$ for all $1 \le j \le k$.*
2. *$\mathrm{IT}_2(X, T) \setminus \Delta_2(X)$ is nonempty if and only if (X, G) is untame.*
3. *$\mathrm{IT}_k(X, G)$ is a closed G-invariant subset of X^k.*
4. *Let $\pi : (X, G) \to (Y, G)$ be a factor map. Then $(\pi \times \cdots \times \pi)(\mathrm{IT}_k(X, G)) = \mathrm{IT}_k(Y, G)$.*
5. *Suppose that Z is a closed G-invariant subset of X. Then $\mathrm{IT}_k(Z, G) \subseteq \mathrm{IT}_k(X, G)$.*

The proof of Proposition 6.4 is similar to that of Proposition 3.9, with (1) following from Lemma 6.3 and (2) following from Proposition 6.6 below.

We remark that (2) and (4) of Proposition 6.4 show that the largest tame factor of (X, G) is obtained from the closed invariant equivalence relation on X generated by the set of IT-pairs.

Corollary 6.5. *For a fixed G, the class of tame G-systems is preserved under taking factors, subsystems, and products.*

Proposition 6.6. *Let $f \in C(X)$. Then f has an infinite ℓ_1-isomorphism set if and only if there is an* IT*-pair (x, y) with $f(x) \neq f(y)$.*

Proof. The "if" part follows as in the analogous situation for entropy by the well-known Rosenthal-Dor argument. For the "only if" part, by Proposition 6.4(1) it suffices to show the existence of a pair of disjoint closed subsets A and B of X which have an infinite independence set and satisfy $f(A) \cap f(B) = \emptyset$. Let $H = \{s_j : j \in \mathbb{N}\} \subseteq G^{\mathrm{op}}$ be an ℓ_1-isomorphism set for f. Then the sequence $\{s_j f\}_{j\in\mathbb{N}}$ has no weakly convergent subsequence. Using Lebesgue's theorem one sees that $\{s_j f\}_{j\in\mathbb{N}}$ has no pointwise convergent subsequence. In Gowers' proof of Rosenthal's ℓ_1 theorem [38, page 1079-1080] it is shown that there exist disjoint closed subsets $Z_1, Z_2 \subseteq \mathbb{C}$ and a subsequence $\{s_{n_k} f\}_{k\in\mathbb{N}}$ such that the sequence of pairs $\{((s_{n_k} f)^{-1}(Z_1), (s_{n_k} f)^{-1}(Z_2))\}_{k\in\mathbb{N}}$ is independent. Therefore $\{s_{n_k} : k \in \mathbb{N}\}$ is an infinite independence set for the pair $(f^{-1}(Z_1), f^{-1}(Z_2))$, yielding the proposition. □

In parallel with the cases of entropy and nullness, we say that (X, G) is *uniformly untame* if $\mathrm{IT}_2(X, G) = X \times X$, or, equivalently, if any pair of nonempty open subsets of X has an infinite independence set. We say that (X, G) is *completely untame* if the maximal tame factor of (X, G) is trivial. Complete untameness is strictly weaker than uniform untameness, as illustrated by Blanchard's example mentioned in the paragraph following Proposition 5.9.

We demonstrate in the following example that untame systems need not be null by constructing a WAP (weakly almost periodic) nonnull subshift. The proof of [49, Corollary 5.7] shows that if (X, G) is HNS (hereditarily nonsensitive [34, Definition 9.1]) then it is tame. Since WAP systems are HNS [34, Section 9], our example is tame.

Example 6.7. Let $0 < m_1 < m_2 < \cdots$ be a sequence in $\mathbb{N}$ with $m_j - m_i > m_i - m_k$ for all $j > i > k$. Let $\{k_n\}_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathbb{N}$ and let $S_j = \sum_{i=1}^{j} k_i$ for $j \in \mathbb{N}$ be the partial sums of $\{k_n\}_{n\in\mathbb{N}}$. Denote by A_j the set of all elements in $\{0, 1\}^{\mathbb{Z}}$ whose support is contained in $\{m_k : S_{j-1} < k \leq S_j\}$. One checks easily that the union X of the orbits of $\bigcup_j A_j$ under the shift T is closed. Thus $(X, \mathbb{Z})$ is a subshift. Denote by Z the set of elements in X supported exactly at one point. It is easy to see that $\mathrm{IN}_2(X, \mathbb{Z}) \setminus \Delta_2(X) = (Z \times \{0\}) \cup (\{0\} \times Z)$. Since the set of elements in $C(X)$ whose $\mathbb{Z}$-orbit is precompact in the weak topology of $C(X)$ is a closed $\mathbb{Z}$-invariant algebra of $C(X)$, to see that $(X, \mathbb{Z})$ is WAP it suffices to check that the orbit of f is precompact in the weak topology, where $f \in C(X)$ is defined by $f(x) = x(0)$ for $x \in X$. However, the union of the zero function and the orbit of f is evidently compact in $\mathbb{C}^X$ and thus is compact in the weak topology by a result of Grothendieck [39] [31, Theorem 1.43.1].

In Section 11 we will see that there exist minimal tame nonnull $\mathbb{Z}$-systems.

Turning finally to tensor products, in analogy with entropy and sequence entropy one can show that the system (X, G) is untame if and only if there is a 2-dimensional operator subsystem $V \subseteq C(X)$ with an infinite independence set for the induced C^*-dynamical system. With $\mathcal{S}_2(X)$ defined as in the paragraph preceding Proposition 4.3, one has the following characterizations of uniform untameness (see the proof of Proposition 8.4).

Proposition 6.8. *For a dynamical system (X, G) the following are equivalent:*

(1) (X, G) *is uniformly untame,*
(2) *the collection of* 2*-dimensional operator subsystems of* $C(X)$ *which have an infinite* 1*-independence set is dense in* $\mathcal{S}_2(X)$*,*
(3) *the collection of* 2*-dimensional operator subsystems of* $C(X)$ *which have an infinite* λ*-independence set for some* $\lambda \geq 1$ *is dense in* $\mathcal{S}_2(X)$*.*

For $n \geq 2$ we say that (X, G) is *uniformly untame of order* n if $\mathrm{IT}_n(X, G) = X^n$, or, equivalently, if every n-tuple of nonempty open subsets of X has an infinite independence set. If (X, G) is uniformly untame of order n for each $n \geq 2$ then we say that it is *uniformly untame of all orders*. For the analogue of Proposition 6.8 for uniform untameness of all orders see Proposition 8.4 and Theorem 8.6.

7. Tame extensions of minimal systems

An extension $\pi : X \to Y$ of dynamical systems with acting group G is said to be *tame* if $\mathrm{IT}_\pi \setminus \Delta_2(X) = \emptyset$, where $R_\pi := \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$ and $\mathrm{IT}_\pi := \mathrm{IT}_2(X, G) \cap R_\pi$. In this section we will analyze the structure of tame extensions of minimal systems. Throughout G will be a group, and we will frequently specialize to the Abelian case, to which the main results (Theorems 7.14, 7.15, and 7.19) apply. Theorems 7.14 and 7.15 address the relation between proximality and equicontinuity within the frame of tame extensions, while Theorem 7.19 asserts that tame minimal systems are uniquely ergodic.

We refer the reader to [3, 78] for general information on extensions of dynamical systems.

As a direct consequence of Proposition 6.4(4) we have:

Proposition 7.1. *The following are true:*

(1) *Let* $\psi : X \to Y$ *and* $\varphi : Y \to Z$ *be extensions. Then* $\varphi \circ \psi$ *is tame if and only if both* φ *and* ψ *are tame.*
(2) *Let* $\pi_j : X_j \to Y_j$ *for* $j \in J$ *be extensions. If every* π_j *is tame, then* $\prod_{j \in J} \pi_j : \prod_{j \in J} X_j \to \prod_{j \in J} Y_j$ *is tame.*
(3) *Let* $\{\pi_\alpha : X_\alpha \to Y\}_{\alpha < \nu}$ *be an inverse system of extensions, where* ν *is an ordinal. Then* $\pi := \varprojlim \pi_\alpha$ *is tame if and only if every* π_α *is tame.*
(4) *For every commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\tau"] & Y \end{tikzcd}$$

of extensions, if X' *is a subsystem of* $Y' \times X$*,* π' *and* σ *are the restrictions to* X' *of the coordinate projections of* $Y' \times X$ *onto* Y' *and* X*, respectively, and* π *is tame, then* π' *is tame.*

A continuous map $f : A \to B$ between topological spaces is said to be *semi-open* if the image of every nonempty open subset of A under f has nonempty interior. For a dynamical system (X, G), denote by $\mathrm{RP}(X, G)$ the *regionally proximal relation* of X, that is, $\mathrm{RP}(X, G) = \bigcap_{U \in \mathcal{U}_X} \overline{GU}$, where $\mathcal{U}_X$ is the collection of open neighbourhoods of the diagonal $\Delta_2(X)$ in X^2. Following [41] we call a pair $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ a *weakly mixing pair* if for any neighbourhoods U_1 and U_2 of x_1 and x_2, respectively, there exists an $s \in G$ such that $sU_1 \cap U_1 \neq \emptyset$ and $sU_1 \cap U_2 \neq \emptyset$. We denote the set of weakly mixing pairs by $\mathrm{WM}(X, G)$. When (X, G) is minimal, $\mathrm{WM}(X, G) = \mathrm{RP}(X, G) \setminus \Delta_2(X)$ [41, Theorem 2.4(1)]. The proofs of Lemma 3.1 and Corollary 4.1 in [41] yield the following two results.

Lemma 7.2. *Suppose that G is Abelian. Let $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ and $A = \overline{G(x_1, x_2)}$. If $\pi_1 : A \to X$ is semi-open, where π_1 is the projection to the first coordinate, then the following are equivalent:*

(1) $(x_1, x_2) \in \mathrm{IT}_2(X, G)$,
(2) $(x_1, x_2) \in \mathrm{IN}_2(X, G)$,
(3) $(x_1, x_2) \in \mathrm{WM}(X, G)$.

Lemma 7.3. *Suppose that G is Abelian. Let $\pi : (X, G) \to (Y, G)$ be a distal extension of minimal systems. Let $(x_1, x_2) \in R_\pi \setminus \Delta_2(X)$. Then the following are equivalent:*

(1) $(x_1, x_2) \in \mathrm{IT}_2(X, G)$,
(2) $(x_1, x_2) \in \mathrm{IN}_2(X, G)$,
(3) $(x_1, x_2) \in \mathrm{WM}(X, G)$,
(4) $(x_1, x_2) \in \mathrm{RP}(X, G)$.

We call an extension $\pi : X \to Y$ a *Bronštein extension* if R_π has a dense subset of almost periodic points. Recall that π is said to be *highly proximal* if for every nonempty open subset U of X and every point $z \in Y$ there exists an $s \in G$ such that $\pi^{-1}(z) \subseteq sU$. In this case (X, G) and (Y, G) are necessarily minimal and π is proximal. We say that π is *strictly PI* if it can be obtained by a transfinite succession of proximal and equicontinuous extensions, and *PI* if there exists a proximal extension $\psi : X' \to X$ such that $\pi \circ \psi$ is strictly PI.

If in these definitions proximality is replaced by high proximality, then we obtain the notions of *strictly HPI* extensions and *HPI* extensions.

Lemma 7.4. *Suppose that G is Abelian. Let $\pi : X \to Y$ be a tame Bronštein extension of minimal systems. Suppose that π has no nontrivial equicontinuous factors. Then π is an isomorphism.*

Proof. For an extension $\psi : X' \to Y'$ denote by RP_ψ the *relative regionally proximal relation* of ψ, that is, $\mathrm{RP}_\psi = \bigcap_{U \in \mathcal{U}_{X'}} \overline{GU \cap R_\psi}$, where $\mathcal{U}_{X'}$ is the collection of open neighbourhoods of the diagonal $\Delta_2(X')$ in $X' \times X'$. As X is minimal, if we denote by S_π the smallest closed G-invariant equivalence relation on X containing RP_π, then the induced extension $X/S_\pi \to Y$ is an equicontinuous factor of π [25] [78, Theorem V.2.21]. Since π has no nontrivial equicontinuous factors, $S_\pi = R_\pi$. It is a theorem of Ellis that for any Bronštein extension ψ of minimal systems, RP_ψ is an equivalence relation [77, Theorem 2.6.2] [78, Theorem VI.3.20]. Therefore $R_\pi = S_\pi = \mathrm{RP}_\pi \subseteq \mathrm{RP}(X, G)$. Since X is minimal, $\mathrm{WM}(X, G) = \mathrm{RP}(X, G) \setminus \Delta_2(X)$ [41, Theorem 2.4(1)]. Thus $R_\pi \setminus \Delta_2(X) \subseteq \mathrm{WM}(X, G)$.

Suppose that π is not an isomorphism. Since π is a Bronštein extension, we can find an almost periodic point (x_1, x_2) in the nonempty open subset $R_\pi \setminus \Delta_2(X)$ of R_π. As extensions of minimal systems are semi-open, we conclude that $(x_1, x_2) \in \mathrm{IT}_2(X, G)$ by Lemma 7.2. This is in contradiction to the tameness of π. Thus π is an isomorphism. □

Lemma 7.5. *Suppose that G is Abelian. Then any tame extension $\pi : X \to Y$ of minimal systems is PI.*

Proof. Consider the canonical PI tower of π [27] [78, Theorem VI.4.20]. This is a commuting diagram of extensions of minimal systems of the form displayed in Proposition 7.1(4), where σ is proximal, τ is strictly PI, π' is RIC (relatively incontractible), and π' has no nontrivial equicontinuous factors. Furthermore, X' is a subsystem of $Y' \times X$, and π' and σ are the restrictions to X' of the coordinate projections of $Y' \times X$ onto Y' and X, respectively (see for example [78, VI.4.22]). Since π is tame, by Proposition 7.1(4) so is π'. As every RIC-extension is a Bronštein extension [27, Corollary 5.12] [78, Corollary VI.2.8], π' is a Bronštein extension. Then π' is an isomorphism by Lemma 7.4. Hence π is PI. □

Lemma 7.5 generalizes a result of Glasner [33, Theorem 2.3], who proved it for tame metrizable (X, G).

A theorem of van der Woude asserts that an extension $\pi : X \to Y$ of minimal systems is HPI if and only if every topologically transitive subsystem W of R_π for which both of the coordinate projections $\pi_i : W \to X$ are semi-open is minimal [79, Theorem 4.8].

Lemma 7.6. *Suppose that G is Abelian. Then any tame extension $\pi : X \to Y$ of minimal metrizable systems is HPI.*

Proof. Let S_π be as in the proof of Lemma 7.4. Then we have the natural extensions $\psi : X \to X/S_\pi$ and $\varphi : X/S_\pi \to Y$ with $\pi = \varphi \circ \psi$, and φ is equicontinuous. To show that π is HPI, it suffices to show that ψ is HPI. Let W be a topologically transitive subsystem of $R_\psi = S_\pi$ for which both of the coordinate projections $\pi_i : W \to X$ are semi-open. By the theorem of van der Woude, the proof will be complete once we show that W is minimal. Since G is Abelian, X has a G-invariant Borel probability measure by a result of Markov and Kakutani [22, Theorem VII.2.1]. Then $\mathrm{RP}(X, G)$ is an equivalence relation by [3, Theorem 9.8] [78, Theorem V.1.17]. Thus $S_\pi \subseteq \mathrm{RP}(X, G)$. Since X is minimal, $\mathrm{WM}(X, G) = \mathrm{RP}(X, G) \setminus \Delta_2(X)$ [41, Theorem 2.4(1)]. Then $R_\psi \setminus \Delta_2(X) = S_\pi \setminus \Delta_2(X) \subseteq \mathrm{WM}(X, G)$. As X is metrizable and W is topologically transitive, we can find a transitive point (x_1, x_2) of W. Since π is tame, by Lemma 7.2 we must have $x_1 = x_2$. Therefore $W = \Delta_2(X)$ is minimal. □

Lemma 7.7. *Suppose that G is Abelian. Let $\pi : X \to Y$ be an open tame extension of minimal systems. If Y is metrizable, then π is HPI.*

To prove Lemma 7.7 we need a variation of the "reduction-to-the-metric-case construction" in [58], which is in turn a relativization of a method of Ellis [26].

Let $\pi : (X, G) \to (Y, G)$ be an open extension of G-systems. Denote by $\mathrm{CP}(X)$ the set of all continuous pseudometrics on X. For a $\rho \in \mathrm{CP}(X)$ and a countable subgroup H of G, define $C_{\rho,H} := \{(x_1, x_2) \in X \times X : \rho(sx_1, sx_2) = 0 \text{ for all } s \in H\}$. Then $C_{\rho,H}$ is a closed H-invariant equivalence relation on X. Denote $X/C_{\rho,H}$ by $X_{\rho,H}$, and denote the quotient map $X \to X_{\rho,H}$ by $\psi_{\rho,H}$. Say $H = \{s_1, s_2, \dots\}$ with $s_1 = e$. Define a pseudometric $d_{\rho,H}$ on X by

$$d_{\rho,H}(x_1, x_2) := \sum_{i=1}^{\infty} 2^{-i}\rho(s_i x_1, s_i x_2) \tag{1}$$

for all $x_1, x_2 \in X$. Then $X_{\rho,H}$ is a metrizable space with a metric $d'_{\rho,H}$ defined by

$$d'_{\rho,H}(\psi_{\rho,H}(x_1), \psi_{\rho,H}(x_2)) = d_{\rho,H}(x_1, x_2) \tag{2}$$

for all $x_1, x_2 \in X$. Let $C'_{\rho,H} = \{(y_1, y_2) \in Y \times Y : \psi_{\rho,H}(\pi^{-1}(y_1)) = \psi_{\rho,H}(\pi^{-1}(y_2))\}$. Then $C'_{\rho,H}$ is a closed H-invariant equivalence relation on Y. Denote $Y/C'_{\rho,H}$ by $Y_{\rho,H}$, and denote the quotient map $Y \to Y_{\rho,H}$ by $\tau_{\rho,H}$. Define a map $\sigma_{\rho,H} : X \to X_{\rho,H} \times Y_{\rho,H}$ by $\sigma_{\rho,H}(x) = (\psi_{\rho,H}(x), \tau_{\rho,H} \circ \pi(x))$, and write $\sigma_{\rho,H}(X)$ as $X^*_{\rho,H}$. Denote by $\pi_{\rho,H}$ the restriction to $X^*_{\rho,H}$ of the projection of $X_{\rho,H} \times Y_{\rho,H}$ onto $Y_{\rho,H}$.

Lemma 7.8. [78, Lemma V.3.3] *For every $\rho \in \mathrm{CP}(X)$ and countable subgroup H of G the diagram*

$$\begin{tikzcd} (X,H) \arrow[r, "\pi"] \arrow[d, "\sigma_{\rho,H}"'] & (Y,H) \arrow[d, "\tau_{\rho,H}"] \\ (X^*_{\rho,H}, H) \arrow[r, "\pi_{\rho,H}"] & (Y_{\rho,H}, H) \end{tikzcd} \tag{3}$$

of extensions of H-systems commutes and $X^*_{\rho,H}$ is metrizable.

For $\rho_1, \rho_2 \in \mathrm{CP}(X)$ we write $\rho_1 \preceq \rho_2$ if $\rho_1(x_1, x_2) = 0$ whenever $\rho_2(x_1, x_2) = 0$.

Lemma 7.9. *Let W be a nonminimal G-subsystem of R_π. Then there exists a $\rho_0 \in \mathrm{CP}(X)$ such that $((\sigma_{\rho,H} \times \sigma_{\rho,H})(W), H)$ is nonminimal for every $\rho \in \mathrm{CP}(X)$ with $\rho_0 \preceq \rho$ and every countable subgroup H of G.*

Proof. Let W' be a minimal G-subsystem of W, and let $(x_1, x_2) \in W \setminus W'$. As in the proof of [78, Lemma VI.4.41], it can be shown that there exists a $\rho_0 \in \mathrm{CP}(X)$ such that $\max(\rho_0(x'_1, x_1), \rho_0(x'_2, x_2)) > 0$ for all $(x'_1, x'_2) \in W'$. Then for every $\rho \in \mathrm{CP}(X)$ with $\rho_0 \preceq \rho$ and every countable subgroup H of G we have $(\sigma_{\rho,H} \times \sigma_{\rho,H})(x_1, x_2) \notin (\sigma_{\rho,H} \times \sigma_{\rho,H})(W')$ so that $((\sigma_{\rho,H} \times \sigma_{\rho,H})(W), H)$ is nonminimal. □

For all $\rho_1 \preceq \rho_2$ in $\mathrm{CP}(X)$ and all countable subgroups $H_1 \subseteq H_2$ of G it is clear that there exist unique maps $\sigma_{21} : X^*_{\rho_2,H_2} \to X^*_{\rho_1,H_1}$ and $\tau_{21} : Y_{\rho_2,H_2} \to Y_{\rho_1,H_1}$ such that $\sigma_{21} \circ \sigma_{\rho_2,H_2} = \sigma_{\rho_1,H_1}$ and $\tau_{21} \circ \tau_{\rho_2,H_2} = \tau_{\rho_1,H_1}$. For $\rho_1 \preceq \rho_2 \preceq \cdots$ in $\mathrm{CP}(X)$ and countable subgroups $H_1 \subseteq H_2 \subseteq \cdots$ of G, we write X^*_∞ and Y_∞ for $\varprojlim X^*_{\rho_n,H_n}$ and $\varprojlim Y_{\rho_n,H_n}$, respectively. Then we have induced maps $\sigma_\infty : X \to X^*_\infty$, $\pi_\infty : X^*_\infty \to Y_\infty$, and $\tau_\infty : Y \to Y_\infty$. The diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\sigma_\infty"'] & Y \arrow[d, "\tau_\infty"] \\ X^*_\infty \arrow[r, "\pi_\infty"] & Y_\infty \end{tikzcd} \tag{4}$$

is easily seen to commute. Moreover, (4) can be identified in a natural way with (3) taking

$$\rho := \sum_{n=1}^{\infty} 2^{-n}\rho_n/(\mathrm{diam}(\rho_n)+1) \quad \text{and} \quad H = \bigcup_{n=1}^{\infty} H_n. \tag{5}$$

Lemma 7.10. *Suppose that (X,G) is minimal. Let W be a topologically transitive G-subsystem of R_π such that the coordinate projections $\pi_i : W \to X$ are both semi-open. Then for every $\rho_0 \in \mathrm{CP}(X)$ and countable subgroup H_0 of G there exist a $\rho \in \mathrm{CP}(X)$ with $\rho_0 \preceq \rho$ and a countable subgroup H of G with $H_0 \subseteq H$ such that*

(a) *$(X^*_{\rho,H}, H)$ is a minimal H-system,*
(b) *$((\sigma_{\rho,H}\times\sigma_{\rho,H})(W), H)$ is a topologically transitive H-system,*
(c) *the coordinate projections $\pi^*_i : (\sigma_{\rho,H}\times\sigma_{\rho,H})(W) \to X^*_{\rho,H}$ for $i = 1,2$ are both semi-open.*

Proof. We shall show by induction that there exist $\rho_0 \preceq \rho_1 \preceq \cdots$ in $\mathrm{CP}(X)$, countable subgroups $H_0 \subseteq H_1 \subseteq \cdots$ of G and a sequence $\{(x_n, x'_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ of elements in W such that for every $n \in \mathbb{Z}_{\geq 0}$ the following conditions are satisfied:

(a') $H_{n+1}\sigma^{-1}_{\rho_n,H_n}(U) = X$ for any nonempty open subset U of $X^*_{\sigma_n,H_n}$,
(b') $(\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})(H_{n+1}(x_n,x'_n))$ is dense in $(\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})(W)$,
(c') for any nonempty open subset V of $(\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})(W)$ there exist $t_1,t_2 \in X$ and $\delta > 0$ such that $B_{\rho_{n+1}}(t_i,\delta) \subseteq \pi_i((\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})^{-1}(V)\cap W)$ for $i = 1,2$, where $B_{\rho_{n+1}}(t_i,\delta) := \{x \in X : \rho_{n+1}(x,t_i) < \delta\}$.

We first indicate how this can be used to prove the lemma. Define ρ and H via (5). Then conditions (a) and (b) follow from (a') and (b'), respectively, as in the proof of [78, Lemma VI.4.43]. We may identify $(\sigma_{\rho,H}\times\sigma_{\rho,H})(W)$ with $(\sigma_\infty\times\sigma_\infty)(W) = \varprojlim(\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})(W)$. Thus for any nonempty open subset $V' \subseteq (\sigma_{\rho,H}\times\sigma_{\rho,H})(W)$ we can find an $n \in \mathbb{N}$ and a nonempty open subset V of $(\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})(W)$ such that $V' \supseteq (\sigma_{\infty,n}\times\sigma_{\infty,n})^{-1}(V)\cap(\sigma_\infty\times\sigma_\infty)(W) = (\sigma_\infty\times\sigma_\infty)((\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})^{-1}(V)\cap W)$, where $\sigma_{\infty,n} : X^*_\infty \to X^*_{\sigma_n,H_n}$ is the natural map. Let t_i and δ be as in (c'). Then

$$\begin{aligned}
\pi^*_i(V') &\supseteq \pi^*_i((\sigma_\infty\times\sigma_\infty)((\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})^{-1}(V)\cap W)) \\
&= \sigma_{\rho,H}(\pi_i((\sigma_{\rho_n,H_n}\times\sigma_{\rho_n,H_n})^{-1}(V)\cap W)) \\
&\supseteq \sigma_{\rho,H}(B_{\rho_{n+1}}(t_i,\delta)) \supseteq \sigma_{\rho,H}(B_\rho(t_i,\delta')) \\
&\supseteq \tilde{\pi}^{-1}_{\rho,H}(B_{d'_{\rho,H}}(\psi_{\rho,H}(t_i),\delta')),
\end{aligned}$$

where $\delta' := 2^{-n-1}\delta/(\mathrm{diam}(\rho_{n+1})+1)$, $d'_{\rho,H}$ is the metric on $X_{\rho,H}$ defined in (2), and $\tilde{\pi}_{\rho,H} : X^*_{\rho,H} \to X_{\rho,H}$ is the coordinate projection. Therefore $\pi^*_i(V')$ has nonempty interior and hence condition (c) follows from (c').

It remains to construct $\{\rho_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, and $\{(x_n,x'_n)\}_{n\geq 0}$ satisfying (a'), (b') and (c'). Note that (a') and (b') do not depend on the choice of ρ_{n+1} while (c') does not depend on the choice of H_{n+1} and (x_n,x'_n). Thus we can choose H_{n+1} and (x_n,x'_n) to satisfy (a') and (b') exactly as in the proof of [78, Lemma VI.4.43]. Now we explain how to choose ρ_1 in order to satisfy (c'). Repeating the procedure we will obtain the desired ρ_n.

Let $\{V_1, V_2, \dots\}$ be a countable nonempty open base for the topology on the metrizable space $(\sigma_{\rho_0,H_0} \times \sigma_{\rho_0,H_0})(W)$. Then $Z_{m,i} := \pi_i((\sigma_{\rho_0,H_0} \times \sigma_{\rho_0,H_0})^{-1}(V_m) \cap W)$ has nonempty interior for each $m \in \mathbb{N}$ and $i = 1, 2$. Pick a $t_{m,i}$ in the interior of $Z_{m,i}$, and take a continuous function $f_{m,i}$ on X such that $0 \leq f_{m,i} \leq 1$, $f_{m,i}(t_{m,i}) = 0$ and $f_{m,i}|_{Z^{\mathrm{c}}_{m,i}} = 1$. Define $\rho_{m,i} \in \mathrm{CP}(X)$ by $\rho_{m,i}(z_1, z_2) = |f_{m,i}(z_1) - f_{m,i}(z_2)|$ for $z_1, z_2 \in X$, and define $\rho_1 \in \mathrm{CP}(X)$ by $\rho_1 = \rho_0 + \sum_{m=1}^{\infty} 2^{-m}(\rho_{m,1} + \rho_{m,2})$. Then $B_{\rho_1}(t_{m,i}, 2^{-m}) \subseteq Z_{m,i}$. One sees immediately that (c') holds for $n = 0$. This completes the proof of Lemma 7.10. □

Lemma 7.11. *Suppose that Y is metrizable. Then there exists a $\rho_1 \in \mathrm{CP}(X)$ such that $\tau_{\rho,H}$ is an isomorphism for every $\rho \in \mathrm{CP}(X)$ with $\rho_1 \preceq \rho$ and every countable subgroup H of G.*

Proof. Let d be a metric on Y inducing its topology. Simply define $\rho_1 \in \mathrm{CP}(X)$ by $\rho_1(x_1, x_2) = d(\pi(x_1), \pi(x_2))$ for all $x_1, x_2 \in X$. □

Proof of Lemma 7.7. Suppose that π is not HPI. By the theorem of van der Woude there exists a topologically transitive nonminimal G-subsystem W of R_π such that the coordinate projections $W \to X$ are both semi-open. Let ρ_0 and ρ_1 be as in Lemmas 7.9 and 7.11, respectively. Take ρ and H as in Lemma 7.10 such that $\rho \succeq \rho_0 + \rho_1$. Then $\tau_{\rho,H}$ is an isomorphism and $((\sigma_{\rho,H} \times \sigma_{\rho,H})(W), H)$ is nonminimal. Since π is tame as an extension of G-systems, it is also tame as an extension of H-systems. By Proposition 7.1(1), $\pi_{\sigma,H}$ is a tame extension of H-systems. By Lemma 7.6 $\pi_{\rho,H}$ is HPI. Thus any topologically transitive H-subsystem of $R_{\pi_{\rho,H}}$ for which both of the coordinate projections to $X^*_{\rho,H}$ are semi-open must be minimal by the theorem of van der Woude. This is a contradiction to Lemma 7.10. Therefore π is HPI. □

The proof of the next lemma follows essentially that of [41, Theorem 4.3].

Lemma 7.12. *Suppose that G is Abelian. Let $\pi : X \to Y$ be a PI (resp. HPI) tame extension of minimal systems such that every equicontinuous factor of (X, G) factors through π. Then π is proximal (resp. highly proximal).*

Proof. Let $\varphi : X' \to X$ be a proximal (resp. highly proximal) extension such that $\pi \circ \varphi$ is strictly PI (resp. strictly HPI). Replacing X' by a minimal G-subsystem we may assume that X' is minimal. Denote by ψ the canonical extension of (X, G) over its maximal equicontinuous factor (X_{eq}, G). Then ψ factors through π by our assumption. Since φ is proximal, (X_{eq}, G) is also the maximal equicontinuous factor of (X', G). Now we need:

Lemma 7.13. *Let (Y'_1, G) and (Y'_2, G) be systems with extensions $X' \to Y'_1$, $\theta : Y'_1 \to Y'_2$, and $Y'_2 \to Y$ such that the composition $X' \to Y'_1 \to Y'_2 \to Y$ is $\pi \circ \varphi$. Suppose that θ is distal. Then θ is an isomorphism.*

Proof. Clearly (X_{eq}, G) is also the maximal equicontinuous factor of (Y'_1, G). Suppose that y'_1 and y'_2 are distinct points in Y'_1 with $\theta(y'_1) = \theta(y'_2)$. Since G is Abelian, Y'_1 has a G-invariant Borel probability measure by a result of Markov and Kakutani [22, Theorem VII.2.1]. Then $(Y'_1/\mathrm{RP}(Y'_1, G), G)$ is the maximal equicontinuous factor of (Y'_1, G) by [3, Theorem 9.8] [78, Theorem V.1.17]. Thus $(y'_1, y'_2) \in \mathrm{RP}(Y'_1, G)$. So $(y'_1, y'_2) \in \mathrm{IT}_2(Y'_1, G)$ by Lemma 7.3. By Proposition 6.4 we can find in X' preimages x'_1 and x'_2 of y'_1 and y'_2, respectively, such that $(x'_1, x'_2) \in \mathrm{IT}_2(X', G)$. Then $(\varphi(x'_1), \varphi(x'_2)) \in \mathrm{IT}_2(X, G) \cap R_\pi = \mathrm{IT}_\pi$.

Since θ is distal, the pair (y_1', y_2') is distal. Then the pair (x_1', x_2') is also distal. As φ is proximal, $\varphi(x_1') \neq \varphi(x_2')$. Thus π is not tame, which contradicts our assumption. Therefore θ is an isomorphism. $\square$

Back to the proof of Lemma 7.12. Since equicontinuous extensions are distal, by Lemma 7.13 we conclude that $\pi \circ \varphi$ can be obtained by a transfinite succession of proximal (resp. highly proximal) extensions. Since proximal (resp. highly proximal) extensions are preserved under transfinite compositions, $\pi \circ \varphi$ is proximal (resp. highly proximal), and hence π is proximal (resp. highly proximal). $\square$

Theorem 7.14. *Suppose that G is Abelian. Let $\pi : X \to Y$ be a tame extension of minimal systems. Consider the following conditions:*

(1) *π is highly proximal,*
(2) *π is proximal,*
(3) *every equicontinuous factor of (X, G) factors through π.*

Then one has (1)$\Rightarrow$(2)$\Leftrightarrow$(3). Moreover, if X is metrizable or π is open and Y is metrizable, then conditions (1) to (3) are all equivalent.

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are trivial, and (3)$\Rightarrow$(2) follows from Lemmas 7.5 and 7.12. When X is metrizable, (3)$\Rightarrow$(1) follows from Lemmas 7.6 and 7.12. When π is open and Y is metrizable, (3)$\Rightarrow$(1) follows from Lemmas 7.7 and 7.12. $\square$

Theorem 7.15. *Suppose that G is Abelian. Let $\pi : X \to Y$ be a tame extension of minimal systems. Then, up to isomorphisms, π has a unique decomposition as $\pi = \varphi \circ \psi$ such that ψ is proximal and φ is equicontinuous. If furthermore X is metrizable or π is open and Y is metrizable, then ψ is highly proximal.*

Proof. When such a decomposition exists, clearly φ must be the maximal equicontinuous factor of π. This proves uniqueness. Let RP_π and S_π be as in the proof of Lemma 7.4. Then we have the natural extensions $\psi : X \to X/S_\pi$ and $\varphi : X/S_\pi \to Y$ with $\pi = \varphi \circ \psi$, and φ is equicontinuous. For any extension $\theta : X \to W$ with (W, G) equicontinuous, since $\mathrm{RP}_\pi \subseteq \mathrm{RP}(X, G) \subseteq R_\theta$ we see that θ factors through ψ. As π is tame, so is ψ. By Theorem 7.14 ψ is proximal. If X is metrizable, then by Theorem 7.14 ψ is highly proximal. If π is open and Y is metrizable, then by Lemma 7.7 π is HPI. Using van der Woude's characterization of HPI extensions one sees immediately that ψ is also HPI. By Lemma 7.12, ψ is highly proximal. $\square$

Corollary 7.16. *Suppose that G is Abelian. Then every tame minimal system (X, G) is a highly proximal extension of an equicontinuous system.*

Corollary 7.16 answers a question of Glasner [33, Problem 2.5], who asked whether every metrizable tame minimal system (X, G) with G Abelian is a proximal extension of an equicontinuous system. It also generalizes [41, Theorem 4.3] in which the conclusion is established for metrizable null minimal systems $(X, \mathbb{Z})$.

Recall that a subset $H \subseteq G$ is called *thick* if for any finite $F \subseteq G$, one has $H \supseteq sF$ for some $s \in G$. We say that $H \subseteq G$ is *Poincaré* if for any measure preserving action of G on a finite measure space $(Y, \mathscr{B}, \mu)$ and any $A \in \mathscr{B}$ with $\mu(A) > 0$, one has $\mu(sA \cap A) > 0$ for some $s \in H$. The argument on page 74 of [29] shows that every thick set is Poincaré.

For a dynamical system (X,G) and a Borel subset $U \subseteq X$, denote by $N(U,U)$ the set $\{s \in G : sU \cap U \neq \emptyset\}$. If $\mu(U) > 0$ for some G-invariant Borel probability measure μ on X, then $N(U,U)$ has nonempty intersection with every Poincaré set. In particular, in this case $N(U,U)$ has nonempty intersection with every thick set, or, equivalently, $N(U,U)$ is syndetic [31, page 16]. Using this fact and Lemma 7.2, one sees that the proof of case 1 in [41, Theorem 3.1] leads to:

Lemma 7.17. *Suppose that G is Abelian. Suppose that a metrizable system (X,G) is nonminimal, has a unique minimal subsystem, and has a G-invariant Borel probability measure with full support. Then (X,G) is untame.*

Using Corollary 7.16 and Lemmas 7.2 and 7.17 one also sees that the proof of [41, Theorem 4.4] works in our context, so that we obtain:

Lemma 7.18. *Suppose that G is Abelian. Then any metrizable tame minimal system (X,G) is uniquely ergodic.*

Theorem 7.19. *Suppose that G is Abelian. Then any tame minimal system (X,G) is uniquely ergodic.*

Proof. Let μ_1 and μ_2 be two G-invariant Borel probability measures on X. We use the notation established after Lemma 7.7. Denote by I the set of all pairs (ρ,H) such that $\rho \in \mathrm{CP}(X)$, H is a countable subgroup of G, and $(X^*_{\rho,H},H)$ is minimal. Take (Y,G) to be the trivial system in Lemma 7.8. For every $(\rho,H) \in I$ we have, by Lemma 7.18, $\sigma_{\rho,H,*}(\mu_1) = \sigma_{\rho,H,*}(\mu_2)$, where $\sigma_{\rho,H,*} : M(X) \to M(X^*_{\rho,H})$ is the map between spaces of Borel probability measures induced by $\sigma_{\rho,H}$. Define a partial order on I by $(\rho_1,H_1) \leq (\rho_2,H_2)$ if $\rho_1 \leq \rho_2$ and $H_1 \subseteq H_2$. As mentioned right after Lemma 7.9, when $(\rho_1,H_1) \leq (\rho_2,H_2)$ there exists a unique map $\sigma_{21} : X^*_{\rho_2,H_2} \to X^*_{\rho_1,H_1}$ such that $\sigma_{21} \circ \sigma_{\rho_2,H_2} = \sigma_{\rho_1,H_1}$. By [78, Lemma V.3.9], I is directed. It is easily checked that $X = \varprojlim_{(\rho,H)\in I} X^*_{\rho,H}$. Thus $M(X) = \varprojlim_{(\rho,H)\in I} M(X^*_{\rho,H})$. Therefore $\mu_1 = \mu_2$. ☐

Theorem 7.19 generalizes [41, Theorem 4.4] in which the conclusion is established for metrizable null minimal systems $(X,\mathbb{Z})$.

8. I-independence

Here we tie together several properties via the notion of I-independence, which, as Theorem 8.6 suggests, can be thought of as an analogue of measure-theoretic weak mixing for C^*-dynamical systems (compare also Theorem 10.4).

Definition 8.1. A C^*-dynamical system (A,G,α) is said to be *I-independent* if for every finite-dimensional operator subsystem $V \subseteq A$ and $\varepsilon > 0$ there is a sequence $\{s_k\}_{k=1}^{\infty}$ in G such that $(s_1,\dots,s_k)$ is a $(1+\varepsilon)$-independence tuple for V for each $k \geq 1$.

Note that I-independence is to be distinguished from $\mathfrak{I}$-independence, although the two turn out to be equivalent, as the next proposition demonstrates.

Proposition 8.2. *Let (A,G,α) be C^*-dynamical system. Let $\mathfrak{S}$ be a collection of finite-dimensional operator subsystems of A with the property that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is a $V \in \mathfrak{S}$ such that $\Omega \subseteq_\varepsilon V$. Then the following are equivalent:*

(1) *α is I-independent,*
(2) *for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set* $\mathrm{Ind}(\alpha, V, \varepsilon)$ *is infinite,*
(3) *for every $V \in \mathfrak{S}$ and $\varepsilon > 0$ the set* $\mathrm{Ind}(\alpha, V, \varepsilon)$ *is nonempty,*
(4) *α is $\mathcal{I}$-independent,*
(5) *α is $\mathcal{N}$-independent.*

Proof. (1)⇒(2)⇒(3) and (4)⇒(5). Trivial.

(2)⇒(4) and (3)⇒(5). Apply Proposition 2.6.

(5)⇒(1). Let V be a finite-dimensional operator subsystem of A and let $\varepsilon > 0$. With the aim of verifying I-independence, we may assume that V does not equal the scalars, so that V has linear dimension at least two. By recursion we will construct a sequence $\{s_1, s_2, \dots\}$ of distinct elements of G such that for each $k \geq 1$ the linear map $\varphi_k : V^{\otimes[1,k]} \to V_k := [\alpha_{s_1}(V) \cdots \alpha_{s_k}(V)] \subseteq A$ determined on elementary tensors by $a_1 \otimes \cdots \otimes a_k \mapsto \alpha_{s_1}(a_1) \cdots \alpha_{s_k}(a_k)$ is a $(1+\varepsilon)^{1-2^{-k+1}}$-c.b.-isomorphism.

To begin with set $s_1 = e$. Now let $k \geq 1$ and suppose that $s_1, s_2, \dots, s_k$ have been defined so that φ_k is a $(1+\varepsilon)^{1-2^{-k+1}}$-c.b.-isomorphism onto its image. By (3) there is an $s_{k+1} \in G$ such that the linear map $\psi : V_k \otimes V_k \to [V_k \alpha_{s_{k+1}}(V_k)]$ determined by $a \otimes b \mapsto a\alpha_{s_{k+1}}(b)$ is a $(1+\varepsilon)^{2^{-k}}$-c.b.-isomorphism. Since V has linear dimension at least two, we must have $s_{k+1} \notin \{s_1, \dots, s_k\}$, for otherwise ψ would not be injective. Set $\gamma = \varphi_k \otimes \mathrm{id}_V : V^{\otimes[1,k]} \otimes V = V^{\otimes[1,k+1]} \to V_k \otimes V$. By the injectivity of the minimal tensor product, we may view $V_k \otimes V$ as a subspace of $V_k \otimes V_k$, in which case we have $\varphi_{k+1} = \psi \circ \gamma$. Then

$$\begin{aligned}
\|\varphi_{k+1}\|_{\mathrm{cb}} \|\varphi_{k+1}^{-1}\|_{\mathrm{cb}} &\leq \|\psi\|_{\mathrm{cb}} \|\varphi_k \otimes \mathrm{id}_V\|_{\mathrm{cb}} \|\psi^{-1}\|_{\mathrm{cb}} \|\varphi_k^{-1} \otimes \mathrm{id}_V\|_{\mathrm{cb}} \\
&= \|\psi\|_{\mathrm{cb}} \|\psi^{-1}\|_{\mathrm{cb}} \|\varphi_k\|_{\mathrm{cb}} \|\varphi_k^{-1}\|_{\mathrm{cb}} \\
&\leq (1+\varepsilon)^{2^{-k}} (1+\varepsilon)^{1-2^{-k+1}} \\
&= (1+\varepsilon)^{1-2^{-k}},
\end{aligned}$$

so that φ_{k+1} is a $(1+\varepsilon)^{1-2^{-k}}$-c.b.-isomorphism, as desired. Since for each $k \geq 1$ the map φ_k is a $(1+\varepsilon)$-c.b.-isomorphism, we obtain (1). □

Proposition 8.3. *A C^*-dynamical system (A, G, α) is I-independent if and only if the product system $(A^{\otimes[1,m]}, G, \alpha^{\otimes[1,m]})$ is I-independent for every $m \in \mathbb{N}$.*

Proof. For the nontrivial direction we can apply Proposition 8.2 using the following two observations: (i) the collection of operator subsystems of $A^{\otimes[1,m]}$ of the form $\bigotimes_{i=1}^m V_i$ for finite-dimensional operator subsystems $V_1, \dots, V_m \subseteq A$ satisfies the property required of $\mathfrak{S}$ in the statement of Proposition 8.2, and (ii) given finite-dimensional operator subsystems $V_1, \dots, V_m \subseteq A$ and a $\lambda \geq 1$, if a tuple in G is a λ-independence tuple for the linear span of $\bigcup_{i=1}^m V_i$ then it is a λ^m-independence tuple for $V_1 \otimes \cdots \otimes V_m$, as follows from the fact that for any c.b. isomorphisms $\varphi_i : E_i \to F_i$ between operator spaces for $i = 1, \dots, m$ the tensor product $\varphi = \bigotimes_{i=1}^m \varphi_i$ is a c.b. isomorphism with $\|\varphi\|_{\mathrm{cb}} \|\varphi^{-1}\|_{\mathrm{cb}} = \prod_{i=1}^m \|\varphi_i\|_{\mathrm{cb}} \|\varphi_i^{-1}\|_{\mathrm{cb}}$. □

Proposition 8.4. *Let (X, G) be a dynamical system. Let $\mathcal{B}$ be a basis for the topology on X which does not contain the empty set. Then the following are equivalent:*

(1) *(X, G) is I-independent,*

(2) *for every finite-dimensional operator subsystem $V \subseteq C(X)$ there are a sequence $\{s_k\}_{k=1}^{\infty}$ in G and a $\lambda \geq 1$ such that $(s_1, \dots, s_k)$ is a λ-independence tuple for V for each $k \geq 1$,*
(3) *every finite-dimensional operator subsystem of $C(X)$ has arbitrarily long λ-independence tuples for some $\lambda \geq 1$,*
(4) *every nonempty finite subcollection of $\mathcal{B}$ has an infinite independence set,*
(5) *every nonempty finite subcollection of $\mathcal{B}$ has arbitrarily large finite independence sets,*
(6) *for every nonempty finite collection $\{U_1, \dots, U_m\} \subseteq \mathcal{B}$ there is an $s \in G$ such that $U_i \cap s^{-1}U_j \neq \emptyset$ for all $i, j = 1, \dots, m$.*

Proof. (1)⇒(2)⇒(3). Trivial.

(2)⇒(4). Let $\{U_1, \dots, U_m\}$ be a nonempty finite subcollection of $\mathcal{B}$. By shrinking the U_i if necessary, we may assume for the purpose of establishing (4) that $U_i \cap U_j = \emptyset$ for $i \neq j$. For each $i = 1, \dots, m$ take a nonempty closed set $W_i \subseteq U_i$. Let $\Theta = \{g_0, g_1, \dots, g_m\}$ be a partition of unity of X such that $\mathrm{supp}(g_i) \subseteq U_i$ for each i and $\mathrm{supp}(g_0) \subseteq X \setminus \bigcup_{i=1}^m W_i$. Let V be the linear span of Θ. By (2) there are a sequence $\{s_1, s_2, \dots\}$ in G and a $\lambda \geq 1$ such that for each $k \geq 1$ the contractive linear map $V^{\otimes[1,k]} \to C(X)$ determined on elementary tensors by $f_1 \otimes \cdots \otimes f_k \mapsto \alpha_{s_1}(f_1) \cdots \alpha_{s_k}(f_k)$ is a λ-isomorphism onto its image. Now suppose we are given a $k \in \mathbb{N}$ and a $\sigma \in \{1, \dots, m\}^{\{1,\dots,k\}}$. Then $\|\alpha_{s_1}(g_{\sigma(1)}) \cdots \alpha_{s_k}(g_{\sigma(k)})\| \geq \lambda^{-1}$. Choose $x \in X$ such that $g_{\sigma(1)}(s_1 x) \cdots g_{\sigma(k)}(s_k x) = \|\alpha_{s_1}(g_{\sigma(1)}) \cdots \alpha_{s_k}(g_{\sigma(k)})\|$. Then for each $i = 1, \dots, k$ we must have $g_{\sigma(i)}(s_i x) > 0$, which implies that $x \in s_i^{-1} U_{\sigma(i)}$. Hence $\bigcap_{i=1}^k s_i^{-1} U_{\sigma(i)} \neq \emptyset$, and so we obtain (4).

(4)⇒(5)⇒(6). Trivial.

(3)⇒(5). This can be proved along the lines of the argument used for (2)⇒(4).

(6)⇒(1). Let $\Theta = \{g_1, \dots, g_m\}$ be a partition of unity of X for which there are elements $U_1, \dots, U_m$ of $\mathcal{B}$ such that $g_i|_{U_j} = \delta_{ij}\chi_{U_j}$, where χ_{U_j} is the characteristic function of U_j. Note that the collection of subspaces of $C(X)$ spanned by such partitions of unity has the property required of $\mathfrak{S}$ in the statement of Proposition 8.2. By (6) there is an $s \in G$ such that $U_i \cap s^{-1}U_j \neq \emptyset$ for all $i, j = 1, \dots, m$. Let V be the subspace of $C(X)$ spanned by Θ and let $\varphi : V \otimes V \to A$ be the linear map determined on elementary tensors by $f \otimes g \mapsto f\alpha_s(g)$. Then $\{g_i \alpha_s(g_j)\}_{1 \leq i,j \leq m}$ is an effective partition of unity of X, and hence is isometrically equivalent to the standard basis of $\ell_\infty^{m^2}$. Since the subset $\{g_i \otimes g_j\}_{1 \leq i,j \leq m}$ of $V \otimes V$ is also isometrically equivalent to standard basis of $\ell_\infty^{m^2}$, we conclude that φ is an isometric isomorphism. In view of Proposition 8.2, this yields (1). $\square$

Remark 8.5. In the case $G = \mathbb{Z}$, the proof of Proposition 8.2 shows that the sequence in the definition of I-independence can be taken to be in $\mathbb{N}$. since $\mathrm{Ind}(\alpha, V, \varepsilon)$ is closed in general under taking inverses (given a c.b. isomorphism $\varphi : V \otimes V \to [V\alpha_s(V)]$ with $\varphi(v \otimes w) = v\alpha_s(w)$, the map $V \otimes V \mapsto [V\alpha_{s^{-1}}(V)]$ defined by $v \otimes w \mapsto v\alpha_{s^{-1}}(w) = \alpha_{s^{-1}}(\varphi(w^* \otimes v^*)^*)$ has the same c.b. isomorphism constant). Thus in Proposition 8.4 the sequence in each of conditions (2) and (4) can be taken to be in $\mathbb{N}$.

The next theorem extends [41, Theorem 2.1]. Recall that (X, G) is said to be *(topologically) transitive* if every nonempty open invariant subset of X is dense, and *(topologically)* *weakly mixing* if the product system $(X \times X, G)$ is transitive. For the definitions of uniform nonnullness and untameness of all orders see Sections 5 and 6, respectively.

Theorem 8.6. *Let (X, G) be a dynamical system and consider the following conditions:*

(1) *(X, G) is I-independent,*
(2) *(X, G) is uniformly untame of all orders,*
(3) *(X, G) is uniformly nonnull of all orders,*
(4) *for every $n \in \mathbb{N}$ the product system (X^n, G) is weakly mixing,*
(5) *for every $n \in \mathbb{N}$ the product system (X^n, G) is transitive,*
(6) *(X, G) is uniformly untame,*
(7) *(X, G) is uniformly nonnull,*
(8) *(X, G) is weakly mixing.*

Then conditions (1) to (5) are equivalent, and (5)⇒(6)⇒(7). When G is Abelian, conditions (1) to (8) are all equivalent.

Proof. The implications (2)⇒(6)⇒(7) are trivial. In the case that G is Abelian, (7)⇒(8) follows as in the proof of [41, Theorem 2.1] using the lemma in [65], while (8)⇒(5) is [28, Proposition II.3]. The implication (1)⇒(2) follows from Proposition 8.4, (2)⇒(3) and (4)⇒(5) are trivial, and (3)⇒(4) is readily seen.

Finally, to show (5)⇒(1), let $\{U_1, \dots, U_n\}$ be a nonempty finite collection of nonempty open subsets of X. Let Λ be the set of all integer pairs (i, j) with $1 \leq i, j \leq n$. Set $V_0 = \prod_{(i,j)\in\Lambda} U_i \subseteq X^\Lambda$ and $V_1 = \prod_{(i,j)\in\Lambda} U_j \subseteq X^\Lambda$. By (5) there is an $s \in G$ such that $V_0 \cap s^{-1}V_1 \neq \emptyset$, so that $U_i \cap s^{-1}U_j \neq \emptyset$ for all $i, j = 1, \dots, n$. It then follows by Proposition 8.4 that (X, G) is I-independent. □

By the above theorem, we can consider I-independence to be the analogue for noncommutative C^*-dynamical systems of, among other properties, uniform untameness of all orders (compare the discussion in the last paragraph of Section 4). On the other hand, the notions of tameness (resp. untameness) and complete untameness make sense for any C^*-dynamical system, the former meaning that no element (resp. some element) has an infinite ℓ_1-isomorphism set and the latter that every nonscalar element has an infinite ℓ_1-isomorphism set. We will end this section by observing that, in the general C^*-dynamical context, I-independence (in fact a weaker independence condition) implies complete untameness.

To verify complete untameness it suffices to check the existence of an infinite ℓ_1-isomorphism set over $\mathbb{R}$ for every self-adjoint nonscalar element, as the following lemma based on Rosenthal-Dor arguments demonstrates.

Lemma 8.7. *Let (A, G, α) be a C^*-dynamical system. Let $a \in A$, and suppose that at least one of* $\mathrm{re}(a)$ *and* $\mathrm{im}(a)$ *has an infinite ℓ_1-isomorphism set over $\mathbb{R}$. Then a has an infinite ℓ_1-isomorphism set.*

Proof. Suppose that $\{\alpha_{s_n}(a)\}_{n\in\mathbb{N}}$ is a sequence in the orbit of a which converges weakly to some $b \in A$. Then $\lim_{n\to\infty} \sigma(\mathrm{re}(\alpha_{s_n}(a))) = \sigma(\mathrm{re}(b))$ and $\lim_{n\to\infty} \sigma(\mathrm{im}(\alpha_{s_n}(a)) = \sigma(\mathrm{im}(b))$ for all self-adjoint $\sigma \in A^*$, so that both $\{\alpha_{s_n}(\mathrm{re}(a))\}_{n\in\mathbb{N}}$ and $\{\alpha_{s_n}(\mathrm{im}(a))\}_{n\in\mathbb{N}}$ are weakly convergent over $\mathbb{R}$. It follows from this observation and [23] that a has an infinite ℓ_1-isomorphism set. □

For a unital C^*-algebra A, we denote by $\mathcal{S}_2(A)$ the collection of 2-dimensional operator subsystems of A equipped with the metric given by Hausdorff distance between unit balls.

Proposition 8.8. *Let (A, G, α) be a C^*-dynamical system. Let $\lambda \geq 1$, and suppose that in $\mathcal{S}_2(A)$ there is a dense collection of V for which there is a sequence $\{s_1, s_2, \dots\}$ in G such that $(s_1, \dots, s_k)$ is a λ-independence tuple for V for each $k \geq 1$. Then α is completely untame. In particular, an I-independent C^*-dynamical system is completely untame.*

Proof. Let a be a nonscalar self-adjoint element of A, and denote by V the 2-dimensional operator system span$\{1, a\}$. Suppose that there is a sequence $\{s_1, s_2, \dots\}$ in G such that for each k the linear map $V^{\otimes[1,k]} \to A$ determined by $a_1 \otimes \cdots \otimes a_k \mapsto \alpha_{s_1}(a_1) \cdots \alpha_{s_k}(a_k)$ is a λ-c.b.-isomorphism onto its image. It then follows by Lemma 4.1 that the set $\{\alpha_{s_k}(a)\}_{k \in \mathbb{N}}$ is λ'-equivalent to the standard basis of $\ell_1^{\mathbb{N}}$ for some λ' depending only on λ and the spectral diameter of a. By a straightforward perturbation argument and Lemma 8.7 we conclude that α is completely untame. $\square$

9. Independence, Abelianness, and weak mixing in C^*-dynamical systems

Once we express independence in terms of minimal tensor products and move into the noncommutative realm, a close connection to Abelianness reveals itself. Indeed one of the goals of this section is to show that, in simple unital nuclear C^*-algebras, independence and Abelianness in the dynamical context essentially amount to the same thing (Theorem 9.6). This provides a conceptual basis for the sense gained from examples that concepts like hyperbolicity and topological K-ness should be interpreted in the simple nuclear case as certain types of asymptotic Abelianness. See for instance [61, 62] and the discussion at the end of Section 4, and compare also [53], where a relationship between tensor product structure and asymptotic Abelianness is established as a tool in the study of derivations and dissipations. The second main result of this section concerns the implications for independence of the existence of weakly mixing invariant states (Theorem 9.10).

Definition 9.1. Let (A, G, α) be a C^*-dynamical system. Let V be a finite-dimensional operator subsystem of A and let $\varepsilon > 0$. We define $\mathrm{Ab}(\alpha, V, \varepsilon)$ to be the set of all $s \in G_0$ such that $\|[v, \alpha_s(w)]\| \leq \varepsilon \|v\| \|w\|$ for all $v, w \in V$. For a collection $\mathcal{C}$ of subsets of G_0 which is closed under taking supersets, we say that the system (A, G, α) or the action α is $\mathcal{C}$-*Abelian* if for every finite-dimensional operator subsystem $V \subseteq A$ and $\varepsilon > 0$ the set $\mathrm{Ab}(\alpha, V, \varepsilon)$ is a member of $\mathcal{C}$.

When G is an infinite group, $\mathcal{I}$-Abelianness can be characterized as follows.

Proposition 9.2. *Let (A, G, α) be a C^*-dynamical system with G an infinite group. Then the following are equivalent:*

1. *α is $\mathcal{I}$-Abelian,*
2. *α is $\mathcal{N}$-Abelian,*
3. *there is a net $\{s_\gamma\}_\gamma$ in G (which can be taken to be a sequence if A is separable) such that $\lim_\gamma \|[a, \alpha_{s_\gamma}(b)]\| = 0$ for all $a, b \in A$.*

Proof. The implication (1)⇒(2) is trivial. Given a net $\{s_\gamma\}_\gamma$ as in (3), if it has a subnet tending to infinity then we obviously obtain (1), and if not then it has convergent subnet and hence A is commutative, so that we again obtain (1). Suppose then that (2) holds and let us prove (3). Let $\mathfrak{D}$ be a set of finite-dimensional operator subsystems of A directed by inclusion such that $\bigcup \mathfrak{D}$ is dense in A. Let Γ be the directed set of all pairs (V, ε) such that $V \in \mathfrak{D}$ and $\varepsilon > 0$, where $(V', \varepsilon') \succ (V, \varepsilon)$ means that $V' \supseteq V$ and $\varepsilon' \leq \varepsilon$. For every $\gamma = (V, \varepsilon) \in \Gamma$ we can find by (2) an $s_\gamma \in G$ such that $\|[v, \alpha_{s_\gamma}(w)]\| \leq \varepsilon \|v\| \|w\|$ for all $v, w \in V$. Then $\lim_\gamma \|[a, \alpha_{s_\gamma}(b)]\| = 0$ for all $a, b \in A$, as desired. In the case that A is separable we can use instead a sequence of pairs $(V_n, 1/n)$ where $V_1 \subseteq V_2 \subseteq \dots$ is an increasing sequence of finite-dimensional operator subsystems of A such that $\bigcup_{n \in \mathbb{N}} V_n$ is dense in A. □

Condition (3) in Proposition 9.2 in the case where G is the entire *-automorphism group of A is of importance in C^*-algebra structure and classification theory. See for example Lemma 5.2.3 in [68].

Lemma 9.3. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever V is an operator system, $\mathcal{H}$ is a Hilbert space, and $\varphi : V \to \mathcal{B}(\mathcal{H})$ is a unital c.b. map with $\|\varphi\|_{\rm cb} \leq 1 + \delta$, there is a complete positive map $\psi : V \to \mathcal{B}(\mathcal{H})$ with $\|\psi - \varphi\| \leq \varepsilon$.*

Proof. Let V be an operator system, $\mathcal{H}$ a Hilbert space, and $\varphi : V \to \mathcal{B}(\mathcal{H})$ a unital c.b. map. We may assume V to be an operator subsystem of a unital C^*-algebra A. Then by the Arveson-Wittstock extension theorem we can extend φ to a c.b. map $\tilde{\varphi} : A \to \mathcal{B}(\mathcal{H})$ with $\|\tilde{\varphi}\|_{\rm cb} = \|\varphi\|_{\rm cb}$. Using the representation theorem for c.b. maps as in the proof of Lemma 3.3 in [67], we then can produce a completely positive map $\rho : A \to \mathcal{B}(\mathcal{H})$ such that $\|\tilde{\varphi} - \rho\|_{\rm cb} \leq \sqrt{2\|\tilde{\varphi}\|_{\rm cb}(\|\tilde{\varphi}\|_{\rm cb} - 1)}$, from which the lemma follows. □

Lemma 9.4. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever A is a unital C^*-algebra, α is a unital *-endomorphism of A, and V is a finite-dimensional operator subsystem of A for which the linear map $V \otimes V \to [V\alpha(V)]$ determined on elementary tensors by $v_1 \otimes v_2 \mapsto v_1\alpha(v_2)$ has c.b. norm at most $1 + \delta$, we have $\|[v_1, \alpha(v_2)]\| \leq \varepsilon \|v_1\| \|v_2\|$ for all $v_1, v_2 \in V$.*

Proof. Let $\varepsilon > 0$, and take δ as given by Lemma 9.3 with respect to $\varepsilon/2$. Let A be a unital C^*-algebra, α a unital *-endomorphism of A, and V an operator subsystem of A, and suppose that the linear map $\varphi : V \otimes V \to [V\alpha(V)]$ determined on elementary tensors by $v_1 \otimes v_2 \mapsto v_1\alpha(v_2)$ has c.b. norm at most $1 + \delta$. We regard A as a unital C^*-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then by our choice of δ there is a completely positive map $\psi : V \otimes V \to \mathcal{B}(\mathcal{H})$ with $\|\psi - \varphi\| \leq \varepsilon/2$. It follows that, for all $v_1, v_2 \in V$,

$$\begin{aligned}
\|v_1\alpha(v_2) - \alpha(v_2)v_1\| &\leq \|\varphi(v_1 \otimes v_2) - \psi(v_1 \otimes v_2)\| + \|\psi(v_1^* \otimes v_2^*)^* - \varphi(v_1^* \otimes v_2^*)^*\| \\
&\leq 2\|\varphi - \psi\| \|v_1\| \|v_2\| \\
&\leq \varepsilon \|v_1\| \|v_2\|,
\end{aligned}$$

yielding the lemma. □

Lemma 9.4 immediately yields:

Proposition 9.5. *Let $\mathcal{C}$ a collection of subsets of G_0 which is closed under taking supersets. Then for a C^*-dynamical system with acting semigroup G, $\mathcal{C}$-contractivity implies $\mathcal{C}$-Abelianness.*

Theorem 9.6. *Let (A, G, α) be a C^*-dynamical system with A nuclear. Let $\mathcal{C}$ be a collection of subsets of G_0 which is closed under taking supersets. Consider the following conditions:*

(1) *α is $\mathcal{C}$-independent,*
(2) *α is $\mathcal{C}$-contractive,*
(3) *α is $\mathcal{C}$-Abelian.*

Then (1)⇒(2)⇔(3), and if A is simple then all three conditions are equivalent.

Proof. The implication (1)⇒(2) is trivial, while (2)⇒(3) is a special case of Proposition 9.5.

Assume that (3) holds and let us prove (2). Suppose that α is not $\mathcal{C}$-contractive. Then there are a finite-dimensional operator subsystem $V \subseteq A$ and an $\varepsilon > 0$ such that no $(1+\varepsilon)$-contraction set for V is a member of $\mathcal{C}$. Denote by Λ the set of pairs (E, δ) where E is a finite-dimensional operator subsystem of A containing V and $\delta > 0$. For every $\lambda = (E, \delta) \in \Lambda$ let H_λ be the set of all $s \in G$ such that $\|[v, \alpha_s(w)]\| \leq \delta\|v\|\|w\|$ for all $v, w \in E$ and the linear map $\varphi_s : V \otimes V \to [V\alpha_s(V)]$ defined on elementary tensors by $\varphi_s(v \otimes w) = v\alpha_s(w)$ has c.b. norm greater than $1+\varepsilon$. Since α is $\mathcal{C}$-Abelian, H_λ is nonempty for every $\lambda \in \Lambda$, and for all $\lambda_1 = (E_1, \delta_1)$ and $\lambda_2 = (E_2, \delta_2)$ in Λ we have $H_\lambda \subseteq H_{\lambda_1} \cap H_{\lambda_2}$ for $\lambda = (E_1 + E_2, \min(\delta_1, \delta_2))$. Thus the collection $\{H_\lambda : \lambda \in \Lambda\}$ forms a filter base over G. Let ω be an ultrafilter containing this collection. We write $\ell_\infty^G(A)/c_\omega(A)$ for the ultrapower of A with respect to ω (see [68, Sect. 6.2]).

Denote by π the quotient map $\ell_\infty^G(A) \to \ell_\infty^G(A)/c_\omega(A)$. Let $\Phi_1, \Phi_2 : A \to \ell_\infty^G(A)/c_\omega(A)$ be the *-homomorphisms given by $\Phi_1(a) = \pi((a)_{s\in G})$ and $\Phi_2(a) = \pi((\alpha_s(a))_{s\in G})$ for all $a \in A$. Then the images of Φ_1 and Φ_2 commute, and thus, since $A \otimes A = A \otimes_{\max} A$ by the nuclearity of A, we obtain a *-homomorphism $\Phi : A \otimes A \to \ell_\infty^G(A)/c_\omega(A)$ such that $\Phi(a_1 \otimes a_2) = \Phi_1(a_1)\Phi_2(a_2)$ for all $a_1, a_2 \in A$ [74, Prop. IV.4.7].

Using the description of nuclearity as a completely positive approximation property we can recursively construct a separable nuclear operator subsystem W of A containing V. By the Choi-Effros lifting theorem [20] there is a unital completely positive map $\psi : W \otimes W \to \ell_\infty^G(A)$ such that $\pi \circ \psi = \Phi|_{W\otimes W}$, viewing $W \otimes W$ as a subspace of $A \otimes A$. Since the unit ball of $V \otimes V$ is compact, ω contains the set H of all $s \in G$ for which $\|\varphi_s - \pi_s \circ \psi|_{V\otimes V}\| \leq \dim(V)^{-2}\varepsilon$, where π_s denotes the coordinate projection $\ell_\infty^G(A) \to A$ associated to s and $V \otimes V$ is viewed as a subspace of $W \otimes W$. Appealing to [24, Cor. 2.2.4], for $s \in H$ we have

$$\begin{aligned}\|\varphi_s\|_{\mathrm{cb}} &\leq \|\pi_s \circ \psi|_{V\otimes V}\|_{\mathrm{cb}} + \|\varphi_s - \pi_s \circ \psi|_{V\otimes V}\|_{\mathrm{cb}} \\ &\leq 1 + \dim(V)^2\|\varphi_s - \pi_s \circ \psi|_{V\otimes V}\| \\ &\leq 1 + \varepsilon,\end{aligned}$$

which yields a contradiction since H intersects every H_λ. We thus obtain (2).

Now suppose that A is simple and let us show (3)⇒(1). Suppose that α is not $\mathcal{C}$-independent. Then there are a finite-dimensional operator subsystem $V \subseteq A$ and an $\varepsilon > 0$ such that no $(1+\varepsilon)^2$-independence set for V is a member of $\mathcal{C}$. As before define Λ to be the set of pairs (E, δ) where E is a finite-dimensional operator subsystem of A containing V and $\delta > 0$. For every $\lambda = (E, \delta) \in \Lambda$ let H_λ be the set of all $s \in G$ such that $\|[v, \alpha_s(w)]\| \leq \delta\|v\|\|w\|$ for all $v, w \in E$ and the linear map $\varphi_s : V \otimes V \to [V\alpha_s(V)]$ defined on elementary tensors by $\varphi_s(v \otimes w) = v\alpha_s(w)$ either has c.b. norm greater than $1+\varepsilon$ or is not invertible or has an inverse with c.b. norm greater than $1+\varepsilon$. As in the previous paragraph we construct a separable nuclear operator subsystem W of A containing V and apply the Choi-Effros lifting theorem to obtain a unital completely positive map $\psi : W \otimes W \to \ell_\infty^G(A)$ such that $\pi \circ \psi = \Phi|_{W\otimes W}$. Since A is simple so is $A \otimes A$, and hence Φ is faithful. Consequently ψ is a complete order isomorphism. Let δ be a positive number to be specified below. Since A is nuclear, V is 1-exact, and thus $V \otimes V$ is 1-exact (see [24, 66]). By a result of Smith [24, Prop. 2.2.2] and the characterization of 1-exactness in terms of almost completely isometric embeddings into matrix algebras (see [66, Lemma 17.8]), it follows that there exists a $k \in \mathbb{N}$ such that whenever E is an operator space and $\rho : E \to V \otimes V$ is a bounded linear map we have $\|\rho\|_{\mathrm{cb}} \leq \sqrt{1+\delta}\|\mathrm{id}_{M_k} \otimes \rho\|$. Under the canonical identification $(M_k \otimes \ell_\infty^G(A))/(M_k \otimes c_\omega(A)) \cong M_k \otimes (\ell_\infty^G(A)/c_\omega(A))$, we regard the complete order embedding $\mathrm{id}_{M_k} \otimes \psi : M_k \otimes W \otimes W \to M_k \otimes \ell_\infty^G(A)$ as a lift of the restriction to $M_k \otimes W \otimes W$ of the *-homomorphism $\mathrm{id}_{M_k} \otimes \Phi : M_k \otimes A \otimes A \to M_k \otimes (\ell_\infty^G(A)/c_\omega(A))$ with respect to the quotient map $M_k \otimes \ell_\infty^G(A) \to (M_k \otimes \ell_\infty^G(A))/(M_k \otimes c_\omega(A))$. Since the unit ball of $M_k \otimes V \otimes V$ is compact, ω contains the set H of all $s \in G$ for which (i) $\|\varphi_s - \pi_s \circ \psi|_{V\otimes V}\| \leq \dim(V)^{-2}\varepsilon$ and (ii) $\pi_s \circ \psi|_{V\otimes V}$ is invertible with $\|\mathrm{id}_{M_k} \otimes (\pi_s \circ \psi|_{V\otimes V})^{-1}\| < \sqrt{1+\delta}$ and $\|\varphi_s - \pi_s \circ \psi|_{V\otimes V}\| < \delta$. From (i) we obtain $\|\varphi_s\|_{\mathrm{cb}} \leq 1+\varepsilon$ as before, while from (ii) we obtain $\|(\pi_s \circ \psi|_{V\otimes V})^{-1}\|_{\mathrm{cb}} < 1+\delta$ so that, by Lemma 2.5, if δ is small enough as a function of $\dim(V)$ and ε then φ_s is invertible with $\|\varphi_s^{-1}\|_{\mathrm{cb}} \leq 1+\varepsilon$. This produces a contradiction since H intersects every H_λ. Thus (3)⇒(1) in the simple case. □

For the remainder of this section G will be a group. Let (A, G, α) be a C^*-dynamical system and σ a G-invariant state on A. We write $\mathcal{H}_\sigma$ for the GNS Hilbert space of σ and Ω_σ for the vector in $\mathcal{H}_\sigma$ associated to the unit of A. We denote by $\mathcal{H}_{\sigma,0}$ the orthogonal complement in $\mathcal{H}_\sigma$ of the one-dimensional subspace spanned by Ω_σ. Note that $\mathcal{H}_{\sigma,0}$ is invariant under the action of G. We denote by $\mathfrak{m}$ the unique invariant mean on the space $\mathrm{WAP}(G)$ of weakly almost periodic bounded uniformly continuous functions on G. A *flight function* is a function $f \in \mathrm{WAP}(G)$ such that $\mathfrak{m}(|f|) = 0$, which is equivalent to the condition that for every $\varepsilon > 0$ the set $\{s \in G : |f(s)| < \varepsilon\}$ is thickly syndetic (see [6, 31]). We write C_σ for the norm closure of the linear span of the functions $s \mapsto \langle U_s\xi, \zeta\rangle$ on G for all $\xi, \zeta \in \mathcal{H}_{\sigma,0}$, which is a subspace of $\mathrm{WAP}(G)$. We can alternatively describe C_σ as the norm closure of the linear span of the functions $s \mapsto \sigma(b\alpha_s(a))$ on G for all $a, b \in A$ such that $\sigma(a) = \sigma(b) = 0$.

Recall that a strongly continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ is said to be *weakly mixing* if for all $\xi, \zeta \in \mathcal{H}$ the function $s \to |\langle \pi(s)\xi, \zeta\rangle|$ on G is a flight function.

Definition 9.7. Let (A, G, α) be a C^*-dynamical system. A G-invariant state σ on A is said to be *weakly mixing* if the representation of G on $\mathcal{H}_{\sigma,0}$ is weakly mixing, i.e., if f is a flight function for every $f \in C_\sigma$.

Definition 9.8. Let (A, G, α) be a C^*-dynamical system. A G-invariant state σ on A is said to be *T-mixing* if for all $a_1, a_2, b_1, b_2 \in A$ and $\varepsilon > 0$ the set of all $s \in G$ such that $|\sigma(a_1\alpha_s(b_1)a_2\alpha_s(b_2)) - \sigma(a_1a_2)\sigma(b_1b_2)| < \varepsilon$ is thickly syndetic.

Since a finite intersection of thickly syndetic sets is thickly syndetic, the G-invariant state σ is T-mixing if and only if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ the set of all $s \in G$ such that $|\sigma(a_1\alpha_s(b_1)a_2\alpha_s(b_2)) - \sigma(a_1a_2)\sigma(b_1b_2)| < \varepsilon$ for all $a_1, a_2, b_1, b_2 \in \Omega$ is thickly syndetic. Similarly, σ is weakly mixing if and only if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ the set of all $s \in G$ such that $|\sigma(a\alpha_s(b)) - \sigma(a)\sigma(b)| < \varepsilon$ for all $a, b \in \Omega$ is thickly syndetic.

T-mixing is a weak form of a clustering property that has been studied in the context of quantum statistical mechanics (see for example [5]). Note that it implies weak mixing. If the system (A, G, α) is $\mathcal{TS}$-Abelian in the sense of Definition 9.1 (in particular if it is commutative) then T-mixing is equivalent to weak mixing (use Lemma 10.1). This is not the case in general however, for if we take a C^*-probability space (A, σ) and consider the shift *-automorphism α on the infinite reduced free product $(B, \omega) = (A, \sigma)^{*\mathbb{Z}}$, then ω is weakly mixing but not T-mixing. See also Example 9.11.

Lemma 9.9. *Let (A, G, α) be C^*-dynamical system with A exact. Let τ be a faithful T-mixing G-invariant tracial state on A. Then α is $\mathcal{TS}$-expansive.*

Proof. Let V be a finite-dimensional operator subsystem of A and let $\varepsilon > 0$. Let $(\pi, \mathcal{H}, \xi)$ be the GNS triple associated to τ. Since τ is faithful, π is a faithful representation, and so via π we can view A as acting on $\mathcal{H}$ and $A \otimes A$ as acting on the Hilbertian tensor product $\mathcal{H} \otimes \mathcal{H}$. By the injectivity of the minimal operator space tensor product we can view $V \otimes V$ as an operator subsystem of $A \otimes A$.

Let $\{v_i\}_{i=1}^r$ be an Auerbach basis for V. Then $\mathcal{S} = \{v_i \otimes v_j\}_{i,j=1}^r$ is an Auerbach basis for $V \otimes V$. By Lemma 2.5 there exists a $\delta > 0$ such that whenever W is an operator space, $\rho : V \otimes V \to W$ is a linear isomorphism onto its image with $\max(\|\rho\|_{\mathrm{cb}}, \|\rho^{-1}\|_{\mathrm{cb}}) \leq 1 + \delta$, and $\{w_{ij}\}_{i,j=1}^r$ is a subset of W with $\|\rho(v_i \otimes v_j) - w_{ij}\| < 4\delta$ for all $1 \leq i, j \leq r$, the linear map $\psi : V \otimes V \to W$ determined on $\mathcal{S}$ by $\psi(v_i \otimes v_j) = w_{ij}$ is an isomorphism onto its image with $\max(\|\psi\|_{\mathrm{cb}}, \|\psi^{-1}\|_{\mathrm{cb}}) \leq 1 + \varepsilon$.

Since A is exact, V is 1-exact as an operator space, and so by Lemma 17.8 of [66] we can find a finite-dimensional unital subspace $E \subseteq A$ such that, with p denoting the orthogonal projection of $\mathcal{H}$ onto $E\xi$, the (completely contractive) compression map $\gamma : V \to pVp$ given by $v \mapsto pvp$ has an inverse with c.b. norm less than $\sqrt[4]{1 + \delta}$. Then the (completely contractive) compression map $\theta = \gamma \otimes \gamma : V \otimes V \to (p \otimes p)(V \otimes V)(p \otimes p)$ given by $z \mapsto (p \otimes p)z(p \otimes p)$ is invertible and $\|\theta^{-1}\|_{\mathrm{cb}} = \|\gamma^{-1} \otimes \gamma^{-1}\|_{\mathrm{cb}} \leq \|\gamma^{-1}\|_{\mathrm{cb}}^2 < \sqrt{1 + \delta}$.

Let $\{a_i\xi\}_{i=1}^m$ be an orthonormal basis for $E\xi$. Then $\mathcal{T} = \{a_i\xi \otimes a_j\xi\}_{i,j=1}^m$ is an orthonormal basis for $E\xi \otimes E\xi$. Set $\delta' = (\dim(E))^{-4}\delta$, and let η be a positive real number to be further specified below. Define K to be the set of all $s \in G \setminus \{e\}$ such that

(1) $|\tau(a_k^* a_i \alpha_s(a_j a_l^*)) - \tau(a_k^* a_i)\tau(a_j a_l^*)| < \eta$ for all $1 \leq i, j, k, l \leq m$, and
(2) $|\tau(c^* v \alpha_s(w) a \alpha_s(bd^*)) - \tau(c^* va)\tau(wbd^*)| < \delta'$ for all $v \otimes w \in \mathcal{S}$ and $a\xi \otimes b\xi, c\xi \otimes d\xi \in \mathcal{T}$.

Since τ is T-mixing, the set K is thickly syndetic.

Let $s \in K$. Define $S : E\xi \otimes E\xi \to [E\alpha_s(E)]\xi$ to be the surjective linear map determined on the basis $\mathcal{T}$ by $S(a_i\xi \otimes a_j\xi) = a_i\alpha_s(a_j)\xi$. For $1 \leq i, j, k, l \leq m$ we have

$$\begin{aligned}\langle S(a_i\xi \otimes a_j\xi), S(a_k\xi \otimes a_l\xi)\rangle &= \langle a_i\alpha_s(a_j)\xi, a_k\alpha_s(a_l)\xi\rangle \\ &= \tau((a_k\alpha_s(a_l))^* a_i\alpha_s(a_j)) = \tau(a_k^* a_i \alpha_s(a_j a_l^*))\end{aligned}$$

$$\approx_\eta \tau(a_k^*a_i)\tau(a_ja_l^*) = \langle a_i\xi \otimes a_j\xi, a_k\xi \otimes a_l\xi\rangle.$$

We thus see by a simple perturbation argument that by taking η small enough we can ensure that S is invertible with $\|S\|\|S^{-1}\| \leq \min(\sqrt{1+\delta}, 2)$ and $\|S^{-1} - S^*\| < \delta'$.

Denote by q the orthogonal projection of $\mathcal{H}$ onto $[E\alpha_s(E)]\xi$, and let $\rho : V\otimes V \to \mathcal{B}(q\mathcal{H})$ be the linear map given by $\rho(z) = S\theta(z)S^{-1}$ for all $z \in V\otimes V$. Then ρ is an isomorphism onto its image with inverse given by $\rho^{-1}(z) = \theta^{-1}(S^{-1}zS)$, and $\|\rho\|_{\rm cb} \leq \|S\|\|S^{-1}\| \leq \sqrt{1+\delta}$ while $\|\rho^{-1}\|_{\rm cb} \leq \|\theta^{-1}\|_{\rm cb}\|S^{-1}\|\|S\| \leq 1+\delta$.

Now suppose we are given $v\otimes w \in \mathcal{S}$. For all $a\xi\otimes b\xi, c\xi\otimes d\xi \in \mathcal{T}$, we have, using the bound $\|S^{-1} - S^*\| < \delta'$ and the definition of the set K,

$$\begin{aligned}
\langle S^{-1}qv\alpha_s(w)qS(a\xi\otimes b\xi), c\xi\otimes d\xi\rangle &\approx_{\delta'} \langle v\alpha_s(w)a\alpha_s(b)\xi, c\alpha_s(d)\xi\rangle \\
&= \tau(\alpha_s(d^*)c^*v\alpha_s(w)a\alpha_s(b)) = \tau(c^*v\alpha_s(w)a\alpha_s(bd^*)) \\
&\approx_{\delta'} \tau(c^*va)\tau(wbd^*) = \tau(c^*va)\tau(d^*wb) \\
&= \langle va\xi, c\xi\rangle\langle wb\xi, d\xi\rangle = \langle va\xi\otimes wb\xi, c\xi\otimes d\xi\rangle \\
&= \langle \theta(v\otimes w)(a\xi\otimes b\xi), c\xi\otimes d\xi\rangle.
\end{aligned}$$

Thus $\|\theta(v\otimes w) - S^{-1}qv\alpha_s(w)qS\| < 2\delta'(\dim(E))^4 = 2\delta$, and so

$$\|\rho(v\otimes w) - qv\alpha_s(w)q\| \leq \|S\|\|\theta(v\otimes w) - S^{-1}qv\alpha_s(w)qS\|\|S^{-1}\| < 4\delta.$$

We conclude by our choice of δ that the linear map $\psi : V\otimes V \to q[V\alpha_s(V)]q$ determined on elementary tensors by $\psi(v\otimes w) = qv\alpha_s(w)q$ is an isomorphism onto its image with $\max(\|\psi\|_{\rm cb}, \|\psi^{-1}\|_{\rm cb}) \leq 1+\varepsilon$.

Letting $\kappa : [V\alpha_s(V)] \to q[V\alpha_s(V)]q$ be the cut-down map $z \mapsto qzq$, we have a commuting diagram

$$\begin{tikzcd}
V\otimes V \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & {[V\alpha_s(V)]} \arrow[d, "\kappa"] \\
& q[V\alpha_s(V)]q.
\end{tikzcd}$$

where φ is the linear map determined on elementary tensors by $v\otimes w \mapsto v\alpha_s(w)$. Since κ is completely contractive, the map φ is invertible and $\|\varphi^{-1}\|_{\rm cb} = \|\psi^{-1}\circ\kappa\|_{\rm cb} \leq \|\psi^{-1}\|_{\rm cb} \leq 1+\varepsilon$. Thus K is a $(1+\varepsilon)$-expansion set for V, and we obtain the result. □

Theorem 9.10. *Let (A, G, α) be a $\mathcal{TS}$-contractive C^*-dynamical system with A exact. Let τ be a faithful weakly mixing G-invariant tracial state on A. Then α is $\mathcal{TS}$-independent.*

Proof. By Proposition 9.5, α is $\mathcal{TS}$-Abelian. Hence τ is T-mixing, in which case we can apply Lemma 9.9 to conclude that α is $\mathcal{TS}$-independent. □

Theorem 9.10 applies in particular to the commutative situation, where the $\mathcal{TS}$-contractivity and exactness hypotheses are automatic. In this case one can also obtain $\mathcal{TS}$-independence in a combinatorial fashion from the characterization of weak mixing in terms of thickly syndetic sets and a partition of unity argument (cf. the proof of Proposition 8.4(6)⇒(1)). It follows, for example, that if X is a compact manifold, possibly with boundary, of dimension at least 2 and μ is a nonatomic Borel probability measure on X with full support which is zero on the boundary, then a generic member of the set $\mathfrak{H}_\mu(X)$

of μ-preserving homeomorphisms from X to itself equipped with the uniform topology is $\mathcal{TS}$-independent, since the elements of $\mathfrak{H}_\mu(X)$ which are weakly mixing for μ form a dense G_δ subset [46] (see also [2]).

The assumption of $\mathcal{TS}$-contractivity in Theorem 9.10 cannot be dropped in general. Indeed certain Bogoliubov automorphisms of the CAR algebra are strongly mixing with respect to the unique tracial state but fail to be $\mathcal{I}$-contractive, as the following example demonstrates.

Example 9.11. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space over the complex numbers. We write $A(\mathcal{H})$ for the CAR algebra over $\mathcal{H}$. This is the unique, up to *-isomorphism, unital C^*-algebra generated by the image of an antilinear map $\xi \mapsto a(\xi)$ from $\mathcal{H}$ to $A(\mathcal{H})$ for which the anticommutation relations

$$\begin{aligned} a(\xi)a(\zeta)^* + a(\zeta)^*a(\xi) &= \langle \xi, \zeta \rangle 1_{A(\mathcal{H})}, \\ a(\xi)a(\zeta) + a(\zeta)a(\xi) &= 0, \end{aligned}$$

hold for all $\xi, \zeta \in \mathcal{H}$ (see [17]). Every U in the unitary group $\mathcal{U}(\mathcal{H})$ gives rise to a *-automorphism α_U of $A(\mathcal{H})$, called a Bogoliubov automorphism, by setting $\alpha_U(a(\xi)) = a(U\xi)$ for every $\xi \in \mathcal{H}$. The unique tracial state τ on $A(\mathcal{H})$ is given on products of the form $a(\zeta_n)^* \cdots a(\zeta_1)^* a(\xi_1) \cdots a(\xi_m)$ by

$$\tau(a(\zeta_n)^* \cdots a(\zeta_1)^* a(\xi_1) \cdots a(\xi_m)) = \delta_{nm} \det[\langle \tfrac{1}{2}\xi_i, \zeta_j \rangle].$$

Let U be a unitary operator on $\mathcal{H}$ such that $\lim_{|n|\to\infty} \langle U^n \xi, \zeta \rangle = 0$ for all $\xi, \zeta \in \mathcal{H}$ (for example, the bilateral shift with respect to some orthonormal basis). It is well known that the corresponding Bogoliubov automorphism α_U of $A(\mathcal{H})$ is strongly mixing for τ, i.e., $\lim_{n\to\infty} |\tau(a\alpha_U^n(b)) - \tau(a)\tau(b)| = 0$ for all $a, b \in A(\mathcal{H})$ (see Example 5.2.21 in [17]). On the other hand, for every $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \|[a(U^n\xi), a(\xi)]\| = 2\|a(U^n\xi)a(\xi)\| &= 2\|a(\xi)^*a(U^n\xi)^*a(U^n\xi)a(\xi))\|^{1/2} \\ &\geq 2|\tau(a(\xi)^*a(U^n\xi)^*a(U^n\xi)a(\xi))|^{1/2} \\ &= \sqrt{\|\xi\|^4 - |\langle U^n\xi, \xi \rangle|^2}, \end{aligned}$$

and this last quantity converges to $\|\xi\|^2$ as $|n| \to \infty$. This shows that α_U fails to be $\mathcal{I}$-Abelian and hence by Proposition 9.5 fails to be $\mathcal{I}$-contractive.

We also remark that τ fails to be T-mixing with respect to α_U. Indeed for $\xi \in \mathcal{H}$ the quantity $\tau(a(\xi)^*a(U^n\xi)^*a(\xi)a(U^n\xi))$ is equal to $\frac{1}{4}(|\langle U^n\xi, \xi \rangle|^2 - \|\xi\|^4)$, which converges to $-\frac{1}{4}\|\xi\|^4 = -\tau(a(\xi)^*a(\xi))^2$ as $n \to \infty$.

10. Independence and weak mixing in UHF algebras

In the previous section we showed that, under certain conditions, the existence of a weakly mixing faithful invariant state implies $\mathcal{TS}$-independence. What can be said in the reverse direction? The transfer in dynamics from topology to measure theory is a subtle one in general; for example, in [43] Huang and Ye exhibited a u.p.e. $\mathbb{Z}$-system which lacks an ergodic invariant measure of full support. On the other hand, the presence of noncommutativity at the C^*-algebra level can give rise to a kind of rigidity that in the commutative setting is more characteristic of measure-theoretic structure. We will illustrate this phenomenon in one of its extreme forms by showing that, for actions on a UHF algebra, I-independence implies weak mixing for the unique tracial state (Theorem 10.3) and, for single *-automorphisms in the type d^∞ case, is point-norm generic (Theorem 10.7). We also obtain characterizations of I-independence for Bogoliubov actions on the even CAR algebra in terms of measure-theoretic weak mixing (Theorem 10.4).

We begin by recording a lemma which reexpresses Corollary 1.6 of [6] in our C^*-dynamical context (see the discussion after Theorem 9.6 for information on weak mixing). Note that although G is generally taken to be σ-compact and locally compact in [6], the arguments on which Corollary 1.6 of [6] are based do not require these assumptions.

Lemma 10.1. *Let (A, G, α) be a C^*-dynamical system and let σ be a G-invariant state on A. Then σ is weakly mixing if and only if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is an $s \in G$ such that $|\sigma(a^*\alpha_s(a)) - \sigma(a^*)\sigma(a)| < \varepsilon$ for all $a \in \Omega$.*

When G is Abelian, it suffices in the above characterization of weak mixing to quantify over singletons in A instead of finite subsets.

Lemma 10.2. *For every $d \in \mathbb{N}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that if A_1 and A_2 are finite-dimensional unital C^*-subalgebras of a unital C^*-algebra A with $\dim(A_2) \leq d$ and $\|[a_1, a_2]\| \leq \delta\|a_1\|\|a_2\|$ for all $a_1 \in A_1$ and $a_2 \in A_2$, then there is a faithful *-homomorphism γ from A_2 to the commutant A_1' such that $\|\gamma - \mathrm{id}_{A_2}\| < \varepsilon$.*

Proof. Let $d \in \mathbb{N}$ and $\varepsilon > 0$, and suppose that A_1 and A_2 are finite-dimensional unital C^*-subalgebras of a unital C^*-algebra A. By Lemma 2.1 of [15] we can find a $\delta > 0$ depending on ε and d such that if the unit ball of A_2 is approximately included to within δ in A_1' then there exists a faithful *-homomorphism $\gamma : A_2 \to A_1'$ such that $\|\gamma - \mathrm{id}_{A_2}\| < \varepsilon$. Let μ be the normalized Haar measure on the unitary group $\mathcal{U}(A_1)$, which is compact by finite-dimensionality. Then setting

$$E(a) = \int_{\mathcal{U}(A_1)} uau^*\, d\mu(u)$$

for $a \in A$ we obtain a conditional expectation E from A onto A_1'. Moreover, if $\|[a_1, a_2]\| \leq \delta\|a_1\|\|a_2\|$ for all $a_1 \in A_1$ and $a_2 \in A_2$, then $\|E(a) - a\| \leq \delta\|a\|$ for all $a \in A_2$, so that the unit ball of A_2 is approximately included to within δ in A_1', yielding the lemma. □

Actually, by a result of Christensen [21], in the above lemma a δ not depending on d can be found.

Theorem 10.3. *Let (A, G, α) be an I-independent C^*-dynamical system with A a UHF algebra. Then α is weakly mixing for the unique (and hence α-invariant) tracial state τ on A.*

Proof. By Propositions 8.2 and 9.5, α is $\mathcal{I}$-Abelian. Let B be a simple finite-dimensional unital C^*-subalgebra of A, and let $\varepsilon > 0$. Suppose we are given an $s \in G$ and a $\delta > 0$ such that $\|[b_1, \alpha_s(b_2)]\| \leq \delta\|b_1\|\|b_2\|$ for all $b_1, b_2 \in B$. By Lemma 10.2, if we assume δ to be sufficiently small we can find a *-homomorphism $\gamma : B \to A$ such that $\|\gamma - \alpha_s|_B\| < \varepsilon$ and the C^*-subalgebras B and $\gamma(B)$ of A commute. Denote by Φ the *-homomorphism from $B \otimes B$ to the C^*-algebra generated by B and $\gamma(B)$ determined on the factors by $b \otimes 1 \mapsto b$

and $1 \otimes b \mapsto \gamma(b)$. Since $B \otimes B$ is simple, the tracial state $\tau \circ \Phi$ is unique and equal to $\tau|_B \otimes \tau|_B$. For all a and b in the unit ball of B, we then have

$$\tau(a\alpha_s(b)) \approx_\varepsilon \tau(a\gamma(b)) = (\tau \circ \Phi)(a \otimes b) = \tau(a)\tau(b),$$

so that the set of all $s \in G$ with $|\tau(a\alpha_s(b)) - \tau(a)\tau(b)| < \varepsilon$ is nonempty. Thus, since A is the closure of an increasing sequence of simple finite-dimensional unital C^*-subalgebras of A, we conclude by Lemma 10.1 that τ is weakly mixing. □

The converse of Theorem 10.3 is false, as Example 9.11 demonstrates.

We will next examine Bogoliubov actions on the even CAR algebra. We refer the reader to Example 9.11 for notation and an outline of the context (see [17] for a general reference). Let $\mathcal{H}$ be a separable Hilbert space. The even CAR algebra $A(\mathcal{H})_{\mathrm{e}}$ is the unital C^*-subalgebra of the CAR algebra $A(\mathcal{H})$ consisting of those elements which are fixed by the Bogoliubov automorphism arising from scalar multiplication by -1 on $\mathcal{H}$, and it is generated by even products of creation and annihilation operators. Both the CAR algebra and the even CAR algebra are *-isomorphic to the type 2^∞ UHF algebra (see [17, Thm. 5.2.5] and [73]). Every Bogoliubov automorphism α_U of $A(\mathcal{H})$ restricts to a *-automorphism of $A(\mathcal{H})_{\mathrm{e}}$. A strongly continuous unitary representation $s \mapsto U_s$ of the group G on $\mathcal{H}$ gives rise via Bogoliubov automorphisms to a C^*-dynamical system on each of $A(\mathcal{H})$ and $A(\mathcal{H})_{\mathrm{e}}$, in which case we will speak of a Bogoliubov action and write α_U.

Theorem 10.4. *Let $\mathcal{H}$ be a separable Hilbert space. Let U be a strongly continuous unitary representation of G on $\mathcal{H}$, and consider the corresponding Bogoliubov action α_U on $A(\mathcal{H})_{\mathrm{e}}$. The following are equivalent:*

(1) *U is a weakly mixing representation of G,*
(2) *the unique tracial state τ on $A(\mathcal{H})_{\mathrm{e}}$ is weakly mixing for α_U,*
(3) *α_U is $\mathcal{TS}$-independent,*
(4) *α_U is I-independent,*
(5) *α_U is $\mathcal{TS}$-Abelian,*
(6) *α_U is $\mathcal{N}$-Abelian.*

Proof. (1)⇒(5). Since the representation U of G is weakly mixing, for every finite set $\Theta \subseteq \mathcal{H}$ and every $\varepsilon > 0$ the set of all $s \in G$ for which $|\langle U_s\xi, \zeta\rangle| < \varepsilon$ for all $\xi, \zeta \in \Theta$ is thickly syndetic. It is then straightforward to check using the anticommutation relations and the fact that the even CAR algebra is generated by even products of creation and annihilation operators that for every finite set $\Omega \subseteq A(\mathcal{H})_{\mathrm{e}}$ and $\varepsilon > 0$ the set of all $s \in G$ for which $\|[a, \alpha_{U_s}(b)]\| < \varepsilon$ for all $a, b \in \Omega$ is thickly syndetic, i.e., α_U is $\mathcal{TS}$-Abelian.

(5)⇒(3). Apply Theorem 9.6.

(3)⇒(4). Apply Proposition 8.2.

(4)⇒(2). Apply Theorem 10.3.

(2)⇒(1). Since τ is weakly mixing for α_U, for all $\xi, \zeta \in \mathcal{H}$ the function

$$s \mapsto \tau(a(\xi)^* a(U_s\zeta)) - \tau(a(\xi)^*)\tau(a(\zeta)) = \frac{1}{2}\langle U_s\zeta, \xi\rangle$$

on G is a flight function, i.e., U is weakly mixing.

(4)⇔(6). This is a consequence of Theorem 9.6 and Proposition 8.2. □

We will now restrict our attention to UHF algebras of the form $M_d^{\otimes\mathbb{Z}}$ for some $d \geq 2$ and establish generic I-independence for *-automorphisms. For a C^*-algebra A we denote by $\mathrm{Aut}(A)$ its *-automorphism group with the point-norm topology, which is a Polish space when A is separable.

Lemma 10.5. *Let A be a separable unital C^*-algebra. Then the I-independent *-automorphisms of A form a G_δ subset of* $\mathrm{Aut}(A)$.

Proof. For every finite-dimensional operator subsystem $V \subseteq A$ and $\varepsilon > 0$ define

$$\Gamma(V,\varepsilon) = \big\{\alpha \in \mathrm{Aut}(A) : \mathrm{Ind}(\alpha, V, \varepsilon') \neq \emptyset \text{ for some } \varepsilon' \in (0,\varepsilon)\big\},$$

which is an open subset of $\mathrm{Aut}(A)$, as can be seen by a straightforward perturbation argument using Lemma 2.13.2 of [66]. Take an increasing sequence $V_1 \subseteq V_2 \subseteq \dots$ of finite-dimensional operator subsystems of A whose union is dense in A. Then by Proposition 8.2 the set of I-independent *-automorphisms of A is equal to $\bigcap_{j=1}^\infty \bigcap_{k=1}^\infty \Gamma(V_j, 1/k)$, which is a G_δ set. $\square$

For the definition of the Rokhlin property in the sense that we use it in the next lemma, see [51].

Lemma 10.6. *Let γ be a *-automorphism of $M_d^{\otimes\mathbb{Z}}$ with the Rokhlin property. Then γ has dense conjugacy class in* $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$.

Proof. Let $\alpha \in \mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$, and let Ω be a finite subset of $M_d^{\otimes\mathbb{Z}}$ and $\varepsilon > 0$. To show that α can be norm approximated to within ε on Ω by a conjugate of γ, we may assume that $\Omega \subseteq M_d^{\otimes I}$ for some finite set $I \subseteq \mathbb{Z}$. By the classification theory for AF algebras, α is approximately inner. Thus, by enlarging I if necessary, we can find a unitary $u \in M_d^{\otimes I}$ such that the *-automorphism $\mathrm{Ad}\, u \otimes \mathrm{id}$ of $M_d^{\otimes I} \otimes M_d^{\otimes\mathbb{Z}\setminus I} = M_d^{\otimes\mathbb{Z}}$ satisfies $\|\alpha(a) - (\mathrm{Ad}\, u \otimes \mathrm{id})(a)\| < \varepsilon$ for all $a \in \Omega$. Pick a *-automorphism γ' of $M_d^{\otimes\mathbb{Z}\setminus I}$ conjugate to γ. Since $\Omega \subseteq M_d^{\otimes I}$, we have $\|\alpha(a) - (\mathrm{Ad}\, u \otimes \gamma')(a)\| < \varepsilon$ for all $a \in \Omega$. Since γ has the Rokhlin property, so does $\mathrm{Ad}\, u \otimes \gamma'$. Hence by Theorem 1.4 of [51] there is a *-automorphism β of $M_d^{\otimes\mathbb{Z}}$ such that $\|\mathrm{Ad}\, u \otimes \gamma' - \beta \circ \gamma \circ \beta^{-1}\| < \varepsilon$. It follows that $\|\alpha(a) - (\beta \circ \gamma \circ \beta^{-1})(a)\| < \varepsilon$ for all $a \in \Omega$, which establishes the lemma. $\square$

Theorem 10.7. *The I-independent *-automorphisms of $M_d^{\otimes\mathbb{Z}}$ form a dense G_δ subset of* $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$.

Proof. The two-sided tensor product shift α on $M_d^{\otimes\mathbb{Z}}$ satisfies the Rokhlin property [18, 52] (note that the Rokhlin property for the one-sided shift implies the Rokhlin property for the two-sided shift) and thus has dense conjugacy class in $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$ by Lemma 10.6. It is also clearly I-independent, and so with an appeal to Lemma 10.5 we obtain the result. $\square$

For systems on UHF algebras, I-independence is equivalent to $\mathfrak{I}$-Abelianness by Proposition 8.2 and Theorem 9.6. The following corollary thus ensues by applying Corollary 4.3.11 of [16].

Corollary 10.8. *The invariant state space of a generic *-automorphism in $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$ is a simplex.*

Proposition 8.8 yields the next corollary.

Corollary 10.9. *A generic* **-automorphism of* $M_d^{\otimes\mathbb{Z}}$ *is completely untame.*

Corollary 10.10. *The set of inner* **-automorphisms of* $M_d^{\otimes\mathbb{Z}}$ *is of first category in* $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$.

Proof. By Corollary 10.9 it suffices to show that no inner *-isomorphism of $M_d^{\otimes\mathbb{Z}}$ is completely untame. Let u be a unitary in $M_d^{\otimes\mathbb{Z}}$. If u is scalar then $\mathrm{Ad}\, u$ is the identity map, which is obviously tame. If u is not scalar then $\mathrm{Ad}\, u$ fails to be completely untame because u, being fixed by $\mathrm{Ad}\, u$, does not have an infinite ℓ_1-isomorphism set. □

It is interesting to compare the generic I-independence in Theorem 10.7 with generic behaviour for homeomorphisms of the Cantor set. Kechris and Rosendal showed by model-theoretic means that the Polish group of homeomorphisms of the Cantor set has a dense G_δ conjugacy class [47], and a description of the generic homeomorphism has been given by Akin, Glasner, and Weiss in [1]. This generic homeomorphism can be seen to be null by examining as follows the construction of Section 1 in [1], to which we refer the reader for notation. The "special" homeomorphism $T(D, C)$ of the Cantor set $X(D, C)$ is defined at the end of Section 1 in [1] and represents the generic conjugacy class. It is a product of the homeomorphism $\tau_{(D,C)}$ of the Cantor set $Z(D, C)$ and an identity map, and so it suffices to show that $\tau_{(D,C)}$ is null. Now $Z(D, C)$ is a closed subset of $q(Z(D, C)) \times \Theta$, and $\tau_{(D,C)}$ is the restriction of the product of an obviously null homeomorphism of $q(Z(D, C))$ and the universal adding machine on Θ, which is an inverse limit of finite systems and hence is also null. It follows that $\tau_{(D,C)}$ is null, as desired.

As a final remark, we point out that the *-automorphisms of $M_d^{\otimes\mathbb{Z}}$ which are weakly mixing for τ form a dense G_δ subset of $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$. This is easily seen using Lemma 10.1 and the fact that tensor product shift on $M_d^{\otimes\mathbb{Z}}$ is weakly mixing for τ and has dense conjugacy class in $\mathrm{Aut}(M_d^{\otimes\mathbb{Z}})$ (cf. the proof of Theorem 10.7).

11. A tame nonnull Toeplitz subshift

We construct in this section a tame nonnull Toeplitz subshift. Toeplitz subshifts were introduced by Jacobs and Keane in [44]. An element $x \in \Omega_m := \{0, 1, \dots, m-1\}^{\mathbb{Z}}$ is called a *Toeplitz sequence* if for any $j \in \mathbb{Z}$ there exists an $n \in \mathbb{N}$ such that $x(j + kn) = x(j)$ for all $k \in \mathbb{Z}$. The subshift generated by x is called a *Toeplitz subshift.* Note that every Toeplitz subshift is minimal.

Set $m = 2$. To construct our Toeplitz sequence x in Ω_2, we choose an increasing sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$ with $n_j | n_{j+1}$ for each $j \in \mathbb{N}$ and $j \cdot 2^j + 1$ distinct elements $y_{j,0}, y_{j,1}, \dots, y_{j,j\cdot 2^j}$ in $\mathbb{Z}/n_j\mathbb{Z}$ for each $j \in \mathbb{N}$ with the following properties:

(i) $y_{j+1,k} \equiv y_{j,0} \mod n_j$ for all $j \in \mathbb{N}$ and all $0 \le k \le (j+1)2^{j+1}$,

(ii) for each $j \in \mathbb{N}$ and $1 \le t \le 2^j$, setting $Y_{j,t} := \{y_{j,k} : (t-1)j < k \le tj\}$, we have, for all $1 \le t < 2^j$, $Y_{j,t+1} = Y_{j,t} + z_{j,t}$ for some $z_{j,t} \in \mathbb{Z}/n_j\mathbb{Z}$,

(iii) there does not exist any $z \in \mathbb{Z}$ with $z \equiv y_{j,0} \mod n_j$ for all $j \in \mathbb{N}$,

(iv) for each $j \in \mathbb{N}$ and any $1 \le k_1, k_2 \le j \cdot 2^j$ and any $0 \le k_3 \le j \cdot 2^j$ we have $y_{j,k_1} - y_{j,0} \ne y_{j,k_3} - y_{j,k_2}$.

Take a map $f : \bigcup_{j\in\mathbb{N},1\leq t\leq 2^j} Y_{j,t} \to \{0,1\}$ such that, for each $j \in \mathbb{N}$, the maps $\{1,2,\dots,j\} \to \{0,1\}$ given by $p \mapsto f(y_{j,(t-1)j+p})$ yield exactly all of the elements in $\{0,1\}^{\{1,2,\dots,j\}}$ as t runs through $1,\dots,2^j$. Now we define our x by

$$x(s) = \begin{cases} f(y_{j,k}), & \text{if } s \equiv y_{j,k} \mod n_j \text{ for some } j \in \mathbb{N} \text{ and some } 1 \leq k \leq 2^j, \\ 0, & \text{otherwise.} \end{cases}$$

Property (i) guarantees that x is well defined. Property (iii) implies that x is a Toeplitz sequence. Denote by X the subshift generated by x. Also denote by A (resp. B) the set of elements in X taking value 1 (resp. 0) at 0. Property (ii) and the condition on f imply that $\tilde{y}_{j,1},\dots,\tilde{y}_{j,j}$ is an independence set for (A,B) for each $j \in \mathbb{N}$, where $\tilde{y}_{j,i}$ is any element in $\mathbb{Z}$ with $\tilde{y}_{j,i} \equiv y_{j,i} \mod n_j$. Thus (A,B) has arbitrarily large finite independence sets and hence the subshift X is nonnull by Proposition 5.4.

Since $\mathrm{IT}_2(X,\mathbb{Z})$ is $\mathbb{Z}$-invariant, to show that X is tame, by Proposition 6.4(2) it suffices to show that (A,B) has no infinite independence set. For each $s \in \mathbb{Z}$ with $x(s) = 1$ let $J(s)$ and $K(s)$ denote the positive integers such that $1 \leq K(s) \leq J(s)2^{J(s)}$ and $s \equiv y_{J(s),K(s)} \mod n_{J(s)}$.

Lemma 11.1. *Suppose that $x(s_1) = x(s_2) = 1$ for some s_1 and s_2 in $\mathbb{Z}$ with $J(s_1) < J(s_2)$. Also suppose that $x(s_1+a) = 0$ and $x(s_2+a) = 1$ for some $a \in \mathbb{Z}$. Then $J(s_2+a) = J(s_1)$.*

Proof. If $J(s_2 + a) > J(s_1)$, then

$$s_1 - s_2 \equiv (s_1 + a) - (s_2 + a) \equiv (s_1 + a) - s_2 \mod n_{J(s_1)}$$

by property (i) and hence $s_1 \equiv s_1 + a \mod n_{J(s_1)}$. Consequently $x(s_1 + a) = x(s_1) = 1$, in contradiction to the fact that $x(s_1 + a) = 0$. Therefore $J(s_2 + a) \leq J(s_1)$.

If $J(s_2+a) < J(s_1)$, then $s_1 \equiv s_2 \mod n_{J(s_2+a)}$ by property (i) and hence $s_1+a \equiv s_2+a \mod n_{J(s_2+a)}$. Consequently, $x(s_1 + a) = x(s_2 + a) = 1$, in contradiction to the fact that $x(s_1 + a) = 0$. Therefore $J(s_2 + a) \geq J(s_1)$ and hence $J(s_2 + a) = J(s_1)$, as desired. □

Lemma 11.2. *Suppose that $\{s_1, s_2, s_3\}$ is an independence set for (A,B) and $x(s_i) = 1$ for all $1 \leq i \leq 3$. Then $J(s_1) = J(s_2) = J(s_3)$.*

Proof. Suppose that the $J(s_i)$ are not all the same. Without loss of generality, we may assume that $J(s_1) < \min(J(s_2), J(s_3))$ or $J(s_1) = J(s_3) < J(s_2)$.

Consider the case $J(s_1) < \min(J(s_2), J(s_3))$ first. Since $\{s_1, s_2, s_3\}$ is an independence set for (A,B), we have $x(s_1 + a) = x(s_3 + a) = 0$ and $x(s_2 + a) = 1$ for some $a \in \mathbb{Z}$. By Lemma 11.1 we have $J(s_2 + a) = J(s_1)$. Note that $s_3 \equiv s_2 \mod n_{J(s_1)}$ by property (i). Consequently, $s_3+a \equiv s_2+a \mod n_{J(s_1)}$ and hence $x(s_3+a) = x(s_2+a) = 1$, contradicting the fact that $x(s_3 + a) = 0$. This rules out the case $J(s_1) < \min(J(s_2), J(s_3))$.

Consider now the case $J(s_1) = J(s_3) < J(s_2)$. Since $\{s_1, s_2, s_3\}$ is an independence set for (A,B), we have $x(s_1 + a) = 0$ and $x(s_2 + a) = x(s_3 + a) = 1$ for some $a \in \mathbb{Z}$. By Lemma 11.1 we have $J(s_2 + a) = J(s_1)$. Note that $s_3 \equiv s_2 \mod n_j$ for any $j < J(s_1) \leq \min(J(s_2), J(s_3))$ by property (i). If $J(s_3 + a) < J(s_1) = J(s_2 + a)$, then $s_3 + a \not\equiv s_2 + a \mod n_{J(s_3+a)}$ leading to a contradiction. Thus $J(s_3 + a) \geq J(s_1)$. We then have

$$\begin{aligned} y_{J(s_3),K(s_3)} - y_{J(s_1),0} &\equiv y_{J(s_3),K(s_3)} - y_{J(s_2),K(s_2)} \\ &\equiv y_{J(s_3+a),K(s_3+a)} - y_{J(s_2+a),K(s_2+a)} \mod n_{J(s_1)} \end{aligned}$$

in contradiction to property (iv). Therefore the case $J(s_1) = J(s_3) < J(s_2)$ is also ruled out, and we obtain the lemma. □

Lemma 11.3. *Suppose that H and H' are disjoint nonempty subsets of $\mathbb{Z}$ and that $H \cup H'$ is an independence set for (A, B). Suppose also that $|H| \geq j \cdot (2^j + 1)$. Then there exist an $H_1 \subseteq H$ and an $a \in \mathbb{Z}$ such that $x(s+a) = 1$ and $J(s+a) > j$ for all $s \in H' \cup (H \setminus H_1)$ and $|H_1| \leq j$.*

Proof. We shall prove the assertion via induction on j. The case $j = 0$ is trivial since $H' \cup H$ is an independence set for (A, B).

Assume that the assertion holds for $j = i$. Suppose that $|H| \geq (i+1) \cdot (2^{i+1}+1)$. By the assumption there exist an $H_1 \subseteq H$ and an $a \in \mathbb{Z}$ such that $x(s + a) = 1$ and $J(s + a) > i$ for all $s \in H' \cup (H \setminus H_1)$ and $|H_1| \leq i$. Note that $|H \setminus H_1| \geq (i + 1)2^{i+1} + 1 \geq 3$. By Lemma 11.2, $J(s + a)$ does not depend on $s \in H' \cup (H \setminus H_1)$. If $J(s + a) > i + 1$ for $s \in H' \cup (H \setminus H_1)$, we are done with the induction step. Suppose then that $J(s+a) = i+1$ for all $s \in H' \cup (H \setminus H_1)$. Then $K(s_1 + a) = K(s_2 + a)$ for some distinct s_1 and s_2 in $H \setminus H_1$. In other words, $s_1 \equiv s_2 \mod n_{i+1}$. Set $H_2 := H_1 \cup \{s_1\}$. Since $H' \cup H$ is an independence set for (A, B), there exists a $b \in \mathbb{Z}$ with $x(s+b) = 1$ for all $s \in H' \cup (H \setminus H_2)$ and $x(s_1 + b) = 0$. Using property (i) one sees easily that $J(s_2 + a) < J(s_2 + b)$. By Lemma 11.2 we have $J(s + b) = J(s_2 + b) > i + 1$ for all $s \in H' \cup (H \setminus H_2)$. This finishes the induction step and proves the lemma. □

Suppose that $H \subseteq \mathbb{Z}$ is an infinite independence set for (A, B). Choose a nonempty finite subset $H' \subset H$. By Lemma 11.3, for any $j \in \mathbb{N}$ there exists an $a_j \in \mathbb{Z}$ such that $x(s + a_j) = 1$ and $J(s + a_j) > j$ for all $s \in H'$. By property (i) we have $s + a_j \equiv s' + a_j \mod n_j$ for all $s, s' \in H'$. Thus $n_j | s - s'$. Since $n_j \to \infty$ as $j \to \infty$, we see that H' cannot contain more than one element, which is a contradiction. Therefore (A, B) has no infinite independence sets and we conclude that X is tame.

12. Convergence groups

Boundary-type actions of free groups are often described as exhibiting chaotic behaviour. The geometric features which account for this description are tied to complexity within the group structure and are suggestive of the free-probabilistic notion of independence. We will see here on the other hand that any convergence group (in particular, any hyperbolic group acting on its Gromov boundary) displays a strong lack of dynamical independence in our sense.

Let (X, G) be a dynamical system with G a group and X containing at least 3 points. We call a net $\{s_i\}_{i \in I}$ in G *wandering* if for any $s \in G$, $s_i \neq s$ for sufficiently large $i \in I$. Recall that G is said to act as a *(discrete) convergence group* on X if for any wandering net $\{s_i\}_{i \in I}$ in G there exist $x, y \in X$ and a subnet $\{s_j\}_{j \in J}$ of $\{s_i\}_{i \in I}$ such that $s_j K \to y$ as $j \to \infty$ for every compact set $K \subseteq X \setminus \{x\}$ [14]. When X is metrizable, one can replace "wandering net" and "subnet" by "sequence of distinct elements" and "subsequence", respectively, in the above definition.

Lemma 12.1. *Let G act as a convergence group on X. Let Z_1 and Z_2 be two disjoint closed subsets of X. Then there exists a finite subset $F \subseteq G$ such that $HH^{-1} \subseteq F$ for every independence set H of Z_1 and Z_2.*

Proof. Suppose that such an F does not exist. Then we can find independence sets $H_1, H_2, \dots$ for the pair (Z_1, Z_2) such that $H_n H_n^{-1} \not\subseteq \bigcup_{k=1}^{n-1} H_k H_k^{-1}$ for all $n \geq 2$. Choose an $s_n \in H_n H_n^{-1} \setminus \bigcup_{k=1}^{n-1} H_k H_k^{-1}$ for each n. Since independence sets are right translation invariant, $\{e, s_n\}$ is an independence set for (Z_1, Z_2) when $n \geq 2$. As G acts as a convergence group on X we can find $x, y \in X$ and a subnet $\{s_j\}_{j \in J}$ of $\{s_n\}_{n \in \mathbb{N}}$ such that $s_j K \to y$ as $j \to \infty$ for every compact set $K \subseteq X \setminus \{x\}$. Without loss of generality we may assume that $x \notin Z_1$. Then $s_j Z_1 \to y$ as $j \to \infty$. We separate the cases $y \notin Z_1$ and $y \in Z_1$. If $y \notin Z_1$, then $s_j Z_1 \cap Z_1 = \emptyset$ when j is large enough, contradicting the fact that $\{e, s_j\}$ is an independence set for (Z_1, Z_2). If $y \in Z_1$, then $s_j Z_1 \cap Z_2 = \emptyset$ when j is large enough, again contradicting the fact that $\{e, s_j\}$ is an independence set for (Z_1, Z_2). Therefore there does exist an F with the desired property. □

Theorem 12.2. *Let G act as a convergence group on X. Then (X, G) is null.*

Proof. Given disjoint closed subsets Z_1 and Z_2 of X let F be as in Lemma 12.1. Then $|H| \leq |HH^{-1}| \leq |F|$ for any independence set H for the pair (Z_1, Z_2). Thus Z_1 and Z_2 do not have arbitrarily large finite independence sets. It follows by Proposition 5.4(2) that (X, G) is null. □

An action of G on a locally compact Hausdorff space Y is said to be *properly discontinuous* if, given any compact subset $Z \subseteq Y$, $sZ \cap Z \neq \emptyset$ for only finitely many $s \in G$. For a general reference on hyperbolic spaces and hyperbolic groups, see [19]. It is a theorem of Tukia that if a group G acts properly discontinuously as isometries on a proper geodesic hyperbolic space Y, then the induced action of G on the Gromov compactification $\overline{Y} = Y \cup \partial Y$ is a convergence group action [76, Theorem 3A]. Thus we get:

Corollary 12.3. *If a group G acts properly discontinuously as isometries on a proper geodesic hyperbolic space Y, then the induced action of G on the Gromov compactification $\overline{Y}$ is null. In particular, the action of a hyperbolic group on its Gromov boundary is null.*

It happens on the other hand that, as the following remark notes, nonelementary convergence group actions fail to be HNS (hereditarily nonsensitive [34, Definition 9.1]), although HNS dynamical systems are tame (see the proof of [49, Corollary 5.7]).

Remark 12.4. For any dynamical system (X, G), denote by $L(X, G)$ its *limit set*, defined as the set of all $x \in X$ such that $|\{s \in G : U \cap sU \neq \emptyset\}| = \infty$ for every neighbourhood U of x. Clearly $L(X, G)$ is a closed G-invariant subset of X. A convergence group action of G on X is *nonelementary* if $|L(X, G)| > 2$, or, equivalently, there is no one- or two-point subset of X fixed setwise by G [76, Theorem 2T]. In this event, $L(X, G)$ is an infinite perfect set, the action of G on $L(X, G)$ is minimal, and there exists a *loxodromic* element $s \in G$ [76, Theorem 2S, Lemmas 2Q, 2D], i.e., s has exactly two distinct fixed points x and y, and $s^n K \to y$ as $n \to \infty$ for any compact set $K \subseteq X \setminus \{x\}$. Notice that $\{x, y\} \subseteq L(X, G)$. Clearly the action of G on $L(X, G)$ is nonequicontinuous in this event. Since minimal HNS dynamical systems are equicontinuous and subsystems of HNS dynamical systems are HNS, we see that no nonelementary convergence group action is HNS.

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