# METRIC ASPECTS OF NONCOMMUTATIVE HOMOGENEOUS SPACES

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*Dedicated to Marc A. Rieffel*
*in honor of his seventieth birthday*

Abstract. For a closed cocompact subgroup $\Gamma$ of a locally compact group $G$, given a compact abelian subgroup $K$ of $G$ and a homomorphism $\rho : \hat{K} \to G$ satisfying certain conditions, Landstad and Raeburn constructed equivariant noncommutative deformations $C^*(\hat{G}/\Gamma, \rho)$ of the homogeneous space $G/\Gamma$, generalizing Rieffel's construction of quantum Heisenberg manifolds. We show that when $G$ is a Lie group and $G/\Gamma$ is connected, given any norm on the Lie algebra of $G$, the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ induced by the derivation map of the canonical $G$-action defines a compact quantum metric. Furthermore, it is shown that this compact quantum metric space depends on $\rho$ continuously, with respect to quantum Gromov-Hausdorff distances.

## 1. Introduction

In recent years, the quantum Heisenberg manifolds have received quite some attention. These interesting $C^*$-algebras were constructed by Rieffel [28] as deformation quantizations of the Heisenberg manifolds, and carry natural actions of the Heisenberg group. The classification of these $C^*$-algebras up to isomorphism (in most cases) and Morita equivalence (in all cases) has been achieved by Abadie and her collaborators [1, 2, 3, 4]. These $C^*$-algebras also appear in the work of Connes and Dubois-Violette on noncommutative 3-spheres [10, 11].

Aiming partly at giving a mathematical foundation for various approximations in the string theory, such as the fuzzy spheres, namely the matrix algebras $M_n(\mathbb{C})$, converging to the 2-sphere $S^2$, Rieffel developed a theory of compact quantum metric spaces and quantum

*Date*: May 21, 2009.

2000 *Mathematics Subject Classification.* Primary 46L87; Secondary 53C23, 46L57.

Partially supported by NSF Grant DMS-0701414.

Gromov-Hausdorff distance between them [31, 32, 33]. As the information of the metric on a compact metric space $X$ is encoded in the Lipschitz seminorm on the algebra of continuous functions on $X$, a quantum metric on (the compact quantum space represented by) a unital $C^*$-algebra $A$ is a (possibly $+\infty$-valued) seminorm on $A$ satisfying suitable conditions (see Section 5 below for detail).

One important class of examples of compact quantum metric spaces comes from ergodic actions of a compact group $G$ on a unital $C^*$-algebra $A$, which should be thought of as the translation action of $G$ on a noncommutative homogeneous space of $G$. Given any length function on $G$, such an ergodic action induces a quantum metric on $A$ [30] (see [25] for a generalization to ergodic actions of co-amenable compact quantum groups). This class of examples includes the (fuzzy) spheres above and the noncommutative tori. When $G$ is a compact connected Lie group and the length function comes from the geodesic distance associated to some bi-invariant Riemannian metric on $G$, this seminorm can also be defined in terms of the derivation map on the space of once differentiable elements of $A$ with respect to the $G$-action [31, Proposition 8.6]. Explicitly, denote by $\sigma_X(b)$ the derivation of a once differentiable element $b$ of $A$ with respect to an element $X$ of the Lie algebra $\mathfrak{g}$ of $G$ (see Section 3 below for detail). Then the seminorm $L(b)$ is defined as the norm of the linear map $\mathfrak{g} \to A$ sending $X$ to $\sigma_X(b)$ when $b$ is once differentiable, or $\infty$ otherwise.

It is natural to ask what conditions are needed to guarantee that $L$ defined above gives rise to a quantum metric when $G$ is not compact. Rieffel raised the question about the quantum Heisenberg manifolds in [33]. In [38] Weaver studied some sub-Riemannian metric on the quantum Heisenberg manifolds, which does not quite fit into the above framework. In [9] Chakraborty showed that certain seminorm associated to some $\ell^1$-norm does define a quantum metric on the quantum Heisenberg manifolds. Since the $\ell^1$-norm is bigger than the $C^*$-norm, this seminorm is bigger than the seminorm $L$ defined above. Thus the result in [9] is weaker than what Riffel's question asks for.

Our first main result in this article is an affirmative answer to Rieffel's question. In fact, we shall deal more generally with Landstad and Raeburn's noncommutative homogeneous spaces. In [22] Landstad and Raeburn generalized Rieffel's construction to obtain equivariant deformations of compact homogeneous spaces $G/\Gamma$, starting from a locally compact group $G$, a closed cocompact subgroup $\Gamma$ of $G$, a compact abelian subgroup $K$ of $G$, and a homomorphism $\rho : \hat{K} \to G$ satisfying certain conditions. These $C^*$-algebras were denoted by $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ and

were further studied in [20]. We shall see in Proposition 2.7 below that these algebras coincide with certain universal $C^*$-algebras, which we denote by $C^*(\hat{G}/\Gamma, \rho)$. For our result to be valid for these algebras, we shall assume conditions (S1)-(S5) (see Sections 2, 3, and 4 below). Among these conditions, (S1)-(S3) are essentially the same but slightly weaker than the conditions of Landstad and Raeburn. The conditions (S4) and (S5) are just that $G$ is a Lie group and $G/\Gamma$ is connected.

**Theorem 1.1.** *Let $G, \Gamma, K$ and $\rho$ satisfy the conditions (S1)-(S5). Fix a norm on the Lie algebra $\mathfrak{g}$ of $G$. Denote by $L_\rho$ the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ defined above for the canonical action $\alpha$ of $G$ on $C^*(\hat{G}/\Gamma, \rho)$. Then $(C^*(\hat{G}/\Gamma, \rho), L_\rho)$ is a $C^*$-algebraic compact quantum metric space.*

Since Rieffel introduced his quantum Gromov-Hausdorff distance in [31], several variations have appeared [18, 19, 24, 25, 26, 35, 39]. Among these quantum distances, probably the most suitable one in our current situation is the distance $\mathrm{dist}_{\mathrm{nu}}$ discussed in [19, Section 5], which is the unital version of the quantum distance introduced in [26, Remark 5.5]. As pointed out in [19, Section 5], this distance is no less than the distances introduced in [18, 31]. It is also no less than the distances in [35] (see Appendix below). Our second main result says that the compact quantum metric spaces $(C^*(\hat{G}/\Gamma, \rho), L_\rho)$ depend on $\rho$ continuously. Let us mention that among the conditions (S1)-(5), only the conditions (S1) and (S2) involve $\rho$.

**Theorem 1.2.** *Fix $G$, $\Gamma$, and $K$ so that there exists $\rho$ satisfying the conditions (S1)-(S5). Denote by $\Omega$ the set of all $\rho$ satisfying the conditions (S1) and (S2), equipped with the weakest topology making the maps $\Omega \to G$ sending $\rho$ to $\rho(s)$ to be continuous for each $s \in \hat{K}$. Then $\Omega$ is a locally compact metrizable space. Fix a norm on the Lie algebra $\mathfrak{g}$ of $G$. Then for any $\rho' \in \Omega$, $\mathrm{dist}_{\mathrm{nu}}(C^*(\hat{G}/\Gamma, \rho), C^*(\hat{G}/\Gamma, \rho')) \to 0$ as $\rho \to \rho'$.*

This paper is organized as follows. In Section 2 we recall Landstad and Raeburn's construction of noncommutative homogeneous spaces, and establish some general properties of these noncommutative spaces. The relation between the derivations coming from two canonical group actions on $C^*(\hat{G}/\Gamma, \rho)$ is established in Section 3. In Section 4 we show that in the nondeformed case $L_\rho$ is essentially the Lipschitz seminorm corresponding to some metric on $G/\Gamma$. A general result of establishing certain seminorm being a quantum metric by the help of a compact group action is proved in Section 5. Theorems 1.1 and 1.2 are proved in Sections 6 and 7 respectively. In an appendix we compare the distance $\mathrm{dist}_{\mathrm{nu}}$ and the proximity Rieffel introduced in [35].

*Acknowledgements.* I am grateful to Wei Wu for comments.

## 2. Noncommutative homogeneous spaces

In this section we recall Landstad and Raeburn's construction of noncommutative deformations of homogeneous spaces, discuss some examples, and establish some general properties of these noncommutative homogeneous spaces. These properties are of independent interest themselves.

Let $G$ be a locally compact group. Throughout this paper, we make the following standard assumptions:

(S1) $K$ is a compact abelian subgroup of $G$, and $\rho : \hat{K} \to G$ is a group homomorphism from its Pontryagin dual $\hat{K}$ into $G$ such that $\rho(\hat{K})$ commutes with $K$.

(S2) $\Gamma$ is a closed subgroup of $G$ commuting with $K$ and satisfies

$$\begin{aligned} \Omega_\gamma(s) &:= \gamma\rho(s)\gamma^{-1}\rho(-s) \text{ is in } K \text{ for all } s \in \hat{K}, \gamma \in \Gamma \text{ and} \\ \langle \Omega_\gamma(s), t\rangle &= \langle \Omega_\gamma(t), s\rangle \text{ for all } s, t \in \hat{K}, \gamma \in \Gamma, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $K$ and $\hat{K}$.

Denote by $C_{\mathrm{b}}(G)$ the Banach algebra of bounded continuous $\mathbb{C}$-valued functions on $G$, equipped with the pointwise multiplication and the supremum norm. Endow $K$ with its normalized Haar measure. Consider the action of $K$ on $C_{\mathrm{b}}(G)$ induced by the right multiplication of $K$ on $G$. For $f \in C_{\mathrm{b}}(G)$, let $f_s \in C_{\mathrm{b}}(G)$ for $s \in \hat{K}$ be the partial Fourier transform defined by $f_s(x) := \int_K \overline{< k, s >} f(xk)\, dk$ for $x \in G$ (this is denoted by $\hat{f}(x, s)$ in (1.3) of [22]). Note that although the action of $K$ on $C_{\mathrm{b}}(G)$ may not be strongly continuous, we do have $f_s \in C_{\mathrm{b}}(G)$. Then

$$C_{\mathrm{b},1}(G) := \{f \in C_{\mathrm{b}}(G) | \quad \|f\|_{\infty,1} := \sum_{s \in \hat{K}} \|f_s\| < \infty\}$$

is a Banach $*$-algebra [21, Proposition 5.2] with norm $\| \cdot \|_{\infty,1}$ and operations

$$f * g(x) = \sum_{s,t} f_s(x\rho(t)) g_t(x\rho(-s)), \tag{1}$$

$$f^*(x) = \overline{f(x)}. \tag{2}$$

Fix a left invariant Haar measure on $G$. For each $s \in \hat{K}$ denote by $P_s$ the projection on $L^2(G)$ corresponding to the restriction of the left

regular representation $L|_K$ of $K$ in $L^2(G)$, i.e.,

$$P_s = \int_K \overline{\langle k, s\rangle} L_k \, dk,$$

where $L_y\xi(x) = \xi(y^{-1}x)$ for $\xi \in L^2(G)$, $x, y \in G$. Then $C_{\mathrm{b},1}(G)$ has a faithful $*$-representation $V$ on $L^2(G)$ [22, Proposition 1.3] given by

$$V(f) = \sum_{s,t} P_t L_{\rho(s)} M(f) L_{\rho(-t)} P_s, \tag{3}$$

where $M$ is the representation of $C_{\mathrm{b}}(G)$ on $L^2(G)$ given by $M(f)\xi(x) = f(x^{-1})\xi(x)$. Denote by $C_0(G/\Gamma)$ the $C^*$-algebra of continuous $\mathbb{C}$-valued functions on $G/\Gamma$ vanishing at $\infty$, and think of it as a $C^*$-subalgebra of $C_{\mathrm{b}}(G)$ via the quotient map $G \to G/\Gamma$. The space $C_{0,1}(G/\Gamma, \rho) := C_0(G/\Gamma) \cap C_{\mathrm{b},1}(G, \rho)$ is a closed $*$-subalgebra of $C_{\mathrm{b},1}(G, \rho)$, and the noncommutative homogeneous space $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ of Landstad and Raeburn is defined as the closure of $V(C_{0,1}(G/\Gamma, \rho))$ [22, Theorem 4.3].

Clearly the left translations $\alpha_y$ defined by $\alpha_y(f)(x) = f(y^{-1}x)$ for $y \in G$ extend to isometric $*$-automorphisms of $C_{0,1}(G/\Gamma, \rho)$. They also extend to $*$-automorphisms of $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ [22, Theorem 4.3]. We shall see later that this action of $G$ on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is strongly continuous.

Before discussing properties of these noncommutative homogeneous spaces, let us look at some examples.

**Example 2.1.** Let $H_1$ be the 3-dimensional Heisenberg group consisting of matrices of the form

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

as a subgroup of $\mathrm{GL}(3, \mathbb{R})$. Denote by $Z$ the subgroup consisting of elements with $x = y = 0$ and $z \in \mathbb{Z}$. Then we can write the elements of $G := H_1/Z$ as $(x, y, e^{2\pi i z})$ for $x, y, z \in \mathbb{R}$. Fix a positive integer $c$. Take

$$\Gamma = \{(x, y, e^{2\pi i z}) \in G | x, y, cz \in \mathbb{Z}\}, \quad K = \{(0, 0, e^{2\pi i z}) \in G | z \in \mathbb{R}\}.$$

Take $\mu, \nu \in \mathbb{R}$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$\rho(s) = (s\mu, s\nu, e^{\pi i s^2 \mu\cdot\nu}).$$

The $C^*$-algebra $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is isomorphic to Rieffel's quantum Heisenberg manifold $D_1$ in [28, Theorem 5.5] (see [22, page 493]).

**Example 2.2.** (cf. [22, Example 4.17]) Let $H_n$ be the $2n+1$-dimensional Heisenberg group consisting of matrices of the form

$$\begin{pmatrix} 1 & y_1 & y_2 & \cdots & y_n & z \\ 0 & 1 & 0 & \cdots & 0 & x_1 \\ 0 & 0 & 1 & \cdots & 0 & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

as a subgroup of $\mathrm{GL}(n+2, \mathbb{R})$. Denote by $Z$ the subgroup consisting of elements with $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ and $z \in \mathbb{Z}$. Then we can write the elements of $G := H_n/Z$ as $(x, y, e^{2\pi i z})$ for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Fix positive integers $b_1, \ldots, b_n, d_1, \ldots, d_n$ and $c$ such that $b_j d_j | c$ for all $j$. Set $b = (b_1, \ldots, b_n)$ and $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$. Take

$$\Gamma = \{(x, y, e^{2\pi i z}) \in G | b \cdot x, d \cdot y, cz \in \mathbb{Z}\}, \quad K = \{(0, 0, e^{2\pi i z}) \in G | z \in \mathbb{R}\}.$$

Take $\mu, \nu \in \mathbb{R}^n$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$\rho(s) = (s\mu, s\nu, e^{\pi i s^2 \mu \cdot \nu}).$$

The $C^*$-algebra $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is a higher-dimensional generalization of Example 2.1.

**Example 2.3.** Let $n \geq 3$. Let $W$ be the subgroup of $\mathrm{GL}(n, \mathbb{Z})$ consisting of upper triangular matrices $(a_{j,l})$ with diagonal entries all being 1. Denote by $Z$ the subgroup consisting of matrices whose entries are all 0 except diagonal ones being 1 and $a_{1,n}$ being an integer. Then we can write the elements of $G := W/Z$ as $(a_{j,l})$ with $a_{1,n} \in \mathbb{T}$. Fix a positive integer $c$. Take

$$\begin{aligned} \Gamma &= \{(a_{j,l}) \in G | a_{1,n}^c = 1 \text{ and } a_{j,l} \in \mathbb{Z} \text{ if } (j,l) \neq (1,n)\}, \\ K &= \{(a_{j,l}) \in G | a_{j,l} = 0 \quad \text{if} \quad j < l \quad \text{and} \quad (j,l) \neq (1,n)\}. \end{aligned}$$

Take $\mu, \nu \in \mathbb{R}$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$(\rho(s))_{j,l} = \begin{cases} s\mu, & \text{if } (j,l) = (2,n), \\ s\nu, & \text{if } (j,l) = (1,n-1), \\ e^{\pi i s^2 \mu \cdot \nu}, & \text{if } (j,l) = (1,n), \\ 0, & \text{for other } j < l. \end{cases}$$

For $n = 3$ we get the quantum Heisenberg manifold in Example 2.1 again.

In the rest of this section we establish some properties of $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$. Denote by $C^*(\hat{G}/\Gamma, \rho)$ the enveloping $C^*$-algebra of the Banach $*$-algebra $C_{0,1}(G/\Gamma, \rho)$ [36, page 42]. By the universality of $C^*(\hat{G}/\Gamma, \rho)$

there is a canonical surjective $*$-homomorphism $C^*(\hat{G}/\Gamma,\rho)\to C^*_{\mathrm{r}}(\hat{G}/\Gamma,\rho)$ such that the diagram

$$\begin{array}{ccc} C_{0,1}(G/\Gamma,\rho) & \longrightarrow & C^*(\hat{G}/\Gamma,\rho) \\ & \searrow & \downarrow \\ & & C^*_{\mathrm{r}}(\hat{G}/\Gamma,\rho) \end{array}$$

commutes.

Clearly the right translations $\beta_k(f)(x)=f(xk)$ for $k\in K$ extend to isometric $*$-automorphisms of $C_{0,1}(G/\Gamma,\rho)$. Recall the action $\alpha$ of $G$ on $C_{0,1}(G/\Gamma,\rho)$ defined before Example 2.1. Then $\alpha$ and $\beta$ induce actions of $G$ and $K$ on $C^*(\hat{G}/\Gamma,\rho)$ respectively, which we still denote by $\alpha$ and $\beta$ respectively. For each $s\in\hat{K}$, set

$$B_s:=\{f\in C_0(G/\Gamma)|f=f_s\}. \tag{4}$$

**Lemma 2.4.** *The actions $\alpha$ and $\beta$ of $G$ and $K$ on $C_{0,1}(G/\Gamma,\rho)$ ($C^*(\hat{G}/\Gamma,\rho)$ resp.) commute with each other and are strongly continuous. The spectral spaces $\{f\in C_{0,1}(G/\Gamma,\rho)|\beta_k(f)=\langle k,s\rangle f$ for all $k\in K\}$ and $\{a\in C^*(\hat{G}/\Gamma,\rho)|\beta_k(a)=\langle k,s\rangle a$ for all $k\in K\}$ of $\beta$ corresponding to $s\in\hat{K}$ are exactly $B_s$, and the norm of $B_s$ in $C_{0,1}(G/\Gamma,\rho)$ and $C^*(\hat{G}/\Gamma,\rho)$ is exactly the supremum norm.*

*Proof.* Clearly $\alpha$ and $\beta$ commute with each other. It is also clear that $B_s=\{f\in C_{0,1}(G/\Gamma,\rho)|\beta_k(f)=\langle k,s\rangle f$ for all $k\in K\}$ and that the norm of $B_s$ in $C_{0,1}(G/\Gamma,\rho)$ is exactly the supremum norm. It follows that the restrictions of the actions $\alpha$ and $\beta$ on $B_s\subseteq C_{0,1}(G/\Gamma,\rho)$ are strongly continuous for each $s\in\hat{K}$. For any $f\in C_{0,1}(G/\Gamma,\rho)$, one has $f_s\in B_s$ for each $s\in\hat{K}$. For any $\varepsilon>0$ take a finite subset $F\subseteq\hat{K}$ such that $\sum_{s\in\hat{K}\setminus F}\|f_s\|<\varepsilon$. Then $\|f-\sum_{s\in F}f_s\|_{\infty,1}=\sum_{s\in\hat{K}\setminus F}\|f_s\|<\varepsilon$. Therefore $\oplus_{s\in\hat{K}}B_s$ is dense in $C_{0,1}(G/\Gamma,\rho)$. It follows that the actions $\alpha$ and $\beta$ are strongly continuous on $C_{0,1}(G/\Gamma,\rho)$. Note that the canonical homomorphism $C_{0,1}(G/\Gamma,\rho)\to C^*(\hat{G}/\Gamma,\rho)$ is contractive [36, Proposition 5.2]. Consequently, the induced actions of $\alpha$ and $\beta$ on $C^*(\hat{G}/\Gamma,\rho)$ are also strongly continuous.

Note that the subalgebra $B_0$ of $C_{0,1}(G/\Gamma,\rho)$ is a $C^*$-algebra, which can be identified with $C_0(G/K\Gamma)$. Since the natural homomorphism $C_{0,1}(G/\Gamma,\rho)\to C^*_{\mathrm{r}}(\hat{G}/\Gamma,\rho)$ is injective, so is the canonical homomorphism $C_{0,1}(G/\Gamma,\rho)\to C^*(\hat{G}/\Gamma,\rho)$. As injective $*$-homomorphisms between $C^*$-algebras are isometric, we conclude that the homomorphism of $B_0$ into $C^*(\hat{G}/\Gamma,\rho)$ is isometric. For any $f\in B_s$ one has $f^**f\in B_0$

and the supremum norm of $f^* * f$ is equal to the square of the supremum norm of $f$. It follows that the homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$ is isometric on $B_s$. In particular, the image of $B_s$ in $C^*(\hat{G}/\Gamma, \rho)$ is closed.

Since the action $\beta$ of $K$ on $C^*(\hat{G}/\Gamma, \rho)$ is strongly continuous, the spectral space $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s \rangle a \text{ for all } k \in K\}$ is the image of the continuous linear operator $C^*(\hat{G}/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$ sending $a$ to $\int_K \overline{\langle k, s \rangle} \beta_k(a) \, dk$. It follows that the image of $B_s = \{f \in C_{0,1}(G/\Gamma, \rho) | \beta_k(f) = \langle k, s \rangle f \text{ for all } k \in K\}$ in $C^*(\hat{G}/\Gamma, \rho)$ is dense in $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s \rangle a \text{ for all } k \in K\}$. Therefore the image of $B_s$ in $C^*(\hat{G}/\Gamma, \rho)$ is exactly $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s \rangle a \text{ for all } k \in K\}$. □

We refer the reader to [8, Chapter 2] for the basics of nuclear $C^*$-algebras.

**Proposition 2.5.** *The $C^*$-algebra $C^*(\hat{G}/\Gamma, \rho)$ is nuclear.*

*Proof.* By Lemma 2.4 the action $\beta$ of $K$ on $C^*(\hat{G}/\Gamma, \rho)$ is strongly continuous, and its fixed-point subalgebra is $B_0$, a commutative $C^*$-algebra, and hence is nuclear [8, Proposition 2.4.2]. For any $C^*$-algebra carrying a strongly continuous action of a compact group, the algebra is nuclear if and only if the fixed-point subalgebra is nuclear [14, Proposition 3.1]. Consequently, $C^*(\hat{G}/\Gamma, \rho)$ is nuclear. □

We shall need the following well-known fact a few times (see for example [8, Proposition 4.5.1]).

**Lemma 2.6.** *Let $H$ be a compact group, and let $\sigma_j$ be a strongly continuous action of $H$ on a $C^*$-algebra $A_j$ for $j = 1, 2$. Let $\varphi : A_1 \to A_2$ be an $H$-equivariant $*$-homomorphism. Then $\varphi$ is injective if and only if the restriction of $\varphi$ on the fixed-point subalgebra $A_1^H$ is injective. In particular, if $\varphi$ is surjective and $\varphi|_{A_1^H}$ is injective, then $\varphi$ is an isomorphism.*

**Proposition 2.7.** *The canonical $*$-homomorphism $C^*(\hat{G}/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is an isomorphism.*

*Proof.* We shall apply Lemma 2.6 to show that the canonical $*$-homomorphism $\varphi : C^*(\hat{G}/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is an isomorphism. By [22, Lemma 4.4] the action $\beta$ on $C_{0,1}(G/\Gamma, \rho)$ extends to an action of $K$ on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$, which we denote by $\beta'$. Clearly $\varphi$ is $K$-equivariant. By Lemma 2.4 $\beta$ is strongly continuous on $C^*(\hat{G}/\Gamma, \rho)$. Since $\varphi$ is contractive, it follows that $\beta'$ is strongly continuous on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$. By Lemma 2.4 the

fixed-point subalgebra $(C^*(\hat{G}/\Gamma,\rho))^K$ is $B_0$. Since the homomorphism $C_{0,1}(G/\Gamma,\rho)\to C^*_{\mathrm{r}}(\hat{G}/\Gamma,\rho)$ is injective, we see that the restriction of $\varphi$ on $(C^*(\hat{G}/\Gamma,\rho))^K$ is injective. Therefore the conditions of Lemma 2.6 are satisfied and we conclude that $\varphi$ is an isomorphism. □

We refer the reader to [15] for a comprehensive treatment of $C^*$-*algebraic bundles*, which are usually called *Fell bundles* now. Notice that for $f_s\in B_s$ and $g_t\in B_t$ the product $f_s*g_t$ is in $B_{s+t}$ and $f_s^*$ is in $B_{-s}$. Also $||f_s^**f_s||=||f_s||^2$. Therefore we have a Fell bundle $\mathcal{B}^\rho=\{B_s\}_{s\in\hat{K}}$ over $\hat{K}$ with operations given by (1) and (2). It is easy to see that $C_{0,1}(G/\Gamma,\rho)$ is exactly the $L^1$-algebra of $\mathcal{B}^\rho$ (cf. the proof of [21, Proposition 5.2]). Thus the $C^*$-algebra $C^*(\hat{G}/\Gamma,\rho)$ is also the enveloping $C^*$-algebra $C^*(\mathcal{B}^\rho)$ of the Fell bundle $\mathcal{B}^\rho$.

Next we discuss what happens if we let $\rho$ vary continuously. We refer the reader to [13, Chapter 10] for the basics of continuous fields of Banach spaces and $C^*$-algebras. On page 505 of [22] Landstad and Raeburn pointed out that it seems reasonable that we shall get a continuous field of $C^*$-algebras, but no proof was given there. This is indeed true, and we give a proof here. To be precise, fix $G$, $\Gamma$ and $K$, let $W$ be a locally compact Hausdorff space and for each $w\in W$ we assign a $\rho_w$ satisfying (S1) and (S2) such that the map $w\mapsto\rho_w(s)$ is continuous for each $s\in\hat{K}$. Notice that $\mathcal{B}^\rho$ as a Banach space bundle over $\hat{K}$ do not depend on $\rho$. For clarity we denote the product and $*$-operation in (1) and (2) by $f_s*_w g_t$ and $f_s^{*_w}$. For any $f_s\in B_s$ and $g_t\in B_t$, clearly the maps $w\mapsto f_s*_w g_t$ and $w\mapsto f_s^{*_w}$ are both continuous. This leads to the next lemma, which is a slight generalization of [5, Proposition 3.3, Theorem 3.5]. The proof of [5, Proposition 3.3, Theorem 3.5], which in turn follows the lines of [29], is easily seen to hold also in our case.

**Lemma 2.8.** *Let $H$ be a discrete group and $A_h$ be a vector space for each $h\in H$. Let $W$ be a locally compact Hausdorff space and for each $w\in W$ assign norms and algebra operations making $\mathcal{A}^w=\{A_h\}_{h\in H}$ into a Fell bundle in such a way that for any $f_s\in A_s$ and $g_t\in A_t$ the map $w\mapsto||f_s||_w\in\mathbb{R}$ is continuous (then we have a continuous field of Banach spaces $(A_s,||\cdot||_w)_{w\in W}$ over $W$ for each $s\in H$) and the sections $w\mapsto f_s*_w g_t\in B_{st}$ and $w\mapsto f_s^{*_w}\in B_{s^{-1}}$ are continuous in the above continuous fields of Banach spaces $(B_{st},||\cdot||_w)_{w\in W}$ and $(B_{s^{-1}},||\cdot||_w)_{w\in W}$ respectively . Then the map $w\mapsto||f||_w$ is upper semi-continuous for each $f\in\oplus_{s\in H}A_s$, where $||\cdot||_w$ is the norm on the enveloping $C^*$-algebra $C^*(\mathcal{A}^w)$ and extends the norm of $A_s$ as part of $\mathcal{A}^w$ for each $s\in H$. Moreover, if $H$ is amenable, then $\{C^*(\mathcal{A}^w)\}_{w\in W}$*

*is a continuous field of $C^*$-algebras with the field structure determined by the continuous sections $w \mapsto f$ for all $f \in \oplus_{s\in H} A_s$.*

Since every discrete abelian group is amenable [27, page 14], from Proposition 2.7 we get

**Proposition 2.9.** *Fix $G, \Gamma$ and $K$. Let $W$ by a locally compact Hausdorff space and for each $w \in W$ let $\rho_w$ satisfy (S1) and (S2) such that the map $w \mapsto \rho_w(s)$ is continuous for each $s \in \hat{K}$. Then $\{C^*(\hat{G}/\Gamma, \rho_w)\}_{w\in W}$ is a continuous field of $C^*$-algebras with the field structure determined by the continuous sections $w \mapsto f$ for all $f \in \oplus_{s\in \hat{K}} B_s$.*

## 3. Derivations

In this section we prove Proposition 3.3, to establish the relation between derivations coming from $\alpha$ and $\beta$.

Throughout the rest of this paper, we assume:

(S3) $G/\Gamma$ is compact.

(S4) $G$ is a Lie group.

The examples in Section 2 all satisfy these conditions.

We refer the reader to [17, Section 1.3] for the discussion about differentiable maps into Fréchet spaces. We just recall that a continuous map $\psi$ from a smooth manifold $M$ into a Fréchet space $A$ is *continuously differentiable* if for any chart $(U, \phi)$ of $G$, where $U$ is an open subset of some Euclidean space $\mathbb{R}^n$ and $\phi$ is a diffeomorphism from $U$ onto an open set of $M$, the derivative

$$D(\psi \circ \phi)(x, h) = \lim_{\mathbb{R}\ni\nu\to 0} \frac{\psi \circ \phi(x + \nu h) - \psi \circ \phi(x)}{\nu}$$

exists for all $(x, h) \in (U, \mathbb{R}^n)$ and is a jointly continuous map from $(U, \mathbb{R}^n)$ into $A$. In such case, $D(\psi \circ \phi)(x, h)$ is linear on $h$, and depends only on $\psi$ and the tangent vector $u := \phi_*(v_{x,h})$ of $M$ at $\phi(x)$, where $v_{x,h}$ denotes the tangent vector $h$ at $x$. Thus we may denote $D(\psi \circ \phi)(x, h)$ by $\partial_u \psi$. Then $\partial_u \psi$ is linear on $u$.

Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$ respectively. For a strongly continuous action $\sigma$ of $G$ on a Banach space $A$ as isometric automorphisms, we say that an element $a \in A$ is *once differentiable* with respect to $\sigma$ if the orbit map $\psi_a$ from $G$ into $A$ sending $x$ to $\sigma_x(a)$ is continuously differentiable. Then the set $A_1$ of once differentiable elements is a linear subspace of $A$. For any $a \in A$ and any compactly supported smooth $\mathbb{C}$-valued function $\varphi$ on $G$, it is easily checked that $\int_G \varphi(x)\sigma_x(a)\, dx$ is in $A_1$. As $a$ can be approximated by such elements, we see that $A_1$ is dense in $A$. Thinking of $\mathfrak{g}$ as the tangent space of

$G$ at the identity element, for each $X \in \mathfrak{g}$ we have the linear map $\sigma_X : A_1 \to A$ sending $a$ to $\partial_X \psi_a$. Fix a norm on $\mathfrak{g}$. We define a seminorm $L$ on $A_1$ by setting $L(a)$ to be the norm of the linear map $\mathfrak{g} \to A$ sending $X$ to $\sigma_X a$.

**Lemma 3.1.** *Let $\sigma$ be a strongly continuous action of $G$ on a Banach space $A$ as isometric automorphisms. For any $a \in A_1$, one has*

$$L(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|}.$$

*Proof.* The proof is similar to that of [31, Proposition 8.6]. Let $X \in \mathfrak{g}$ with $\|X\| = 1$. One has

$$\sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} \geq \lim_{\nu \to 0^+} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} = \|\sigma_X(a)\|.$$

For any $\nu > 0$, one also has

$$\begin{aligned} \|\sigma_{e^{\nu X}}(a) - a\| &= \| \int_0^\nu \sigma_{e^{zX}}(\sigma_X(a))\, dz\| \leq \int_0^\nu \|\sigma_{e^{zX}}(\sigma_X(a))\|\, dz \\ &= \int_0^\nu \|\sigma_X(a)\|\, dz = \nu \|\sigma_X(a)\|. \end{aligned}$$

Therefore

$$\sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} = \|\sigma_X(a)\|.$$

Thus

$$\begin{aligned} \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|} &= \sup_{X \in \mathfrak{g}, \|X\| = 1} \sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} \\ &= \sup_{X \in \mathfrak{g}, \|X\| = 1} \|\sigma_X(a)\| = L(a). \end{aligned}$$

□

**Lemma 3.2.** *Let $\sigma$ be a strongly continuous action of $G$ on a Banach space $A$ as isometric automorphisms. Then $A_1$ is a Banach space with the norm $\mathfrak{p}(a) := L(a) + \|a\|$. Suppose that $\sigma'$ is a strongly continuous isometric action of a topological group $H$ on $A$, commuting with $\sigma$. Then $H$ preserves $A_1$, and the restriction of $\sigma'$ on $A_1$ preserves the norm $\mathfrak{p}$ and is strongly continuous with respect to $\mathfrak{p}$.*

*Proof.* Let $\{a_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $A_1$ under the norm $\mathfrak{p}$. Then as $n$ goes to infinity, $a_n$ converges to some $a \in A$, and $\sigma_X(a_n)$ converge to some $b_X$ in $A$ uniformly on $X$ in bounded subsets of $\mathfrak{g}$.

Let $\varrho : [0,1] \to G$ be a continuously differentiable curve in $G$. Then $\lim_{z\to 0} \frac{\sigma_{\varrho_{\nu+z}}(a_n)-\sigma_{\varrho_\nu}(a_n)}{z} = \sigma_{\varrho_\nu}(\sigma_{\varrho'_\nu}(a_n))$ for all $\nu \in [0,1]$. Thus

$$\sigma_{\varrho_\nu}(a_n) - \sigma_{\varrho_0}(a_n) = \int_0^\nu \sigma_{\varrho_z}(\sigma_{\varrho'_z}(a_n))\, dz.$$

Letting $n \to \infty$ we get

$$\sigma_{\varrho_\nu}(a) - \sigma_{\varrho_0}(a) = \int_0^\nu \sigma_{\varrho_z}(b_{\varrho'_z})\, dz.$$

Therefore $\lim_{z\to 0} \frac{\sigma_{\varrho_z}(a)-\sigma_{\varrho_0}(a)}{z} = \sigma_{\varrho_0}(b_{\varrho'_0})$. It follows easily that $a \in A_1$ and $\sigma_X(a) = b_X$ for all $X \in \mathfrak{g}$. Consequently, $a_n$ converges to $a$ in $A_1$ under the norm $\mathfrak{g}$, and hence $A_1$ is a Banach space under the norm $\mathfrak{g}$.

Clearly $\sigma'$ preserves $A_1$ and the norm $\mathfrak{p}$. For any $a \in A_1$, the set of $\sigma_X(a)$ for $X$ in the unit ball of $\mathfrak{g}$ is compact. Then for any $h \in H$ and $\varepsilon > 0$, when $h' \in H$ is close enough to $h$, one has $\|\sigma'_h(a) - \sigma'_{h'}(a)\| < \varepsilon$ and $\|\sigma_X(\sigma'_h(a)) - \sigma_X(\sigma'_{h'}(a))\| = \|\sigma'_h(\sigma_X(a)) - \sigma'_{h'}(\sigma_X(a))\| < \varepsilon$ for all $X$ in the unit ball of $\mathfrak{g}$. Consequently, $\mathfrak{p}(\sigma'_h(a) - \sigma'_{h'}(a)) = L(\sigma'_h(a) - \sigma'_{h'}(a)) + \|\sigma'_h(a) - \sigma'_{h'}(a)\| < 2\varepsilon$. Therefore the restriction of $\sigma'$ on $A_1$ is strongly continuous with respect to $\mathfrak{p}$. □

By Lemma 2.4 the actions $\alpha$ and $\beta$ on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other and are strongly continuous. Denote by $C^1(\hat{G}/\Gamma, \rho)$ the space of once differentiable elements of $C^*(\hat{G}/\Gamma, \rho)$ with respect to the action $\alpha$. Recall the $B_s$ defined in (4).

**Proposition 3.3.** *Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$. For $Y \in \mathfrak{k}$ say*

$$\mathrm{Ad}_x(Y) = \sum_j F_{j,Y}(x) X_j,$$

*where* Ad *denotes the adjoint action of $G$ on $\mathfrak{g}$. Then $F_{j,Y} \in B_0$. Any $f \in C^1(\hat{G}/\Gamma, \rho)$ is once differentiable with respect to the action $\beta$ and*

$$\beta_Y(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f). \tag{5}$$

*Proof.* Clearly $F_{j,Y}$ is a smooth function on $G$. Since the subgroups $\Gamma$, $K$ and $\rho(\hat{K})$ commute with $K$, if $y$ is in any of these subgroups, then $\mathrm{Ad}_y(Y) = Y$, and hence

$$\sum_j F_{j,Y}(x) X_j = \mathrm{Ad}_x(Y) = \mathrm{Ad}_x(\mathrm{Ad}_y(Y)) = \mathrm{Ad}_{xy}(Y) = \sum_j F_{j,Y}(xy) X_j,$$

which means that $F_{j,Y}$ is invariant under the right translation of $y$. Thus $F_{j,Y} \in C(G/K\Gamma) = C_0(G/K\Gamma) = B_0$. For each $X \in \mathfrak{g}$ denote

by $X^{\#}$ ($X_{\#}$ resp.) the corresponding right (left resp.) translation invariant vector field on $G$. Then $Y_{\#} = \sum_j F_{j,Y} X_j^{\#}$.

Let $f \in C^1(\hat{G}/\Gamma, \rho) \cap B_s$ for some $s \in \hat{K}$. By Lemma 2.4 the norm on $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$ is exactly the supremum norm. Thus $f$ belongs to the space $C^1(G)$ of continuously differentiable functions on $G$. For any continuous vector field $Z$ on $G$ denote by $\partial_Z$ the corresponding derivation map $C^1(G) \to C(G)$. Then

$$\partial_{Y_{\#}}(f) = \sum_j F_{j,Y} \partial_{X_j^{\#}}(f) = -\sum_j F_{j,Y} \alpha_{X_j}(f).$$

Since $F_{j,Y}$ is invariant under the right translation of $\Gamma$ and $\rho(K)$, we have $F_{j,Y}(x) g_t(x) = F_{j,Y} * g_t(x)$ for any $g_t \in B_t$ and $x \in G$. By Lemma 2.4 the actions $\alpha$ and $\beta$ on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other. Thus $\alpha$ preserves $B_s$, and hence $\alpha_X(f) \in B_s$ for every $X \in \mathfrak{g}$. Therefore $\partial_{Y_{\#}}(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f)$.

Let $\varrho : [0,1] \to K$ be a continuously differentiable curve in $K$. Then

$$\lim_{z \to 0} \frac{f(x\varrho_{\nu+z}) - f(x\varrho_\nu)}{z} = (\partial_{(\varrho'_\nu)_{\#}}(f))(x\varrho_\nu) = (-\sum_j F_{j,\varrho'_\nu} * \alpha_{X_j}(f))(x\varrho_\nu)$$

for all $\nu \in [0,1]$ and $x \in G$, and hence we have the integral form

$$f(x\varrho_\nu) - f(x\varrho_0) = \int_0^\nu (-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))(x\varrho_z)\, dz \tag{6}$$

for all $\nu \in [0,1]$ and $x \in G$. The left hand side of (6) is the value of $\beta_{\varrho_\nu}(f) - \beta_{\varrho_0}(f)$ at $x$, while the right hand side of (6) is the value of $\int_0^\nu \beta_{\varrho_z}(-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))\, dz$ at $x$, where the integral is taken in $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$. Therefore

$$\beta_{\varrho_\nu}(f) - \beta_{\varrho_0}(f) = \int_0^\nu \beta_{\varrho_z}(-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))\, dz \tag{7}$$

for all $\nu \in [0,1]$.

Clearly (7) also holds for $f \in \oplus_{s \in \hat{K}} (C^1(\hat{G}/\Gamma, \rho) \cap B_s)$. By Lemma 3.2 $C^1(\hat{G}/\Gamma, \rho)$ is a Banach space with norm $\mathfrak{p}(\cdot) = L(\cdot) + \|\cdot\|$, $\beta$ preserves $C^1(\hat{G}/\Gamma, \rho)$ and $\mathfrak{p}$, and the restriction of $\beta$ on $C^1(\hat{G}/\Gamma, \rho)$ is strongly continuous on $C^1(\hat{G}/\Gamma, \rho)$ with respect to $\mathfrak{p}$. By Lemma 2.4 the spectral subspace of $C^*(\hat{G}/\Gamma, \rho)$ corresponding to $s \in \hat{K}$ for the action $\beta$ is equal to $B_s$. It follows that the spectral subspace of $C^1(\hat{G}/\Gamma, \rho)$ corresponding to $s \in \hat{K}$ for the restriction of $\beta$ on $C^1(\hat{G}/\Gamma, \rho)$ is exactly $C^1(\hat{G}/\Gamma, \rho) \cap B_s$. Then standard techniques tell us that $\oplus_{s \in \hat{K}} (C^1(\hat{G}/\Gamma, \rho) \cap B_s)$ is dense in $C^1(\hat{G}/\Gamma, \rho)$ with respect to $\mathfrak{p}$. Notice that both sides of

(7) define continuous maps from $C^1(\hat{G}/\Gamma, \rho)$ to $C^*(\hat{G}/\Gamma, \rho)$. Therefore (7) holds for all $f \in C^1(\hat{G}/\Gamma, \rho)$. Consequently,

$$\lim_{z\to 0} \frac{\beta_{\varrho_z}(f) - \beta_{\varrho_0}(f)}{z} = \beta_{\varrho_0}(-\sum_j F_{j,\varrho_0'} * \alpha_{X_j}(f))$$

for all $f \in C^1(\hat{G}/\Gamma, \rho)$. It follows easily that $f$ is once differentiable with respect to $\beta$ and $\beta_Y(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f)$ for all $f \in C^1(\hat{G}/\Gamma, \rho)$ and $Y \in \mathfrak{k}$. □

We shall need the following lemma (compare [34, Proposition 2.5]).

**Lemma 3.4.** *Let $\sigma$ be a strongly continuous action of $G$ on a Banach space $A$ as isometric automorphisms. Let $a \in A$. Then for any $\varepsilon > 0$, there is some $b \in A$ such that $b$ is smooth with respect to $\sigma$, $\|b\| \leq \|a\|$, $\|b - a\| \leq \varepsilon$, and $\sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_X(b)\|}{\|X\|} \leq \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{eX}(a)-a\|}{\|X\|}$. If $A$ has an isometric involution being invariant under $\sigma$, then when $a$ is self-adjoint, we can choose $b$ also to be self-adjoint.*

*Proof.* Endow $G$ with a left-invariant Haar measure. Let $U$ be a small open neighborhood of the identity element in $G$ with compact closure, which we shall determine later. Let $\varphi$ be a non-negative smooth function on $G$ with support contained in $U$ such that $\int_G \varphi(x)\, dx = 1$. Set $b = \int_G \varphi(x)\sigma_x(a)\, dx$. Then $b$ is smooth with respect to $\sigma$, and $\|b\| \leq \|a\|$. When $U$ is small enough, we have $\|a - b\| \leq \varepsilon/2$. For any $X \in \mathfrak{g}$, setting $\psi(x) = \mathrm{Ad}_{x^{-1}}(X)$, we have

$$\begin{aligned}
\|\sigma_{e^X}(b) - b\| &= \|\int_G \varphi(x)(\sigma_{e^X x}(a) - \sigma_x(a))\, dx\| \\
&= \|\int_G \varphi(x)\sigma_x(\sigma_{e^{\psi(x)}}(a) - a)\, dx\| \\
&\leq \int_G \varphi(x)\|\sigma_x(\sigma_{e^{\psi(x)}}(a) - a)\|\, dx \\
&\leq \sup_{x\in U} \|\sigma_{e^{\psi(x)}}(a) - a\| \\
&\leq \sup_{0\neq Y\in\mathfrak{g}} \frac{\|\sigma_{e^Y}(a) - a\|}{\|Y\|} \cdot \sup_{x\in U} \|\psi(x)\|.
\end{aligned}$$

Set $\delta = \varepsilon/(2+2\|a\|)$. When $U$ is small enough, we have $\|\mathrm{Ad}_{x^{-1}}(X)\| \leq (1+\delta)\|X\|$ for all $X \in \mathfrak{g}$ and $x \in U$. Then $\|\sigma_{e^X}(b) - b\| \leq (1+\delta)\|X\| \sup_{0\neq Y\in\mathfrak{g}} \frac{\|\sigma_{e^Y}(a)-a\|}{\|Y\|}$ for all $X \in \mathfrak{g}$. By Lemma 3.1 we get

$$\sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_X(b)\|}{\|X\|} = \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{e^X}(b) - b\|}{\|X\|} \leq (1+\delta) \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|}.$$

Now it is clear that $b' = b/(1+\delta)$ satisfies the requirement. Note that $b'$ is self-adjoint if $a$ is so. □

## 4. Nondeformed case

In this section we consider the nondeformed case, i.e., the case $\rho$ is the trivial homomorphism $\rho_0$ sending the whole $\hat{K}$ to the identity element of $G$. In Proposition 4.2 we identify $L_{\rho_0}$ on $C^1(\hat{G}/\Gamma, \rho_0)$ with the Lipschitz seminorm for certain metric on $G/\Gamma$.

Note that $C_{0,1}(G/\Gamma, \rho_0)$ is sub-$*$-algebra of $C_0(G/\Gamma) = C(G/\Gamma)$. By the universality of $C^*(\hat{G}/\Gamma, \rho_0)$ we have a natural $*$-homomorphism $\psi$ of $C^*(\hat{G}/\Gamma, \rho_0)$ into $C(G/\Gamma)$, extending the inclusion $C_{0,1}(G/\Gamma, \rho_0) \hookrightarrow C(G/\Gamma)$. The right translation of $K$ on $G$ induces a strongly continuous action $\beta''$ of $K$ on $C(G/\Gamma)$, and clearly $\psi$ intertwines $\beta$ and $\beta''$. An application of Lemmas 2.6 and 2.4 tells us that $\psi$ is injective. By definition $B_s$ is the spectral subspace of $C(G/\Gamma)$ corresponding to $s \in \hat{K}$. Thus $\oplus_{s\in\hat{K}} B_s$ is dense in $C(G/\Gamma)$. As $\oplus_{s\in\hat{K}} B_s$ is in the image of $\psi$, we see that $\psi$ is surjective and hence is an isomorphism. We shall identify $C^*(\hat{G}/\Gamma, \rho_0)$ and $C(G/\Gamma)$ via $\psi$.

The seminorm $L_{\rho_0}$ describes the size of derivatives of $f \in C^1(\hat{G}/\Gamma, \rho_0)$. If it corresponds to some metric on $G/\Gamma$, this metric should be kind of geodesic distance. In order for the geodesic distance to be defined, throughout the rest of this paper we assume:

(S5) $G/\Gamma$ is connected.

The examples in Section 2 all satisfy this condition.

Fix an inner product on $\mathfrak{g}$. Then we obtain a right translation invariant Riemannian metric on $G$ in the usual way. Denote by $d_G$ the geodesic distance on connected components of $G$. We extend $d_G$ to a semi-distance on $G$ via setting $d_G(x, y) = \infty$ if $x$ and $y$ lie in different connected components of $G$.

**Lemma 4.1.** *The function $d$ on $G/\Gamma \times G/\Gamma$ defined by $d(x\Gamma, y\Gamma) := \inf_{x'\in x\Gamma, y'\in y\Gamma} d_G(x', y')$ is equal to $\inf_{y'\in y\Gamma} d_G(x, y')$. It is a metric on $G/\Gamma$ and induces the quotient topology on $G/\Gamma$.*

*Proof.* Let $V$ be a connected component of $G$. Then $V\Gamma$ is clopen in $G$, and hence $V\Gamma/\Gamma$ is clopen in $G/\Gamma$ for the quotient topology. As $G/\Gamma$ is connected, we conclude that $V\Gamma/\Gamma = G/\Gamma$. Therefore $d$ is finite valued.

Since $d_G$ is right translation invariant, we have $\inf_{x'\in x\Gamma, y'\in y\Gamma} d_G(x', y') = \inf_{y'\in y\Gamma} d_G(x, y')$. It follows easily that $d$ is a metric on $G/\Gamma$.

Let $x \in G$. Let $W$ be a neighborhood of $x\Gamma$ in $G/\Gamma$ for the quotient topology. Then there exists $\varepsilon > 0$ such that if $d_G(x, y) < \varepsilon$, then

$y\Gamma \in W$. It follows that if $d(x\Gamma, y\Gamma) < \varepsilon$, then $y\Gamma \in W$. Therefore the topology induced by $d$ on $G/\Gamma$ is finer than the quotient topology. For any $\varepsilon' > 0$, set $U = \{y \in G | d_G(x, y) < \varepsilon'\}$. Then $U$ is an open neighborhood of $x$. Thus $U\Gamma/\Gamma$ is an open neighborhood of $x\Gamma$ for the quotient topology. For any $z\Gamma \in U\Gamma/\Gamma$, we can find $z' \in z\Gamma \cap U$ and hence $d(x\Gamma, z\Gamma) \leq d_G(x, z') < \varepsilon'$. Therefore the quotient topology on $G/\Gamma$ is finer than the topology induced by $d$. We conclude that $d$ induces the quotient topology. □

**Proposition 4.2.** *For any $f \in C^1(\hat{G}/\Gamma, \rho_0) \subseteq C^*(\hat{G}/\Gamma, \rho_0) = C(G/\Gamma)$, we have*

$$L_{\rho_0}(f) = \sup_{x\Gamma \neq y\Gamma} \frac{|f(x\Gamma) - f(y\Gamma)|}{d(x\Gamma, y\Gamma)}.$$

*Proof.* The right hand side of the above equation is equal to $\sup_{x \neq y} \frac{|f(x)-f(y)|}{d_G(x,y)}$. So it suffices to show

$$L_{\rho_0}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_G(x, y)}. \tag{8}$$

The proof is similar to that of [31, Proposition 8.6]. Let $\varrho : [0, 1] \to G$ be a continuously differentiable curve. Denote by $\ell(\varrho)$ the length of $\varrho$. Then $(f \circ \varrho)'(\nu) = (\alpha_{-\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f)(\varrho_\nu)$ for all $\nu \in [0, 1]$, and hence

$$\begin{aligned} |f(\varrho_1) - f(\varrho_0)| &= |\int_0^1 (f \circ \varrho)'(\nu)\, d\nu| \leq \int_0^1 |(f \circ \varrho)'(\nu)|\, d\nu \\ &= \int_0^1 |(\alpha_{-\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f)(\varrho_\nu)|\, d\nu \leq \int_0^1 \|\alpha_{\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f\|\, d\nu \\ &\leq L_{\rho_0}(f) \int_0^1 \|\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)\|\, d\nu = L_{\rho_0}(f)\ell(\varrho), \end{aligned}$$

where in the last equality we use the fact that the Riemannian metric on $G$ is right translation invariant. It follows easily that $|f(\varrho_1) - f(\varrho_0)| \leq L_{\rho_0}(f)\ell(\varrho)$ holds if $\varrho$ is only piecewise continuously differentiable. Considering all piecewise continuously differentiable curves connecting $x$ and $y$ we obtain $|f(x) - f(y)| \leq L_{\rho_0}(f) d_G(x, y)$ for all $x, y \in G$.

Denote by $e_G$ the identity element of $G$. For any $0 \neq X \in \mathfrak{g}$, we have

$$\begin{aligned}
\sup_{x\neq y} \frac{|f(x)-f(y)|}{d_G(x,y)} &\geq \sup_x \sup_{\nu\neq 0} \frac{|f(x)-f(e^{\nu X}x)|}{d_G(x,e^{\nu X}x)} \\
&= \sup_{\nu\neq 0} \sup_x \frac{|f(x)-f(e^{\nu X}x)|}{d_G(e_G,e^{\nu X})} \\
&\geq \sup_{\nu\neq 0} \sup_x \frac{|f(x)-f(e^{\nu X}x)|}{|\nu|\|X\|} \\
&= \sup_{\nu\neq 0} \frac{\|f-\alpha_{e^{-\nu X}}f\|}{|\nu|\|X\|} \\
&= \sup_{\nu\neq 0} \frac{\|\alpha_{e^{\nu X}}f-f\|}{|\nu|\|X\|} \geq \frac{\|\alpha_X(f)\|}{\|X\|}.
\end{aligned}$$

Therefore $\sup_{x\neq y} \frac{|f(x)-f(y)|}{d_G(x,y)} \geq L_{\rho_0}(f)$. This proves (8). □

## 5. Lip-norms and compact group actions

In this section we recall the definition of compact quantum metric spaces and prove Theorem 5.2, which enables one to show that certain seminorm defines a quantum metric, via the help of a compact group action.

Rieffel has set up the theory of compact quantum metric spaces in the general framework of order-unit spaces [31, Defintion 2.1]. We shall need it only for $C^*$-algebras. By a *$C^*$-algebraic compact quantum metric space* we mean a pair $(A, L)$ consisting of a unital $C^*$-algebra $A$ and a (possibly $+\infty$-valued) seminorm $L$ on $A$ satisfying the *reality condition*

$$L(a) = L(a^*) \tag{9}$$

for all $a \in A$, such that $L$ vanishes exactly on $\mathbb{C}$ and the metric $d_L$ on the state space $S(A)$ defined by

$$d_L(\psi,\phi) = \sup_{L(a)\leq 1} |\psi(a)-\phi(a)| \tag{10}$$

induces the weak$^*$-topology. The *radius* of $(A, L)$, denote by $r_A$, is defined to be the radius of $(S(A), d_L)$. We say that $L$ is a *Lip-norm*.

Let $A$ be a unital $C^*$-algebra and let $L$ be a (possibly $+\infty$-valued) seminorm on $A$ vanishing on $\mathbb{C}$. Then $L$ and $\|\cdot\|$ induce (semi)norms $\tilde{L}$ and $\|\cdot\|^\sim$ respectively on the quotient space $\tilde{A} = A/\mathbb{C}$.

Recall that a *character* of a compact group is the trace function of a finite-dimensional complex representation of the group [7, Section II.4].

**Lemma 5.1.** *Let $H$ be a compact group and $H_0$ be a closed normal subgroup of $H$ of finite index. Then for any linear combination of finitely many characters of $H$, its multiplication with the characteristic function of $H_0$ is also a linear combination of finitely many characters of $H$.*

*Proof.* The products and sums of characters of $H$ are still characters [7, Proposition II.4.10]. Thus it suffices to show that the characteristic function of $H_0$ on $H$ is a linear combination of finitely many characters of $H$.

Since $H/H_0$ is finite, every $\mathbb{C}$-valued class function on $H/H_0$, i.e., functions being constant on conjugate classes, is a linear combination of characters of $H/H_0$ [16, Proposition 2.30]. Thus the characteristic function of $\{e_{H/H_0}\}$ on $H/H_0$, where $e_{H/H_0}$ denotes the identity element of $H/H_0$, is a linear combination of characters of $H/H_0$. Then the characteristic function $H_0$ on $H$ is a linear combination of characters of $H$. □

Recall that a *length function* on a topological group $H$ is a continuous $\mathbb{R}_{\geq 0}$-valued function, $\ell$, on $H$ such that $\ell(h) = 0$ if and only if $h$ is equal to the identity element $e_H$ of $H$, that $\ell(h_1h_2) \leq \ell(h_1) + \ell(h_2)$ for all $h_1, h_2 \in H$, and that $\ell(h^{-1}) = \ell(h)$ for all $h \in H$.

Suppose that a compact group $H$ has a strongly continuous action $\sigma$ on a Banach space $A$ as isometric automorphisms. Endow $H$ with its normalized Haar measure. For any continuous $\mathbb{C}$-valued function $\varphi$ on $H$, define a linear map $\sigma_\varphi : A \to A$ by

$$\sigma_\varphi(a) = \int_H \varphi(h)\sigma_h(a)\, dh$$

for $a \in A$. Denote by $\hat{H}$ the set of isomorphism classes of irreducible representations of $H$. For each $s \in \hat{H}$, denote by $A_s$ the spectral subspace of $A$ corresponding to $s$. For a finite subset $J$ of $\hat{H}$, set $A_J = \sum_{s\in J} A_s$.

The main tool we use for the proof of Theorem 1.1 will be the following slight generalization of [23, Theorem 4.1].

**Theorem 5.2.** *Let $A$ be a unital $C^*$-algebra, let $L$ be a (possibly $+\infty$-valued) seminorm on $A$ satisfying the reality condition (9), and let $\sigma$ be a strongly continuous action of a compact group $H$ on $A$ by automorphisms. Assume that $L$ takes finite values on a dense subspace of $A$, and that $L$ vanishes on $\mathbb{C}$. Suppose that the following conditions are satisfied:*

(1) *there are some length function $\ell$ on a closed normal subgroup $H_0$ of $H$ of finite index and some constant $C > 0$ such that $L^\ell \leq C \cdot L$ on $A$, where $L^\ell$ is the (possibly $+\infty$-valued) seminorm on $A$ defined by*

$$L^\ell(a) = \sup\{\frac{\|\sigma_h(a) - a\|}{\ell(h)} | h \in H_0, h \neq e_H\}. \tag{11}$$

(2) *for any linear combination $\varphi$ of finitely many characters on $H$ we have $L \circ \sigma_\varphi \leq \|\varphi\|_1 \cdot L$ on $A$, where $\|\varphi\|_1$ denotes the $L^1$ norm of $\varphi$;*

(3) *for each $s \in \hat{H}$ not being the trivial representation $s_0$ of $H$, the set $\{a \in A_s | L(a) \leq 1, \|a\| \leq r\}$ is totally bounded for some $r > 0$, and the only element in $A_s$ vanishing under $L$ is $0$;*

(4) *there is a unital $C^*$-algebra $\mathcal{A}$ containing the fixed-point subalgebra $A^\sigma$, with a Lip-norm $L_\mathcal{A}$, such that $L_\mathcal{A}$ extends the restriction of $L$ to $A^\sigma$;*

(5) *for each $s \in \widehat{H/H_0} \subseteq \hat{H}$ not equal to $s_0$, there exists some constant $C_s > 0$ such that $\|\cdot\| \leq C_s L$ on $A_s$.*

*Then $(A, L)$ is a $C^*$-algebraic compact quantum metric space with $r_A \leq C \int_{H_0} \ell(h)\, dh + \sum_{s_0 \neq s \in \widehat{H/H_0}} C_s (\dim(s))^2 + r_\mathcal{A}$, where $H_0$ is endowed with its normalized Haar measure.*

We need some preparation for the proof of Theorem 5.2. The following lemma generalizes [23, Lemma 3.4].

**Lemma 5.3.** *Let $H$ be a compact group, and let $H_0$ be a closed normal subgroup of $H$ of finite index. Let $f$ be a continuous $\mathbb{C}$-valued function on $H$ with $f(e_H) = 0$. Then for any $\varepsilon > 0$ there is a nonnegative function $\varphi$ on $H$ with support contained in $H_0$ such that $\varphi$ is a linear combination of finitely many characters of $H$, $\|\varphi\|_1 = 1$, and $\|\varphi \cdot f\|_1 < \epsilon$.*

*Proof.* Denote by $\chi$ the characteristic function of $H_0$ on $H$. Set $g = f\chi + \varepsilon(1 - \chi)$. Then $g \in C(H)$ and $g(e_H) = 0$. By [23, Lemma 3.4] we can find a nonnegative function $\phi$ on $H$ such that $\phi$ is a linear combination of finitely many characters, $\|\phi\|_1 = 1$, and $\|\phi \cdot g\|_1 < \varepsilon/2$. Then $\varepsilon \int_{H \setminus H_0} \phi(h)\, dh \leq \|\phi \cdot g\|_1 < \varepsilon/2$, and hence

$$\|\chi\phi\|_1 = \|\phi\|_1 - \int_{H \setminus H_0} \phi(h)\, dh > 1 - 1/2 = 1/2.$$

Set $\varphi = \chi\phi / \|\chi\phi\|_1$. By Lemma 5.1 $\varphi$ is a linear combination of finitely many characters of $H$. One has

$$\|\varphi \cdot f\|_1 = \|\chi\phi f\|_1 / \|\chi\phi\|_1 = \|\chi\phi g\|_1 / \|\chi\phi\|_1 < (\varepsilon/2)/(1/2) = \varepsilon.$$

□

For a compact group $H$ and a finite subset $J$ of $\hat{H}$, set $\bar{J} = \{\bar{s}|s \in J\}$, where $\bar{s}$ denotes the contragradient representation. Replacing [23, Lemma 3.4] by Lemma 5.3 in the proof of [23, Lemma 4.4], we get:

**Lemma 5.4.** *Let $H$ be a compact group. For any $\varepsilon > 0$ there is a finite subset $J = \bar{J}$ in $\hat{H}$, containing the trivial representation $s_0$, depending only on $\ell$ and $\varepsilon/C$, such that for any strongly continuous isometric action $\sigma$ of $H$ on a complex Banach space $A$ with a (possibly $+\infty$-valued) seminorm $L$ on $A$ satisfying conditions (1) and (2) in Theorem 5.2, and any $a \in A$, there is some $a' \in A_J$ with*

$$\|a'\| \leq \|a\|, \quad L(a') \leq L(a), \quad \text{and } \|a - a'\| \leq \varepsilon L(a).$$

*If $A$ has an isometric involution being invariant under $\sigma$, then when $a$ is self-adjoint we can choose $a'$ also to be self-adjoint.*

We are ready to prove Theorem 5.2.

*Proof of Theorem 5.2.* Most part of the proof of [23, Theorem 4.1] carries over here. In fact, conditions (2)-(4) here are the same as the conditions (2)-(4) in [23, Theorem 4.1]. Since the proof of Lemma 4.5 in [23] does not involve condition (1) there, this lemma still holds in our current situation. Replacing [23, Lemma 4.4] by Lemma 5.4 in the proof of Lemma 4.6 of [23], we see that the latter also holds in our current situation. To finish the proof of Theorem 5.2, we only need to prove the following analogue of Lemma 4.7 of [23]:

**Lemma 5.5.** *We have*

$$\|\cdot\|^{\sim} \leq (C \int_{H_0} \ell(h)\, dh + \sum_{s_0 \neq s \in \widehat{H/H_0}} C_s (\dim(s))^2 + r_{\mathcal{A}}) L^{\sim}$$

*on $(\tilde{A})_{\mathrm{sa}}$, where $H_0$ is endowed with its normalized Haar measure.*

*Proof.* By Lemma 5.1 the characteristic function $\varphi$ of $H_0$ on $H$ is a linear combination of characters of $H$. Set $n = |H/H_0|$. Let $a \in A_{\mathrm{sa}}$. Then $\sigma_{n\varphi}(a)$ belongs to $A_{\mathrm{sa}}$ and is fixed by $\sigma|_{H_0}$. We have

$$\begin{aligned} \|a - \sigma_{n\varphi}(a)\| &= \|\int_{H_0} a\, dh - \int_{H_0} \sigma_h(a)\, dh\| \leq \int_{H_0} \|a - \sigma_h(a)\|\, dh \\ &\leq L^{\ell}(a) \int_{H_0} \ell(h)\, dh \leq C \cdot L(a) \int_{H_0} \ell(h)\, dh, \end{aligned}$$

where the last inequality comes from the condition (1). By the condition (2) we have

$$L(\sigma_{n\varphi}(a)) \leq \|n\varphi\|_1 \cdot L(a) = L(a).$$

Note that $A^{\sigma|_{H_0}} = \oplus_{s\in\widehat{H/H_0}} A_s$. Say, $\sigma_{n\varphi}(a) = \sum_{s\in\widehat{H/H_0}} a_s$ with $a_s \in A_s$. For each $s \in \widehat{H/H_0}$, denote by $\chi_s$ the corresponding character of $H/H_0$, thought of as a character of $H$. Then $a_s = \sigma_{\dim(s)\overline{\chi_s}}(\sigma_{n\varphi}(a))$ [23, Lemma 3.2]. Thus

$$L(a_s) = L(\sigma_{\dim(s)\overline{\chi_s}}(\sigma_{n\varphi}(a))) \leq \|\dim(s)\overline{\chi_s}\|_1 L(\sigma_{n\varphi}(a)) \leq (\dim(s))^2 L(a),$$

where the first inequality comes from the condition (2). Note that $a_{s_0} \in A_{\mathrm{sa}}$. By the condition (5) we have

$$\|a_s\| \leq C_s L(a_s) \leq C_s(\dim(s))^2 L(a)$$

for each $s \in \widehat{H/H_0}$ not equal to $s_0$. By the condition (4), we have

$$\|b\|^{\sim} \leq r_{\mathcal{A}} L^{\sim}(b)$$

for all $b \in (A_{s_0})_{\mathrm{sa}} = (A^{\sigma})_{\mathrm{sa}}$ [30, Proposition 1.6, Theorem 1.9] [23, Proposition 2.11]. Thus

$$\|a_{s_0}\|^{\sim} \leq r_{\mathcal{A}} L^{\sim}(a_{s_0}) = r_{\mathcal{A}} L(a_{s_0}) \leq r_{\mathcal{A}} L(a).$$

Therefore we have

$$\begin{aligned} \|a\|^{\sim} &\leq \|a - \sigma_{n\varphi}(a)\| + \|a_{s_0}\|^{\sim} + \sum_{s_0 \neq s \in \widehat{H/H_0}} \|a_s\| \\ &\leq C \cdot L(a) \int_{H_0} \ell(h)\, dh + r_{\mathcal{A}} L(a) + \sum_{s_0 \neq s \in \widehat{H/H_0}} C_s(\dim(s))^2 L(a) \end{aligned}$$

as desired. □

This finishes the proof of Theorem 5.2. □

## 6. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Denote by $K_0$ the connected component of $K$ containing the identity element $e_K$. Take an inner product on $\mathfrak{k}$ and use it to get a translation invariant Riemannian metric on $K$ in the usual way. For each $x \in K_0$ set $\ell(x)$ to be the geodesic distance form $e_K$ to $x$. Then $\ell$ is a length function on $K_0$.

In order to prove Theorem 1.1, we just need to verify the conditions in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$. Recall that we are given a norm on $\mathfrak{g}$, and

$$L_\rho(a) = \begin{cases} \sup_{0\neq X\in\mathfrak{g}} \frac{\|\alpha_X(a)\|}{\|X\|}, & \text{if } a \in C^1(\hat{G}/\Gamma, \rho); \\ \infty, & \text{otherwise,} \end{cases} \tag{12}$$

for $a \in C^*(\hat{G}/\Gamma, \rho)$.

By Lemma 2.4 the actions $\alpha$ and $\beta$ on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other. Thus $\beta$ preserves $C^1(\hat{G}/\Gamma, \rho)$ and $L_\rho$.

Choose the basis $X_1, \dots, X_{\dim(G)}$ of $\mathfrak{g}$ in Proposition 3.3 to be of norm 1. Denote by $C_1$ the supremum of $||F_{j,Y}||$ for all $1 \leq j \leq \dim(G)$ and $Y$ in the unit sphere of $\mathfrak{k}$ (with respect to the inner product on $\mathfrak{k}$ above) in Proposition 3.3.

**Lemma 6.1.** *We have $L^\ell \leq (dim(G)C_1) \cdot L_\rho$ on $C^*(\hat{G}/\Gamma, \rho)$.*

*Proof.* It suffices to show $L^\ell \leq (\dim(G)C_1) \cdot L_\rho$ on $C^1(\hat{G}/\Gamma, \rho)$. By Proposition 3.3 every $a \in C^1(\hat{G}/\Gamma, \rho)$ is once differentiable with respect to the action $\beta$. By [31, Proposition 8.6] we have $L^\ell(a) = \sup_{Y \in \mathfrak{k}, ||Y||=1} ||\beta_Y(a)||$. Then from (5) in Proposition 3.3 we get $L^\ell(a) \leq (\dim(G)C_1)L_\rho(a)$. □

**Lemma 6.2.** *For any linear combination $\varphi$ of finitely many characters of $K$ we have $L_\rho \circ \beta_\varphi \leq ||\varphi||_1 \cdot L_\rho$ on $C^*(\hat{G}/\Gamma, \rho)$.*

*Proof.* We have remarked above that $\beta$ preserves $L_\rho$. By Lemma 3.1 one has

$$L_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{e^X}(a) - a\|}{\|X\|} \tag{13}$$

for every $a \in C^1(\hat{G}/\Gamma, \rho)$. It follows that $L_\rho$ is lower semi-continuous on $C^1(\hat{G}/\Gamma, \rho)$ equipped with the relative topology from $C^1(\hat{G}/\Gamma, \rho) \subseteq C^*(\hat{G}/\Gamma, \rho)$. By Lemma 3.2 the action $\beta$ is also strongly continuous on $C^1(\hat{G}/\Gamma, \rho)$ with respect to the norm defined in Lemma 3.2. Then $\beta_\psi$ is also well-defined on $C^1(\hat{G}/\Gamma, \rho)$ for any continuous $\mathbb{C}$-valued function $\psi$ on $K$. By [23, Remark 4.2.(3)] we get Lemma 6.2. □

The conditions (1) and (2) in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ follow from Lemmas 6.1 and 6.2 respectively.

Fix an inner product on $\mathfrak{g}$, and denote by $L'_\rho$ the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ defined by (12) but using this inner product norm instead. Since $\mathfrak{g}$ is finite dimensional, any two norms on $\mathfrak{g}$ are equivalent. Therefore there exists some constant $C_2 > 0$ not depending on $\rho$ such that $L'_\rho \leq C_2 L_\rho$.

By Lemma 4.1 and Proposition 4.2 the restriction of $L'_{\rho_0}$ on $C^1(\hat{G}/\Gamma, \rho_0) \subseteq C(G/\Gamma)$ is the Lipschitz seminorm associated to some metric $d$ on $G/\Gamma$. The Arzela-Ascoli theorem [12, Theorem VI.3.8] tells us that the set $\{a \in C^*(\hat{G}/\Gamma, \rho_0) | L_{\rho_0}(a) \leq r_1, \|a\| \leq r_2\}$ is totally bounded for any $r_1, r_2 > 0$. Since for each $s \in \hat{K}$ neither the seminorm $L_\rho$ nor the

$C^*$-norm on $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$ depends on $\rho$, the condition (3) in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ follows.

From the criterion of Lip-norms in [30, Proposition 1.6, Theorem 1.9] (see also [23, Proposition 2.11]) one sees that the Lipschitz seminorm associated to the metric on any compact metric space is a Lip-norm on the $C^*$-algebra of continuous functions on this space. Since $L'_{\rho_0}$ on $C^*(\hat{G}/\Gamma, \rho_0) = C(G/\Gamma)$ is no less than the Lipschitz seminorm associated to the metric $d$ on $G/\Gamma$, from [30, Proposition 1.6, Theorem 1.9] one concludes that $L_{\rho_0}$ is also a Lip-norm on $C(G/\Gamma)$. Therefore we may take $(\mathcal{A}, L_{\mathcal{A}})$ in condition (4) of Theorem 5.2 to be $(C(G/\Gamma), L_{\rho_0})$ for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$.

Let $s \in \hat{K}$ not being the trivial representation of $K$, and let $a \in B_s$. Then $L'_{\rho_0}(a) \leq C_2 L_{\rho_0}(a) = C_2 L_\rho(a)$. Thus for any $\lambda$ in the range of $a$ on $G/\Gamma$ one has $\|a - \lambda 1_{C(G/\Gamma)}\|_{C(G/\Gamma)} \leq C_2 C_3 L_\rho(a)$, where $C_3$ denotes the diameter of $G/\Gamma$ under the metric $d$. We have

$$\begin{aligned} \|a\|_{C^*(\hat{G}/\Gamma,\rho)} &= \|a\|_{C(G/\Gamma)} = \| \int_K \overline{\langle k, s\rangle} \beta_k(a - \lambda 1_{C(G/\Gamma)})\, dk\|_{C(G/\Gamma)} \\ &\leq \|a - \lambda 1_{C(G/\Gamma)}\|_{C(G/\Gamma)} \leq C_2 C_3 L_\rho(a). \end{aligned}$$

This establishes the condition (5) of Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$.

We have shown that the conditions in Theorem 5.2 hold for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$. Thus Theorem 1.1 follows from Theorem 5.2.

## 7. Quantum Gromov-Hausdorff distance

In this section we prove Theorem 1.2.

We recall first the definition of the distance $\mathrm{dist}_{\mathrm{nu}}$ from [19, Section 5]. To simplify the notation, for fixed unital $C^*$-algebras $A_1$ and $A_2$, when we take infimum over unital $C^*$-algebras $B$ containing both $A_1$ and $A_2$, we mean to take infimum over all unital isometric $*$-homomorphisms of $A_1$ and $A_2$ into some unital $C^*$-algebra $B$. Denote by $\mathrm{dist}_{\mathrm{H}}^B$ the Hausdorff distance between subsets of $B$. For a $C^*$-algebraic compact quantum metric spaces $(A, L_A)$, set

$$\mathcal{E}(A) = \{a \in A_{\mathrm{sa}} | L_A(a) \leq 1\}.$$

For any $C^*$-algebraic compact quantum metric spaces $(A_1, L_{A_1})$ and $(A_2, L_{A_2})$, the distance $\mathrm{dist}_{\mathrm{nu}}(A_1, A_2)$ is defined as

$$\mathrm{dist}_{\mathrm{nu}}(A_1, A_2) = \inf \mathrm{dist}_{\mathrm{H}}^B(\mathcal{E}(A_1), \mathcal{E}(A_2)),$$

where the infimum is taken over all unital $C^*$-algebras $B$ containing $A_1$ and $A_2$.

Throughout the rest of this section, we fix $G$, $\Gamma$, $K$ such that there exits $\rho$ satisfying the conditions (S1)-(S5). We also fix a norm on $\mathfrak{g}$. Denote by $\Omega$ the set of all $\rho$ satisfying the conditions (S1) and (S2), equipped with the weakest topology making the maps $\Omega \to G$ sending $\rho$ to $\rho(s)$ to be continuous for each $s \in \hat{K}$.

Every closed subgroup of a Lie group is also a Lie group [37, Theorem 3.42]. Thus $K$ is a compact abelian Lie group. Then $K$ is the product of a torus and a finite abelian group [7, Corollary 3.7]. Therefore $\hat{K}$ is finitely generated. Let $s_1, \dots, s_n$ be a finite subset of $\hat{K}$ generating $\hat{K}$. Then the map $\varphi : \Omega \to \prod_{j=1}^n G$ sending $\rho$ to $(\rho(s_1), \dots, \rho(s_n))$ is injective, and its image is closed. Furthermore, it is easily checked that the topology on $\Omega$ is exactly the pullback of the relative topology of $\varphi(\Omega)$ in $\prod_{j=1}^n G$. Since $G$ is a Lie group, it is locally compact metrizable. Thus $\prod_{j=1}^n G$ and $\Omega$ are also locally compact metrizable.

For clarity and convenience, we shall denote the actions $\alpha$ and $\beta$ on $C^*(\hat{G}/\Gamma, \rho)$ by $\alpha_\rho$ and $\beta_\rho$ respectively, and denote the $C^*$-norm on $C^*(\hat{G}/\Gamma, \rho)$ by $\| \cdot \|_\rho$. Consider the (possibly $+\infty$-valued) auxiliary seminorm $L''_\rho$ on $C^*(\hat{G}/\Gamma, \rho)$ defined by

$$L''_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{\rho, e^X}(a) - a\|_\rho}{\|X\|}.$$

**Lemma 7.1.** *Let $W$ be a locally compact Hausdorff space with a continuous map $W \to \Omega$ sending $w$ to $\rho_w$. Let $f$ be a continuous section of the continuous field of $C^*$-algebras over $W$ in Proposition 2.9. Then the function $w \mapsto L''_{\rho_w}(f_w)$ is lower semi-continuous on $W$.*

*Proof.* Let $w' \in W$. To show that the above function is lower semi-continuous at $w'$, we consider the case $L''_{\rho_{w'}}(f_{w'}) < \infty$. The case $L''_{\rho_{w'}}(f_{w'}) = \infty$ can be dealt with similarly. Let $\varepsilon > 0$. Take $0 \neq X \in \mathfrak{g}$ such that

$$L''_{\rho_{w'}}(f_{w'})\|X\| < \|\alpha_{\rho_{w'}, e^X}(f_{w'}) - f_{w'}\|_{\rho_{w'}} + \varepsilon\|X\|.$$

It is easily checked that $w \mapsto \alpha_{\rho_w, e^X}(f_w)$ is also a continuous section of the continuous field. Then when $w$ is close enough to $w'$, we have

$$\|\alpha_{\rho_{w'}, e^X}(f_{w'}) - f_{w'}\|_{\rho_{w'}} < \|\alpha_{\rho_w, e^X}(f_w) - f_w\|_{\rho_w} + \varepsilon\|X\|$$

and hence

$$L''_{\rho_{w'}}(f_{w'})\|X\| < \|\alpha_{\rho_w, e^X}(f_w) - f_w\|_{\rho_w} + 2\varepsilon\|X\| \leq (L''_{\rho_w}(f_w) + 2\varepsilon)\|X\|.$$

Therefore $L''_{\rho_{w'}}(f_{w'}) \leq L''_{\rho_w}(f_w) + 2\varepsilon$. □

Note that although the $*$-algebra structure of $C_{0,1}(G/\Gamma,\rho)$ ($C_{\mathrm{b},1}(G,\rho)$ resp.) depends on $\rho$, the Banach space structure, the left translation action of $G$ and the right translation action of $K$ on $C_{0,1}(G/\Gamma,\rho)$ ($C_{\mathrm{b},1}(G,\rho)$ resp.) do not depend on $\rho$. Thus we may denote by $C_{0,1}(G/\Gamma)$, $\alpha$ and $\beta$ this Banach space and these actions respectively. Also denote by $C^1_{0,1}(G/\Gamma)$ the set of once differentiable elements of $C_{0,1}(G/\Gamma)$ with respect to $\alpha$.

**Lemma 7.2.** *For any $a$ in $\oplus_{s\in\hat{K}}(B_s\cap C^1_{0,1}(G/\Gamma))$, the function $\rho\mapsto L_\rho(a)$ is continuous on $\Omega$.*

*Proof.* Say, $a=\sum_{s\in F}a_s$ for some finite subset $F$ of $\hat{K}$ and $a_s\in B_s\cap C^1_{0,1}(G/\Gamma)$ for each $s\in F$. Then $L_\rho(a)=\sup_{X\in\mathfrak{g},\|X\|=1}\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ for each $\rho\in\Omega$. Since $\alpha$ commutes with $\beta$, we have $\alpha_X(a_s)\in B_s$. By Proposition 2.9 the function $\rho\mapsto\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on $\Omega$ for each $X\in\mathfrak{g}$. Since $\mathfrak{g}$ is a finite-dimensional vector space and $\alpha_X(a_s)$ depends on $X$ linearly, it follows easily that the function $(X,\rho)\mapsto\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on $\mathfrak{g}\times\Omega$. As the unit sphere of $\mathfrak{g}$ is compact, one concludes that the function $\rho\mapsto\sup_{X\in\mathfrak{g},\|X\|=1}\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on $\Omega$. □

Fix $\rho'\in\Omega$. Let $Z$ be a compact neighborhood of $\rho'$ in $\Omega$.

Note that the linear span of $\rho\mapsto f(\rho)a\in C^*(\hat{G}/\Gamma,\rho)$ for $a$ in some $B_s$ and $f\in C(Z)$ is dense in the $C^*$-algebra of continuous sections of the continuous field over $Z$ in Proposition 2.9. Since $Z$ is a compact metrizable space, $C(Z)$ is separable. As $G$ is a Lie group, it is separable. Then $G/\Gamma$ is separable, and hence is a compact metrizable space. Thus $C(G/\Gamma)$ is separable, and hence $B_s$ is separable for each $s\in\hat{K}$. On the other hand, since $\hat{K}$ is finitely generated, $\hat{K}$ is countable. Therefore the $C^*$-algebra of continuous sections of the continuous field over $Z$ in Proposition 2.9 is separable.

By Proposition 2.5 each $C^*(\hat{G}/\Gamma,\rho)$ is nuclear. Every separable continuous field of unital nuclear $C^*$-algebras over a compact metric space can be subtrivialized [6, Theorem 3.2]. Thus we can find a unital $C^*$-algebra $B$ and unital embeddings $C^*(\hat{G}/\Gamma,\rho)\to B$ for all $\rho\in Z$ such that, via identifying each $C^*(\hat{G}/\Gamma,\rho)$ with its image in $B$, the continuous sections of the continuous field over $Z$ in Proposition 2.9 are exactly the continuous maps $Z\to B$ whose images at each $\rho$ are in $C^*(\hat{G}/\Gamma,\rho)$.

For any $C^*$-algebraic compact quantum metric space $(A,L_A)$ and any constant $R$ no less than the radius of $(A,L_A)$, the set $D_R(A):=\{a\in A_{\mathrm{sa}}|L_A(a)\leq 1,\|a\|\leq R\}$ is totally bounded and every $a\in\mathcal{E}(A)$ can be written as $x+\lambda$ for some $x\in D_R(A)$ and $\lambda\in\mathbb{R}$ [30, Proposition

1.6, Theorem 1.9]. In Section 6 we have seen that the conditions in Theorem 5.2 hold for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ with some $C, C_s$ and $(\mathcal{A}, L_\mathcal{A})$ not depending on $\rho$. Thus, by Theorem 5.2 there is some constant $R$ such that the radius of $(C^*(\hat{G}/\Gamma, \rho_\rho), L_\rho)$ is no bigger than $R$ for all $\rho \in \Omega$. For any $\varepsilon > 0$, by Lemmas 5.4 and 2.4 there is a finite subset $F \subseteq \hat{K}$ satisfying that for any $\rho \in \Omega$ and any $x \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ there is some $y \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho)) \cap \sum_{s\in F} B_s$ with $\|y\|_\rho \leq \|x\|_\rho$ and $\|x - y\|_\rho < \varepsilon$.

**Lemma 7.3.** *Let $\varepsilon > 0$. Then there is a neighborhood $U$ of $\rho'$ in $Z$ such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ with $\|a - b\|_B < \varepsilon$.*

*Proof.* According to the discussion above we can find a finite subset $Y$ of $\mathcal{E}(C^*(\hat{G}/\Gamma, \rho')) \cap \sum_{s\in F} B_s$ such that for every $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ there are some $z \in Y$ and $\lambda \in \mathbb{R}$ with $\|a - (z + \lambda)\|_{\rho'} < \varepsilon$. For each $y \in Y$, write $y$ as $\sum_{s\in F} y_s$ with $y_s \in B_s$. Since $L_{\rho'}(y) < \infty$, $y$ is once differentiable with respect to $\alpha_{\rho'}$. It is easy to see that each $y_s$ is once differentiable with respect to $\alpha_{\rho'}$. Thus, by Lemma 7.2 the function $\rho \mapsto L_\rho(y)$ is continuous on $\Omega$. Then we can find a constant $\delta > 0$ and a neighborhood $U$ of $\rho'$ in $Z$ such that $\delta\|y_\rho\|_\rho < \varepsilon$, $\|y_{\rho'} - y_\rho\|_B < \varepsilon$, and $L_\rho(y_\rho) < 1 + \delta$ for all $y \in Y$ and $\rho \in U$, where $y_\rho$ denotes $y$ as an element in $C^*(\hat{G}/\Gamma, \rho)$. Fix $\rho \in U$. Set $b = z_\rho/(1 + \delta)$. Then $L_\rho(b + \lambda) = L_\rho(b) < 1$, and

$$\begin{aligned} \|a - (b + \lambda)\|_B &\leq \|a - (z_{\rho'} + \lambda)\|_{\rho'} + \|z_{\rho'} - z_\rho\|_B + \|z_\rho - b\|_\rho \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

□

**Lemma 7.4.** *Let $\varepsilon > 0$. Then there is a neighborhood $U$ of $\rho'$ in $Z$ such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ with $\|a - b\|_B < \varepsilon$.*

*Proof.* According to the discussion before Lemma 7.3, it suffices to show that there is a neighborhood $U$ of $\rho'$ in $Z$ such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho)) \cap \oplus_{s\in F} B_s$ satisfying $\|a\|_\rho \leq R$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ with $\|a - b\|_B < \varepsilon$. Suppose that this fails. Then we can find a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $Z$ converging to $\rho'$ and an $a_n \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho_n)) \cap \oplus_{s\in F} B_s$ satisfying $\|a_n\|_{\rho_n} \leq R$ for each $n \in \mathbb{N}$ such that $\|a_n - b\|_B \geq \varepsilon$ for all $n \in \mathbb{N}$ and $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$. Write $a_n$ as $\sum_{s\in F} a_{n,s}$ with $a_{n,s} \in B_s$. Then $a_{n,s} = \int_K \overline{\langle k, s\rangle}\beta_{\rho_n,k}(a_n)\, dk$. Thus $\|a_{n,s}\|_{\rho_n} \leq \|a_n\|_{\rho_n} \leq R$ and $L_{\rho_n}(a_{n,s}) \leq L_{\rho_n}(a_n) \leq 1$ by Lemma 6.2. Since the restriction of $L_\rho$ on $B_s$ does not depend on $\rho$, and the set $\{a \in$

$B_s | L_\rho(a) \leq 1, \|a\| \leq R\}$ is totally bounded, passing to a subsequence if necessary, we may assume that $a_{n,s}$ converges to some $a_s$ in $B_s$ when $n \to \infty$ for each $s \in F$. Set $a = \sum_{s \in F} a_s$. Then $(a_n)_{\rho_n}$ converges to $a_{\rho'}$ in $B$ as $n \to \infty$, where $(a_n)_{\rho_n}$ and $a_{\rho'}$ denote $a_n$ and $a$ as elements in $C^*(\hat{G}/\Gamma, \rho_n)$ and $C^*(\hat{G}/\Gamma, \rho')$ respectively. In particular, $a$ is self-adjoint and $\|a\|_{\rho'} \leq \lim_{n\to\infty} \|a_n\|_{\rho_n} \leq R$.

By Lemma 3.1 we have $L''_{\rho_n}(a_n) = L_{\rho_n}(a_n) \leq 1$ for all $n \in \mathbb{N}$. On the one-point compactification $W = \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$, consider the continuous map $W \to \Omega$ sending $n \in \mathbb{N}$ to $\rho_n$ and $\infty$ to $\rho'$. Then the section $f$ defined as $f_n = a_n \in C^*(\hat{G}/\Gamma, \rho_n)$ for $n \in \mathbb{N}$ and $f_\infty = a \in C^*(\hat{G}/\Gamma, \rho')$ is a continuous section of the continuous field on $W$ in Proposition 2.9. Thus, by Lemma 7.1 we have $L''_{\rho'}(a) \leq \liminf_{n\to\infty} L''_{\rho_n}(a_n) \leq 1$. By Lemma 3.4 we can find some self-adjoint $b \in C^1(\hat{G}/\Gamma, \rho')$ with $\|b\|_{\rho'} \leq \|a\|_{\rho'} \leq R$, $\|b - a\|_{\rho'} \leq \varepsilon/2$, and $L_{\rho'}(b) \leq L''_{\rho'}(a) \leq 1$. Then $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$, and

$$\|b - a_n\|_B \to \|b - a\|_{\rho'} \leq \varepsilon/2$$

as $n \to \infty$. Therefore, when $n$ is large enough, we have $\|b - a_n\|_B < \varepsilon$, contradicting our assumption. This finishes the proof of the lemma. □

From Lemmas 7.3 and 7.4 we conclude that Theorem 1.2 holds.

## Appendix A. Comparison of $\mathrm{dist}_{\mathrm{nu}}$ and prox

In this appendix we compare the distance $\mathrm{dist}_{\mathrm{nu}}$ and the proximity Rieffel introduced in [35].

A (possibly $+\infty$-valued) seminorm $L$ on a unital (possibly incomplete) $C^*$-norm algebra $A$ is called a *$C^*$-metric* [35, Definition 4.1] if

(1) $L$ is lower semi-continuous, satisfies the reality condition (9), and is *strongly-Leibniz* in the sense that $L(ab) \leq L(a)\|b\| + \|a\|L(b)$ for all $a, b \in A$, $L(1_A) = 0$, and $L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$ for all $a$ being invertible in $A$,
(2) $L$ extended to the completion $\bar{A}$ of $A$ by $L(a) = \infty$ for $a \in \bar{A} \setminus A$ is a Lip-norm on $\bar{A}$,
(3) the algebra $\{a \in A | L(a) < \infty\}$ is spectrally stable in $\bar{A}$.

In such case, the pair $(A, L)$ is called a *compact $C^*$-metric space*.

The seminorm $L_\rho$ in Theorem 1.1 may fail to be a $C^*$-metric since it may fail to be lower semi-continuous. However, it is lower semi-continuous on $C^1(\hat{G}/\Gamma, \rho)$ by Lemma 3.1. Thus its restriction on the algebra of smooth elements in $C^*(\hat{G}/\Gamma, \rho)$ with respect to $\alpha$ is a $C^*$-metric. By [35, Proposition 3.2] its closure $\bar{L}_\rho$ is a $C^*$-metric on

$C^*(\hat{G}/\Gamma, \rho)$. Lemma 3.4 tells us that

$$\bar{L}_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{e^X}(a) - a\|}{\|X\|}$$

for all $a \in C^*(\hat{G}/\Gamma, \rho)$.

In [35, Definition 5.6, Section 14] Rieffel introduced the notions of *proximity* $\mathrm{prox}(A, B)$ and *complete proximity* $\mathrm{prox}_s(A, B)$ between two compact $C^*$-metric spaces $(A, L_A)$ and $(B, L_B)$. In general, one has $\mathrm{prox}_s(A, B) \geq \mathrm{prox}(A, B)$. For each $q \in \mathbb{N}$, denote by $\mathrm{UCP}_q(A)$ the set of unital completely positive linear maps from the completion $\bar{A}$ of $A$ to $M_q(\mathbb{C})$. Define $\mathrm{prox}^q(A, B)$ as the infimum of the Hausdorff distance of $\mathrm{UCP}_q(A)$ and $\mathrm{UCP}_q(B)$ in $\mathrm{UCP}_q(A \oplus B)$ under the metric $d_L^q$, for $L$ running through $C^*$-metrics $L$ on $A \oplus B$ whose quotients on $A$ and $B$ agree with $L_A$ and $L_B$ on $A_{\mathrm{sa}}$ and $B_{\mathrm{sa}}$ respectively. Here the metric $d_L^q$ is defined as

$$d_L^q(\varphi, \psi) = \sup_{L(a,b) \leq 1} \|\varphi(a, b) - \psi(a, b)\|.$$

Then $\mathrm{prox}_s(A, B)$ is defined as $\sup_q \mathrm{prox}^q(A, B)$.

Note that the definition of $\mathrm{dist}_{\mathrm{nu}}$ extends to compact $C^*$-metric spaces $(A, L_A)$ and $(B, L_B)$ directly.

**Theorem A.1.** *For any compact $C^*$-metric spaces $(A, L_A)$ and $(B, L_B)$, one has*

$$\mathrm{dist}_{\mathrm{nu}}(A, B) \geq \mathrm{prox}_s(A, B).$$

*Proof.* The proof is similar to those of [24, Proposition 4.7] and [19, Theorem 3.7]. Let $\mathcal{A}$ be a unital $C^*$-algebra containing $\bar{A}$ and $\bar{B}$. Set $c = \mathrm{dist}_{\mathrm{H}}^{\mathcal{A}}(\mathcal{E}(A), \mathcal{E}(B))$. Let $\varepsilon > 0$. Define a seminorm $L$ on $A \oplus B$ by

$$L(a, b) = \max(L_A(a), L_B(b), \frac{\|a - b\|}{c + \varepsilon}).$$

It was pointed in the proof of [24, Proposition 4.7] that $L$ extended to $\overline{A \oplus B} = \bar{A} \oplus \bar{B}$ as in the condition (2) of the definition of $C^*$-metrics above is a Lip-norm, and that the quotients of $L$ on $A$ and $B$ agree with $L_A$ and $L_B$ on $A_{\mathrm{sa}}$ and $B_{\mathrm{sa}}$ respectively. It is readily checked that $L$ satisfies the conditions (1) and (3) in the definition of $C^*$-metrics. Thus $L$ is a $C^*$-metric on $A \oplus B$. For any $q \in \mathbb{N}$ and $\varphi \in \mathrm{UCP}_q(A)$, by Arveson's extension theorem [8, Theorem 1.6.1] extend $\varphi$ to a $\phi$ in $\mathrm{UCP}_q(\mathcal{A})$. Set $\psi$ to be the restriction of $\phi$ on $\bar{B}$. For any $(a, b) \in \mathcal{E}(A \oplus B)$ one has

$$\|\varphi(a, b) - \psi(a, b)\| = \|\varphi(a) - \psi(b)\| = \|\phi(a - b)\| \leq \|a - b\| \leq c + \varepsilon.$$

Thus $d_L^q(\varphi,\psi) \leq c+\varepsilon$. Similarly, for any $\psi' \in \mathrm{UCP}_q(B)$, we can find some $\varphi' \in \mathrm{UCP}_q(A)$ with $d_L^q(\varphi',\psi') \leq c+\varepsilon$. Therefore $\mathrm{prox}^q(A,B) \leq c+\varepsilon$. It follows that $\mathrm{prox}^q(A,B) \leq \mathrm{dist}_{\mathrm{nu}}(A,B)$, and hence $\mathrm{prox}_s(A,B) \leq \mathrm{dist}_{\mathrm{nu}}(A,B)$ as desired. □

It was pointed out in Section 5 of [19] that one has continuity of quantum tori and $\theta$-deformation, convergence of matrix algebras to integral coadjoint orbits of compact connected semisimple Lie groups, and approximation of quantum tori by finite quantum tori with respect to $\mathrm{dist}_{\mathrm{nu}}$. It follows from Theorem A.1 that we also have such continuity, convergence and approximation with respect to $\mathrm{prox}_s$ and prox. In particular, this yields a new proof for [35, Theorem 14.1].

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