METRIC ASPECTS OF NONCOMMUTATIVE HOMOGENEOUS SPACES

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Dedicated to Marc A. Rieffel
in honor of his seventieth birthday

Abstract. For a closed cocompact subgroup Γ of a locally compact group G, given a compact abelian subgroup K of G and a homomorphism $\rho : \hat{K} \to G$ satisfying certain conditions, Landstad and Raeburn constructed equivariant noncommutative deformations $C^*(\hat{G}/\Gamma, \rho)$ of the homogeneous space G/Γ, generalizing Rieffel's construction of quantum Heisenberg manifolds. We show that when G is a Lie group and G/Γ is connected, given any norm on the Lie algebra of G, the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ induced by the derivation map of the canonical G-action defines a compact quantum metric. Furthermore, it is shown that this compact quantum metric space depends on ρ continuously, with respect to quantum Gromov-Hausdorff distances.

1. Introduction

In recent years, the quantum Heisenberg manifolds have received quite some attention. These interesting C^*-algebras were constructed by Rieffel [28] as deformation quantizations of the Heisenberg manifolds, and carry natural actions of the Heisenberg group. The classification of these C^*-algebras up to isomorphism (in most cases) and Morita equivalence (in all cases) has been achieved by Abadie and her collaborators [1, 2, 3, 4]. These C^*-algebras also appear in the work of Connes and Dubois-Violette on noncommutative 3-spheres [10, 11].

Aiming partly at giving a mathematical foundation for various approximations in the string theory, such as the fuzzy spheres, namely the matrix algebras $M_n(\mathbb{C})$, converging to the 2-sphere S^2, Rieffel developed a theory of compact quantum metric spaces and quantum

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Gromov-Hausdorff distance between them [31, 32, 33]. As the information of the metric on a compact metric space X is encoded in the Lipschitz seminorm on the algebra of continuous functions on X, a quantum metric on (the compact quantum space represented by) a unital C^*-algebra A is a (possibly $+\infty$-valued) seminorm on A satisfying suitable conditions (see Section 5 below for detail).

One important class of examples of compact quantum metric spaces comes from ergodic actions of a compact group G on a unital C^*-algebra A, which should be thought of as the translation action of G on a noncommutative homogeneous space of G. Given any length function on G, such an ergodic action induces a quantum metric on A [30] (see [25] for a generalization to ergodic actions of co-amenable compact quantum groups). This class of examples includes the (fuzzy) spheres above and the noncommutative tori. When G is a compact connected Lie group and the length function comes from the geodesic distance associated to some bi-invariant Riemannian metric on G, this seminorm can also be defined in terms of the derivation map on the space of once differentiable elements of A with respect to the G-action [31, Proposition 8.6]. Explicitly, denote by $\sigma_X(b)$ the derivation of a once differentiable element b of A with respect to an element X of the Lie algebra $\mathfrak{g}$ of G (see Section 3 below for detail). Then the seminorm $L(b)$ is defined as the norm of the linear map $\mathfrak{g} \to A$ sending X to $\sigma_X(b)$ when b is once differentiable, or ∞ otherwise.

It is natural to ask what conditions are needed to guarantee that L defined above gives rise to a quantum metric when G is not compact. Rieffel raised the question about the quantum Heisenberg manifolds in [33]. In [38] Weaver studied some sub-Riemannian metric on the quantum Heisenberg manifolds, which does not quite fit into the above framework. In [9] Chakraborty showed that certain seminorm associated to some ℓ^1-norm does define a quantum metric on the quantum Heisenberg manifolds. Since the ℓ^1-norm is bigger than the C^*-norm, this seminorm is bigger than the seminorm L defined above. Thus the result in [9] is weaker than what Riffel's question asks for.

Our first main result in this article is an affirmative answer to Rieffel's question. In fact, we shall deal more generally with Landstad and Raeburn's noncommutative homogeneous spaces. In [22] Landstad and Raeburn generalized Rieffel's construction to obtain equivariant deformations of compact homogeneous spaces G/Γ, starting from a locally compact group G, a closed cocompact subgroup Γ of G, a compact abelian subgroup K of G, and a homomorphism $\rho : \hat{K} \to G$ satisfying certain conditions. These C^*-algebras were denoted by $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ and were further studied in [20]. We shall see in Proposition 2.7 below that these algebras coincide with certain universal C^*-algebras, which we denote by $C^*(\hat{G}/\Gamma, \rho)$. For our result to be valid for these algebras, we shall assume conditions (S1)-(S5) (see Sections 2, 3, and 4 below). Among these conditions, (S1)-(S3) are essentially the same but slightly weaker than the conditions of Landstad and Raeburn. The conditions (S4) and (S5) are just that G is a Lie group and G/Γ is connected.

Theorem 1.1. *Let G, Γ, K and ρ satisfy the conditions (S1)-(S5). Fix a norm on the Lie algebra $\mathfrak{g}$ of G. Denote by L_ρ the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ defined above for the canonical action α of G on $C^*(\hat{G}/\Gamma, \rho)$. Then $(C^*(\hat{G}/\Gamma, \rho), L_\rho)$ is a C^*-algebraic compact quantum metric space.*

Since Rieffel introduced his quantum Gromov-Hausdorff distance in [31], several variations have appeared [18, 19, 24, 25, 26, 35, 39]. Among these quantum distances, probably the most suitable one in our current situation is the distance $\mathrm{dist}_{\mathrm{nu}}$ discussed in [19, Section 5], which is the unital version of the quantum distance introduced in [26, Remark 5.5]. As pointed out in [19, Section 5], this distance is no less than the distances introduced in [18, 31]. It is also no less than the distances in [35] (see Appendix below). Our second main result says that the compact quantum metric spaces $(C^*(\hat{G}/\Gamma, \rho), L_\rho)$ depend on ρ continuously. Let us mention that among the conditions (S1)-(5), only the conditions (S1) and (S2) involve ρ.

Theorem 1.2. *Fix G, Γ, and K so that there exists ρ satisfying the conditions (S1)-(S5). Denote by Ω the set of all ρ satisfying the conditions (S1) and (S2), equipped with the weakest topology making the maps $\Omega \to G$ sending ρ to $\rho(s)$ to be continuous for each $s \in \hat{K}$. Then Ω is a locally compact metrizable space. Fix a norm on the Lie algebra $\mathfrak{g}$ of G. Then for any $\rho' \in \Omega$, $\mathrm{dist}_{\mathrm{nu}}(C^*(\hat{G}/\Gamma, \rho), C^*(\hat{G}/\Gamma, \rho')) \to 0$ as $\rho \to \rho'$.*

This paper is organized as follows. In Section 2 we recall Landstad and Raeburn's construction of noncommutative homogeneous spaces, and establish some general properties of these noncommutative spaces. The relation between the derivations coming from two canonical group actions on $C^*(\hat{G}/\Gamma, \rho)$ is established in Section 3. In Section 4 we show that in the nondeformed case L_ρ is essentially the Lipschitz seminorm corresponding to some metric on G/Γ. A general result of establishing certain seminorm being a quantum metric by the help of a compact group action is proved in Section 5. Theorems 1.1 and 1.2 are proved in Sections 6 and 7 respectively. In an appendix we compare the distance $\mathrm{dist}_{\mathrm{nu}}$ and the proximity Rieffel introduced in [35].

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2. Noncommutative homogeneous spaces

In this section we recall Landstad and Raeburn's construction of noncommutative deformations of homogeneous spaces, discuss some examples, and establish some general properties of these noncommutative homogeneous spaces. These properties are of independent interest themselves.

Let G be a locally compact group. Throughout this paper, we make the following standard assumptions:

(S1) K is a compact abelian subgroup of G, and $\rho : \hat{K} \to G$ is a group homomorphism from its Pontryagin dual $\hat{K}$ into G such that $\rho(\hat{K})$ commutes with K.

(S2) Γ is a closed subgroup of G commuting with K and satisfies

$$\begin{aligned} \Omega_\gamma(s) &:= \gamma\rho(s)\gamma^{-1}\rho(-s) \text{ is in } K \text{ for all } s \in \hat{K}, \gamma \in \Gamma \text{ and} \\ \langle \Omega_\gamma(s), t\rangle &= \langle \Omega_\gamma(t), s\rangle \text{ for all } s, t \in \hat{K}, \gamma \in \Gamma, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between K and $\hat{K}$.

Denote by $C_{\rm b}(G)$ the Banach algebra of bounded continuous $\mathbb{C}$-valued functions on G, equipped with the pointwise multiplication and the supremum norm. Endow K with its normalized Haar measure. Consider the action of K on $C_{\rm b}(G)$ induced by the right multiplication of K on G. For $f \in C_{\rm b}(G)$, let $f_s \in C_{\rm b}(G)$ for $s \in \hat{K}$ be the partial Fourier transform defined by $f_s(x) := \int_K \overline{< k, s >} f(xk)\, dk$ for $x \in G$ (this is denoted by $\hat{f}(x, s)$ in (1.3) of [22]). Note that although the action of K on $C_{\rm b}(G)$ may not be strongly continuous, we do have $f_s \in C_{\rm b}(G)$. Then

$$C_{{\rm b},1}(G) := \{ f \in C_{\rm b}(G) | \quad \|f\|_{\infty,1} := \sum_{s\in\hat{K}} \|f_s\| < \infty \}$$

is a Banach $*$-algebra [21, Proposition 5.2] with norm $\|\cdot\|_{\infty,1}$ and operations

$$f * g(x) = \sum_{s,t} f_s(x\rho(t)) g_t(x\rho(-s)), \tag{1}$$

$$f^*(x) = \overline{f(x)}. \tag{2}$$

Fix a left invariant Haar measure on G. For each $s \in \hat{K}$ denote by P_s the projection on $L^2(G)$ corresponding to the restriction of the left regular representation $L|_K$ of K in $L^2(G)$, i.e.,

$$P_s = \int_K \overline{\langle k, s\rangle} L_k \, dk,$$

where $L_y\xi(x) = \xi(y^{-1}x)$ for $\xi \in L^2(G)$, $x, y \in G$. Then $C_{\mathrm{b},1}(G)$ has a faithful $*$-representation V on $L^2(G)$ [22, Proposition 1.3] given by

$$V(f) = \sum_{s,t} P_t L_{\rho(s)} M(f) L_{\rho(-t)} P_s, \tag{3}$$

where M is the representation of $C_{\mathrm{b}}(G)$ on $L^2(G)$ given by $M(f)\xi(x) = f(x^{-1})\xi(x)$. Denote by $C_0(G/\Gamma)$ the C^*-algebra of continuous $\mathbb{C}$-valued functions on G/Γ vanishing at ∞, and think of it as a C^*-subalgebra of $C_{\mathrm{b}}(G)$ via the quotient map $G \to G/\Gamma$. The space $C_{0,1}(G/\Gamma, \rho) := C_0(G/\Gamma) \cap C_{\mathrm{b},1}(G, \rho)$ is a closed $*$-subalgebra of $C_{\mathrm{b},1}(G, \rho)$, and the noncommutative homogeneous space $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ of Landstad and Raeburn is defined as the closure of $V(C_{0,1}(G/\Gamma, \rho))$ [22, Theorem 4.3].

Clearly the left translations α_y defined by $\alpha_y(f)(x) = f(y^{-1}x)$ for $y \in G$ extend to isometric $*$-automorphisms of $C_{0,1}(G/\Gamma, \rho)$. They also extend to $*$-automorphisms of $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ [22, Theorem 4.3]. We shall see later that this action of G on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is strongly continuous.

Before discussing properties of these noncommutative homogeneous spaces, let us look at some examples.

Example 2.1. Let H_1 be the 3-dimensional Heisenberg group consisting of matrices of the form

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

as a subgroup of $\mathrm{GL}(3, \mathbb{R})$. Denote by Z the subgroup consisting of elements with $x = y = 0$ and $z \in \mathbb{Z}$. Then we can write the elements of $G := H_1/Z$ as $(x, y, e^{2\pi i z})$ for $x, y, z \in \mathbb{R}$. Fix a positive integer c. Take

$$\Gamma = \{(x, y, e^{2\pi i z}) \in G | x, y, cz \in \mathbb{Z}\}, \quad K = \{(0, 0, e^{2\pi i z}) \in G | z \in \mathbb{R}\}.$$

Take $\mu, \nu \in \mathbb{R}$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$\rho(s) = (s\mu, s\nu, e^{\pi i s^2 \mu \cdot \nu}).$$

The C^*-algebra $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is isomorphic to Rieffel's quantum Heisenberg manifold D_1 in [28, Theorem 5.5] (see [22, page 493]).

Example 2.2. (cf. [22, Example 4.17]) Let H_n be the $2n+1$-dimensional Heisenberg group consisting of matrices of the form

$$\begin{pmatrix} 1 & y_1 & y_2 & \cdots & y_n & z \\ 0 & 1 & 0 & \cdots & 0 & x_1 \\ 0 & 0 & 1 & \cdots & 0 & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

as a subgroup of $\mathrm{GL}(n+2,\mathbb{R})$. Denote by Z the subgroup consisting of elements with $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ and $z \in \mathbb{Z}$. Then we can write the elements of $G := H_n/Z$ as $(x, y, e^{2\pi i z})$ for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Fix positive integers $b_1, \ldots, b_n, d_1, \ldots, d_n$ and c such that $b_j d_j | c$ for all j. Set $b = (b_1, \ldots, b_n)$ and $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$. Take

$$\Gamma = \{(x, y, e^{2\pi i z}) \in G | b \cdot x, d \cdot y, cz \in \mathbb{Z}\}, \quad K = \{(0, 0, e^{2\pi i z}) \in G | z \in \mathbb{R}\}.$$

Take $\mu, \nu \in \mathbb{R}^n$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$\rho(s) = (s\mu, s\nu, e^{\pi i s^2 \mu \cdot \nu}).$$

The C^*-algebra $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is a higher-dimensional generalization of Example 2.1.

Example 2.3. Let $n \geq 3$. Let W be the subgroup of $\mathrm{GL}(n,\mathbb{Z})$ consisting of upper triangular matrices $(a_{j,l})$ with diagonal entries all being 1. Denote by Z the subgroup consisting of matrices whose entries are all 0 except diagonal ones being 1 and $a_{1,n}$ being an integer. Then we can write the elements of $G := W/Z$ as $(a_{j,l})$ with $a_{1,n} \in \mathbb{T}$. Fix a positive integer c. Take

$$\begin{aligned} \Gamma &= \{(a_{j,l}) \in G | a_{1,n}^c = 1 \text{ and } a_{j,l} \in \mathbb{Z} \text{ if } (j,l) \neq (1,n)\}, \\ K &= \{(a_{j,l}) \in G | a_{j,l} = 0 \quad \text{if} \quad j < l \quad \text{and} \quad (j,l) \neq (1,n)\}. \end{aligned}$$

Take $\mu, \nu \in \mathbb{R}$ and define $\rho : \mathbb{Z} = \hat{K} \to G$ by

$$(\rho(s))_{j,l} = \begin{cases} s\mu, & \text{if } (j,l) = (2,n), \\ s\nu, & \text{if } (j,l) = (1,n-1), \\ e^{\pi i s^2 \mu \cdot \nu}, & \text{if } (j,l) = (1,n), \\ 0, & \text{for other } j < l. \end{cases}$$

For $n = 3$ we get the quantum Heisenberg manifold in Example 2.1 again.

In the rest of this section we establish some properties of $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$. Denote by $C^*(\hat{G}/\Gamma, \rho)$ the enveloping C^*-algebra of the Banach $*$-algebra $C_{0,1}(G/\Gamma, \rho)$ [36, page 42]. By the universality of $C^*(\hat{G}/\Gamma, \rho)$

there is a canonical surjective $*$-homomorphism $C^*(\hat{G}/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ such that the diagram

$$\begin{tikzcd} C_{0,1}(G/\Gamma, \rho) \arrow[r] \arrow[dr] & C^*(\hat{G}/\Gamma, \rho) \arrow[d] \\ & C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho) \end{tikzcd}$$

commutes.

Clearly the right translations $\beta_k(f)(x) = f(xk)$ for $k \in K$ extend to isometric $*$-automorphisms of $C_{0,1}(G/\Gamma, \rho)$. Recall the action α of G on $C_{0,1}(G/\Gamma, \rho)$ defined before Example 2.1. Then α and β induce actions of G and K on $C^*(\hat{G}/\Gamma, \rho)$ respectively, which we still denote by α and β respectively. For each $s \in \hat{K}$, set

$$B_s := \{f \in C_0(G/\Gamma) | f = f_s\}. \tag{4}$$

Lemma 2.4. *The actions α and β of G and K on $C_{0,1}(G/\Gamma, \rho)$ ($C^*(\hat{G}/\Gamma, \rho)$ resp.) commute with each other and are strongly continuous. The spectral spaces $\{f \in C_{0,1}(G/\Gamma, \rho) | \beta_k(f) = \langle k, s\rangle f$ for all $k \in K\}$ and $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s\rangle a$ for all $k \in K\}$ of β corresponding to $s \in \hat{K}$ are exactly B_s, and the norm of B_s in $C_{0,1}(G/\Gamma, \rho)$ and $C^*(\hat{G}/\Gamma, \rho)$ is exactly the supremum norm.*

Proof. Clearly α and β commute with each other. It is also clear that $B_s = \{f \in C_{0,1}(G/\Gamma, \rho) | \beta_k(f) = \langle k, s\rangle f$ for all $k \in K\}$ and that the norm of B_s in $C_{0,1}(G/\Gamma, \rho)$ is exactly the supremum norm. It follows that the restrictions of the actions α and β on $B_s \subseteq C_{0,1}(G/\Gamma, \rho)$ are strongly continuous for each $s \in \hat{K}$. For any $f \in C_{0,1}(G/\Gamma, \rho)$, one has $f_s \in B_s$ for each $s \in \hat{K}$. For any $\varepsilon > 0$ take a finite subset $F \subseteq \hat{K}$ such that $\sum_{s \in \hat{K} \setminus F} \|f_s\| < \varepsilon$. Then $\|f - \sum_{s \in F} f_s\|_{\infty,1} = \sum_{s \in \hat{K} \setminus F} \|f_s\| < \varepsilon$. Therefore $\oplus_{s \in \hat{K}} B_s$ is dense in $C_{0,1}(G/\Gamma, \rho)$. It follows that the actions α and β are strongly continuous on $C_{0,1}(G/\Gamma, \rho)$. Note that the canonical homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$ is contractive [36, Proposition 5.2]. Consequently, the induced actions of α and β on $C^*(\hat{G}/\Gamma, \rho)$ are also strongly continuous.

Note that the subalgebra B_0 of $C_{0,1}(G/\Gamma, \rho)$ is a C^*-algebra, which can be identified with $C_0(G/K\Gamma)$. Since the natural homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is injective, so is the canonical homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$. As injective $*$-homomorphisms between C^*-algebras are isometric, we conclude that the homomorphism of B_0 into $C^*(\hat{G}/\Gamma, \rho)$ is isometric. For any $f \in B_s$ one has $f^* * f \in B_0$

and the supremum norm of $f^* * f$ is equal to the square of the supremum norm of f. It follows that the homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$ is isometric on B_s. In particular, the image of B_s in $C^*(\hat{G}/\Gamma, \rho)$ is closed.

Since the action β of K on $C^*(\hat{G}/\Gamma, \rho)$ is strongly continuous, the spectral space $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s\rangle\, a \text{ for all } k \in K\}$ is the image of the continuous linear operator $C^*(\hat{G}/\Gamma, \rho) \to C^*(\hat{G}/\Gamma, \rho)$ sending a to $\int_K \overline{\langle k, s\rangle} \beta_k(a)\, dk$. It follows that the image of $B_s = \{f \in C_{0,1}(G/\Gamma, \rho) | \beta_k(f) = \langle k, s\rangle\, f \text{ for all } k \in K\}$ in $C^*(\hat{G}/\Gamma, \rho)$ is dense in $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s\rangle\, a \text{ for all } k \in K\}$. Therefore the image of B_s in $C^*(\hat{G}/\Gamma, \rho)$ is exactly $\{a \in C^*(\hat{G}/\Gamma, \rho) | \beta_k(a) = \langle k, s\rangle\, a \text{ for all } k \in K\}$. □

We refer the reader to [8, Chapter 2] for the basics of nuclear C^*-algebras.

Proposition 2.5. *The C^*-algebra $C^*(\hat{G}/\Gamma, \rho)$ is nuclear.*

Proof. By Lemma 2.4 the action β of K on $C^*(\hat{G}/\Gamma, \rho)$ is strongly continuous, and its fixed-point subalgebra is B_0, a commutative C^*-algebra, and hence is nuclear [8, Proposition 2.4.2]. For any C^*-algebra carrying a strongly continuous action of a compact group, the algebra is nuclear if and only if the fixed-point subalgebra is nuclear [14, Proposition 3.1]. Consequently, $C^*(\hat{G}/\Gamma, \rho)$ is nuclear. □

We shall need the following well-known fact a few times (see for example [8, Proposition 4.5.1]).

Lemma 2.6. *Let H be a compact group, and let σ_j be a strongly continuous action of H on a C^*-algebra A_j for $j = 1, 2$. Let $\varphi : A_1 \to A_2$ be an H-equivariant $*$-homomorphism. Then φ is injective if and only if the restriction of φ on the fixed-point subalgebra A_1^H is injective. In particular, if φ is surjective and $\varphi|_{A_1^H}$ is injective, then φ is an isomorphism.*

Proposition 2.7. *The canonical $*$-homomorphism $C^*(\hat{G}/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is an isomorphism.*

Proof. We shall apply Lemma 2.6 to show that the canonical $*$-homomorphism $\varphi : C^*(\hat{G}/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is an isomorphism. By [22, Lemma 4.4] the action β on $C_{0,1}(G/\Gamma, \rho)$ extends to an action of K on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$, which we denote by β'. Clearly φ is K-equivariant. By Lemma 2.4 β is strongly continuous on $C^*(\hat{G}/\Gamma, \rho)$. Since φ is contractive, it follows that β' is strongly continuous on $C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$. By Lemma 2.4 the fixed-point subalgebra $(C^*(\hat{G}/\Gamma, \rho))^K$ is B_0. Since the homomorphism $C_{0,1}(G/\Gamma, \rho) \to C^*_{\mathrm{r}}(\hat{G}/\Gamma, \rho)$ is injective, we see that the restriction of φ on $(C^*(\hat{G}/\Gamma, \rho))^K$ is injective. Therefore the conditions of Lemma 2.6 are satisfied and we conclude that φ is an isomorphism. $\square$

We refer the reader to [15] for a comprehensive treatment of C^*-*algebraic bundles*, which are usually called *Fell bundles* now. Notice that for $f_s \in B_s$ and $g_t \in B_t$ the product $f_s * g_t$ is in B_{s+t} and f_s^* is in B_{-s}. Also $||f_s^* * f_s|| = ||f_s||^2$. Therefore we have a Fell bundle $\mathcal{B}^\rho = \{B_s\}_{s\in\hat{K}}$ over $\hat{K}$ with operations given by (1) and (2). It is easy to see that $C_{0,1}(G/\Gamma, \rho)$ is exactly the L^1-algebra of $\mathcal{B}^\rho$ (cf. the proof of [21, Proposition 5.2]). Thus the C^*-algebra $C^*(\hat{G}/\Gamma, \rho)$ is also the enveloping C^*-algebra $C^*(\mathcal{B}^\rho)$ of the Fell bundle $\mathcal{B}^\rho$.

Next we discuss what happens if we let ρ vary continuously. We refer the reader to [13, Chapter 10] for the basics of continuous fields of Banach spaces and C^*-algebras. On page 505 of [22] Landstad and Raeburn pointed out that it seems reasonable that we shall get a continuous field of C^*-algebras, but no proof was given there. This is indeed true, and we give a proof here. To be precise, fix G, Γ and K, let W be a locally compact Hausdorff space and for each $w \in W$ we assign a ρ_w satisfying (S1) and (S2) such that the map $w \mapsto \rho_w(s)$ is continuous for each $s \in \hat{K}$. Notice that $\mathcal{B}^\rho$ as a Banach space bundle over $\hat{K}$ do not depend on ρ. For clarity we denote the product and $*$-operation in (1) and (2) by $f_s *_w g_t$ and $f_s^{*_w}$. For any $f_s \in B_s$ and $g_t \in B_t$, clearly the maps $w \mapsto f_s *_w g_t$ and $w \mapsto f_s^{*_w}$ are both continuous. This leads to the next lemma, which is a slight generalization of [5, Proposition 3.3, Theorem 3.5]. The proof of [5, Proposition 3.3, Theorem 3.5], which in turn follows the lines of [29], is easily seen to hold also in our case.

Lemma 2.8. *Let H be a discrete group and A_h be a vector space for each $h \in H$. Let W be a locally compact Hausdorff space and for each $w \in W$ assign norms and algebra operations making $\mathcal{A}^w = \{A_h\}_{h\in H}$ into a Fell bundle in such a way that for any $f_s \in A_s$ and $g_t \in A_t$ the map $w \mapsto ||f_s||_w \in \mathbb{R}$ is continuous (then we have a continuous field of Banach spaces $(A_s, ||\cdot||_w)_{w\in W}$ over W for each $s \in H$) and the sections $w \mapsto f_s *_w g_t \in B_{st}$ and $w \mapsto f_s^{*_w} \in B_{s^{-1}}$ are continuous in the above continuous fields of Banach spaces $(B_{st}, ||\cdot||_w)_{w\in W}$ and $(B_{s^{-1}}, ||\cdot||_w)_{w\in W}$ respectively . Then the map $w \mapsto ||f||_w$ is upper semi-continuous for each $f \in \oplus_{s\in H} A_s$, where $||\cdot||_w$ is the norm on the enveloping C^*-algebra $C^*(\mathcal{A}^w)$ and extends the norm of A_s as part of $\mathcal{A}^w$ for each $s \in H$. Moreover, if H is amenable, then $\{C^*(\mathcal{A}^w)\}_{w\in W}$*

is a continuous field of C^*-algebras with the field structure determined by the continuous sections $w \mapsto f$ for all $f \in \oplus_{s\in H} A_s$.

Since every discrete abelian group is amenable [27, page 14], from Proposition 2.7 we get

Proposition 2.9. *Fix G, Γ and K. Let W by a locally compact Hausdorff space and for each $w \in W$ let ρ_w satisfy (S1) and (S2) such that the map $w \mapsto \rho_w(s)$ is continuous for each $s \in \hat{K}$. Then $\{C^*(\hat{G}/\Gamma, \rho_w)\}_{w\in W}$ is a continuous field of C^*-algebras with the field structure determined by the continuous sections $w \mapsto f$ for all $f \in \oplus_{s\in\hat{K}} B_s$.*

3. Derivations

In this section we prove Proposition 3.3, to establish the relation between derivations coming from α and β.

Throughout the rest of this paper, we assume:

(S3) G/Γ is compact.
(S4) G is a Lie group.

The examples in Section 2 all satisfy these conditions.

We refer the reader to [17, Section 1.3] for the discussion about differentiable maps into Fréchet spaces. We just recall that a continuous map ψ from a smooth manifold M into a Fréchet space A is *continuously differentiable* if for any chart (U, ϕ) of G, where U is an open subset of some Euclidean space $\mathbb{R}^n$ and ϕ is a diffeomorphism from U onto an open set of M, the derivative

$$D(\psi\circ\phi)(x,h) = \lim_{\mathbb{R}\ni\nu\to 0} \frac{\psi\circ\phi(x+\nu h) - \psi\circ\phi(x)}{\nu}$$

exists for all $(x, h) \in (U, \mathbb{R}^n)$ and is a jointly continuous map from $(U, \mathbb{R}^n)$ into A. In such case, $D(\psi \circ \phi)(x, h)$ is linear on h, and depends only on ψ and the tangent vector $u := \phi_*(v_{x,h})$ of M at $\phi(x)$, where $v_{x,h}$ denotes the tangent vector h at x. Thus we may denote $D(\psi \circ \phi)(x, h)$ by $\partial_u\psi$. Then $\partial_u\psi$ is linear on u.

Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. For a strongly continuous action σ of G on a Banach space A as isometric automorphisms, we say that an element $a \in A$ is *once differentiable* with respect to σ if the orbit map ψ_a from G into A sending x to $\sigma_x(a)$ is continuously differentiable. Then the set A_1 of once differentiable elements is a linear subspace of A. For any $a \in A$ and any compactly supported smooth $\mathbb{C}$-valued function φ on G, it is easily checked that $\int_G \varphi(x)\sigma_x(a)\,dx$ is in A_1. As a can be approximated by such elements, we see that A_1 is dense in A. Thinking of $\mathfrak{g}$ as the tangent space of

G at the identity element, for each $X \in \mathfrak{g}$ we have the linear map $\sigma_X : A_1 \to A$ sending a to $\partial_X \psi_a$. Fix a norm on $\mathfrak{g}$. We define a seminorm L on A_1 by setting $L(a)$ to be the norm of the linear map $\mathfrak{g} \to A$ sending X to $\sigma_X a$.

Lemma 3.1. *Let σ be a strongly continuous action of G on a Banach space A as isometric automorphisms. For any $a \in A_1$, one has*

$$L(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|}.$$

Proof. The proof is similar to that of [31, Proposition 8.6]. Let $X \in \mathfrak{g}$ with $\|X\| = 1$. One has

$$\sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} \geq \lim_{\nu \to 0^+} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} = \|\sigma_X(a)\|.$$

For any $\nu > 0$, one also has

$$\begin{aligned} \|\sigma_{e^{\nu X}}(a) - a\| &= \| \int_0^\nu \sigma_{e^{zX}}(\sigma_X(a))\, dz\| \leq \int_0^\nu \|\sigma_{e^{zX}}(\sigma_X(a))\|\, dz \\ &= \int_0^\nu \|\sigma_X(a)\|\, dz = \nu \|\sigma_X(a)\|. \end{aligned}$$

Therefore

$$\sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} = \|\sigma_X(a)\|.$$

Thus

$$\begin{aligned} \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|} &= \sup_{X \in \mathfrak{g}, \|X\| = 1} \sup_{\nu > 0} \frac{\|\sigma_{e^{\nu X}}(a) - a\|}{\nu} \\ &= \sup_{X \in \mathfrak{g}, \|X\| = 1} \|\sigma_X(a)\| = L(a). \end{aligned}$$

∎

Lemma 3.2. *Let σ be a strongly continuous action of G on a Banach space A as isometric automorphisms. Then A_1 is a Banach space with the norm $\mathfrak{p}(a) := L(a) + \|a\|$. Suppose that σ' is a strongly continuous isometric action of a topological group H on A, commuting with σ. Then H preserves A_1, and the restriction of σ' on A_1 preserves the norm $\mathfrak{p}$ and is strongly continuous with respect to $\mathfrak{p}$.*

Proof. Let $\{a_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in A_1 under the norm $\mathfrak{p}$. Then as n goes to infinity, a_n converges to some $a \in A$, and $\sigma_X(a_n)$ converge to some b_X in A uniformly on X in bounded subsets of $\mathfrak{g}$.

Let $\varrho : [0,1] \to G$ be a continuously differentiable curve in G. Then $\lim_{z\to 0} \frac{\sigma_{\varrho_{\nu+z}}(a_n)-\sigma_{\varrho_\nu}(a_n)}{z} = \sigma_{\varrho_\nu}(\sigma_{\varrho'_\nu}(a_n))$ for all $\nu \in [0,1]$. Thus

$$\sigma_{\varrho_\nu}(a_n) - \sigma_{\varrho_0}(a_n) = \int_0^\nu \sigma_{\varrho_z}(\sigma_{\varrho'_z}(a_n))\, dz.$$

Letting $n \to \infty$ we get

$$\sigma_{\varrho_\nu}(a) - \sigma_{\varrho_0}(a) = \int_0^\nu \sigma_{\varrho_z}(b_{\varrho'_z})\, dz.$$

Therefore $\lim_{z\to 0} \frac{\sigma_{\varrho_z}(a)-\sigma_{\varrho_0}(a)}{z} = \sigma_{\varrho_0}(b_{\varrho'_0})$. It follows easily that $a \in A_1$ and $\sigma_X(a) = b_X$ for all $X \in \mathfrak{g}$. Consequently, a_n converges to a in A_1 under the norm $\mathfrak{g}$, and hence A_1 is a Banach space under the norm $\mathfrak{g}$.

Clearly σ' preserves A_1 and the norm $\mathfrak{p}$. For any $a \in A_1$, the set of $\sigma_X(a)$ for X in the unit ball of $\mathfrak{g}$ is compact. Then for any $h \in H$ and $\varepsilon > 0$, when $h' \in H$ is close enough to h, one has $\|\sigma'_h(a) - \sigma'_{h'}(a)\| < \varepsilon$ and $\|\sigma_X(\sigma'_h(a)) - \sigma_X(\sigma'_{h'}(a))\| = \|\sigma'_h(\sigma_X(a)) - \sigma'_{h'}(\sigma_X(a))\| < \varepsilon$ for all X in the unit ball of $\mathfrak{g}$. Consequently, $\mathfrak{p}(\sigma'_h(a) - \sigma'_{h'}(a)) = L(\sigma'_h(a) - \sigma'_{h'}(a)) + \|\sigma'_h(a) - \sigma'_{h'}(a)\| < 2\varepsilon$. Therefore the restriction of σ' on A_1 is strongly continuous with respect to $\mathfrak{p}$. $\square$

By Lemma 2.4 the actions α and β on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other and are strongly continuous. Denote by $C^1(\hat{G}/\Gamma, \rho)$ the space of once differentiable elements of $C^*(\hat{G}/\Gamma, \rho)$ with respect to the action α. Recall the B_s defined in (4).

Proposition 3.3. *Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$. For $Y \in \mathfrak{k}$ say*

$$\mathrm{Ad}_x(Y) = \sum_j F_{j,Y}(x) X_j,$$

where Ad *denotes the adjoint action of G on $\mathfrak{g}$. Then $F_{j,Y} \in B_0$. Any $f \in C^1(\hat{G}/\Gamma, \rho)$ is once differentiable with respect to the action β and*

$$\beta_Y(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f). \tag{5}$$

Proof. Clearly $F_{j,Y}$ is a smooth function on G. Since the subgroups Γ, K and $\rho(\hat{K})$ commute with K, if y is in any of these subgroups, then $\mathrm{Ad}_y(Y) = Y$, and hence

$$\sum_j F_{j,Y}(x) X_j = \mathrm{Ad}_x(Y) = \mathrm{Ad}_x(\mathrm{Ad}_y(Y)) = \mathrm{Ad}_{xy}(Y) = \sum_j F_{j,Y}(xy) X_j,$$

which means that $F_{j,Y}$ is invariant under the right translation of y. Thus $F_{j,Y} \in C(G/K\Gamma) = C_0(G/K\Gamma) = B_0$. For each $X \in \mathfrak{g}$ denote by $X^{\#}$ ($X_{\#}$ resp.) the corresponding right (left resp.) translation invariant vector field on G. Then $Y_{\#} = \sum_j F_{j,Y} X_j^{\#}$.

Let $f \in C^1(\hat{G}/\Gamma, \rho) \cap B_s$ for some $s \in \hat{K}$. By Lemma 2.4 the norm on $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$ is exactly the supremum norm. Thus f belongs to the space $C^1(G)$ of continuously differentiable functions on G. For any continuous vector field Z on G denote by ∂_Z the corresponding derivation map $C^1(G) \to C(G)$. Then

$$\partial_{Y_{\#}}(f) = \sum_j F_{j,Y} \partial_{X_j^{\#}}(f) = -\sum_j F_{j,Y} \alpha_{X_j}(f).$$

Since $F_{j,Y}$ is invariant under the right translation of Γ and $\rho(K)$, we have $F_{j,Y}(x) g_t(x) = F_{j,Y} * g_t(x)$ for any $g_t \in B_t$ and $x \in G$. By Lemma 2.4 the actions α and β on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other. Thus α preserves B_s, and hence $\alpha_X(f) \in B_s$ for every $X \in \mathfrak{g}$. Therefore $\partial_{Y_{\#}}(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f)$.

Let $\varrho : [0,1] \to K$ be a continuously differentiable curve in K. Then

$$\lim_{z \to 0} \frac{f(x\varrho_{\nu+z}) - f(x\varrho_\nu)}{z} = (\partial_{(\varrho'_\nu)_{\#}}(f))(x\varrho_\nu) = (-\sum_j F_{j,\varrho'_\nu} * \alpha_{X_j}(f))(x\varrho_\nu)$$

for all $\nu \in [0,1]$ and $x \in G$, and hence we have the integral form

$$f(x\varrho_\nu) - f(x\varrho_0) = \int_0^\nu (-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))(x\varrho_z)\, dz \tag{6}$$

for all $\nu \in [0,1]$ and $x \in G$. The left hand side of (6) is the value of $\beta_{\varrho_\nu}(f) - \beta_{\varrho_0}(f)$ at x, while the right hand side of (6) is the value of $\int_0^\nu \beta_{\varrho_z}(-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))\, dz$ at x, where the integral is taken in $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$. Therefore

$$\beta_{\varrho_\nu}(f) - \beta_{\varrho_0}(f) = \int_0^\nu \beta_{\varrho_z}(-\sum_j F_{j,\varrho'_z} * \alpha_{X_j}(f))\, dz \tag{7}$$

for all $\nu \in [0,1]$.

Clearly (7) also holds for $f \in \oplus_{s \in \hat{K}}(C^1(\hat{G}/\Gamma, \rho) \cap B_s)$. By Lemma 3.2 $C^1(\hat{G}/\Gamma, \rho)$ is a Banach space with norm $\mathfrak{p}(\cdot) = L(\cdot) + \|\cdot\|$, β preserves $C^1(\hat{G}/\Gamma, \rho)$ and $\mathfrak{p}$, and the restriction of β on $C^1(\hat{G}/\Gamma, \rho)$ is strongly continuous on $C^1(\hat{G}/\Gamma, \rho)$ with respect to $\mathfrak{p}$. By Lemma 2.4 the spectral subspace of $C^*(\hat{G}/\Gamma, \rho)$ corresponding to $s \in \hat{K}$ for the action β is equal to B_s. It follows that the spectral subspace of $C^1(\hat{G}/\Gamma, \rho)$ corresponding to $s \in \hat{K}$ for the restriction of β on $C^1(\hat{G}/\Gamma, \rho)$ is exactly $C^1(\hat{G}/\Gamma, \rho) \cap B_s$. Then standard techniques tell us that $\oplus_{s \in \hat{K}}(C^1(\hat{G}/\Gamma, \rho) \cap B_s)$ is dense in $C^1(\hat{G}/\Gamma, \rho)$ with respect to $\mathfrak{p}$. Notice that both sides of

(7) define continuous maps from $C^1(\hat{G}/\Gamma, \rho)$ to $C^*(\hat{G}/\Gamma, \rho)$. Therefore (7) holds for all $f \in C^1(\hat{G}/\Gamma, \rho)$. Consequently,

$$\lim_{z\to 0} \frac{\beta_{\varrho_z}(f) - \beta_{\varrho_0}(f)}{z} = \beta_{\varrho_0}(-\sum_j F_{j,\varrho_0'} * \alpha_{X_j}(f))$$

for all $f \in C^1(\hat{G}/\Gamma, \rho)$. It follows easily that f is once differentiable with respect to β and $\beta_Y(f) = -\sum_j F_{j,Y} * \alpha_{X_j}(f)$ for all $f \in C^1(\hat{G}/\Gamma, \rho)$ and $Y \in \mathfrak{k}$. $\square$

We shall need the following lemma (compare [34, Proposition 2.5]).

Lemma 3.4. *Let σ be a strongly continuous action of G on a Banach space A as isometric automorphisms. Let $a \in A$. Then for any $\varepsilon > 0$, there is some $b \in A$ such that b is smooth with respect to σ, $\|b\| \le \|a\|$, $\|b - a\| \le \varepsilon$, and $\sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_X(b)\|}{\|X\|} \le \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{e^X}(a)-a\|}{\|X\|}$. If A has an isometric involution being invariant under σ, then when a is self-adjoint, we can choose b also to be self-adjoint.*

Proof. Endow G with a left-invariant Haar measure. Let U be a small open neighborhood of the identity element in G with compact closure, which we shall determine later. Let φ be a non-negative smooth function on G with support contained in U such that $\int_G \varphi(x)\, dx = 1$. Set $b = \int_G \varphi(x)\sigma_x(a)\, dx$. Then b is smooth with respect to σ, and $\|b\| \le \|a\|$. When U is small enough, we have $\|a - b\| \le \varepsilon/2$. For any $X \in \mathfrak{g}$, setting $\psi(x) = \mathrm{Ad}_{x^{-1}}(X)$, we have

$$\begin{aligned}
\|\sigma_{e^X}(b) - b\| &= \|\int_G \varphi(x)(\sigma_{e^X x}(a) - \sigma_x(a))\, dx\| \\
&= \|\int_G \varphi(x)\sigma_x(\sigma_{e^{\psi(x)}}(a) - a)\, dx\| \\
&\le \int_G \varphi(x)\|\sigma_x(\sigma_{e^{\psi(x)}}(a) - a)\|\, dx \\
&\le \sup_{x\in U} \|\sigma_{e^{\psi(x)}}(a) - a\| \\
&\le \sup_{0\neq Y\in\mathfrak{g}} \frac{\|\sigma_{e^Y}(a) - a\|}{\|Y\|} \cdot \sup_{x\in U} \|\psi(x)\|.
\end{aligned}$$

Set $\delta = \varepsilon/(2+2\|a\|)$. When U is small enough, we have $\|\mathrm{Ad}_{x^{-1}}(X)\| \le (1 + \delta)\|X\|$ for all $X \in \mathfrak{g}$ and $x \in U$. Then $\|\sigma_{e^X}(b) - b\| \le (1 + \delta)\|X\| \sup_{0\neq Y\in\mathfrak{g}} \frac{\|\sigma_{e^Y}(a)-a\|}{\|Y\|}$ for all $X \in \mathfrak{g}$. By Lemma 3.1 we get

$$\sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_X(b)\|}{\|X\|} = \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{e^X}(b) - b\|}{\|X\|} \le (1 + \delta) \sup_{0\neq X\in\mathfrak{g}} \frac{\|\sigma_{e^X}(a) - a\|}{\|X\|}.$$

Now it is clear that $b' = b/(1+\delta)$ satisfies the requirement. Note that b' is self-adjoint if a is so. $\square$

4. Nondeformed case

In this section we consider the nondeformed case, i.e., the case ρ is the trivial homomorphism ρ_0 sending the whole $\hat{K}$ to the identity element of G. In Proposition 4.2 we identify L_{ρ_0} on $C^1(\hat{G}/\Gamma, \rho_0)$ with the Lipschitz seminorm for certain metric on G/Γ.

Note that $C_{0,1}(G/\Gamma, \rho_0)$ is sub-$*$-algebra of $C_0(G/\Gamma) = C(G/\Gamma)$. By the universality of $C^*(\hat{G}/\Gamma, \rho_0)$ we have a natural $*$-homomorphism ψ of $C^*(\hat{G}/\Gamma, \rho_0)$ into $C(G/\Gamma)$, extending the inclusion $C_{0,1}(G/\Gamma, \rho_0) \hookrightarrow C(G/\Gamma)$. The right translation of K on G induces a strongly continuous action β'' of K on $C(G/\Gamma)$, and clearly ψ intertwines β and β''. An application of Lemmas 2.6 and 2.4 tells us that ψ is injective. By definition B_s is the spectral subspace of $C(G/\Gamma)$ corresponding to $s \in \hat{K}$. Thus $\oplus_{s\in\hat{K}} B_s$ is dense in $C(G/\Gamma)$. As $\oplus_{s\in\hat{K}} B_s$ is in the image of ψ, we see that ψ is surjective and hence is an isomorphism. We shall identify $C^*(\hat{G}/\Gamma, \rho_0)$ and $C(G/\Gamma)$ via ψ.

The seminorm L_{ρ_0} describes the size of derivatives of $f \in C^1(\hat{G}/\Gamma, \rho_0)$. If it corresponds to some metric on G/Γ, this metric should be kind of geodesic distance. In order for the geodesic distance to be defined, throughout the rest of this paper we assume:

(S5) G/Γ is connected.

The examples in Section 2 all satisfy this condition.

Fix an inner product on $\mathfrak{g}$. Then we obtain a right translation invariant Riemannian metric on G in the usual way. Denote by d_G the geodesic distance on connected components of G. We extend d_G to a semi-distance on G via setting $d_G(x, y) = \infty$ if x and y lie in different connected components of G.

Lemma 4.1. *The function d on $G/\Gamma \times G/\Gamma$ defined by $d(x\Gamma, y\Gamma) := \inf_{x'\in x\Gamma, y'\in y\Gamma} d_G(x', y')$ is equal to $\inf_{y'\in y\Gamma} d_G(x, y')$. It is a metric on G/Γ and induces the quotient topology on G/Γ.*

Proof. Let V be a connected component of G. Then $V\Gamma$ is clopen in G, and hence $V\Gamma/\Gamma$ is clopen in G/Γ for the quotient topology. As G/Γ is connected, we conclude that $V\Gamma/\Gamma = G/\Gamma$. Therefore d is finite valued.

Since d_G is right translation invariant, we have $\inf_{x'\in x\Gamma, y'\in y\Gamma} d_G(x', y') = \inf_{y'\in y\Gamma} d_G(x, y')$. It follows easily that d is a metric on G/Γ.

Let $x \in G$. Let W be a neighborhood of $x\Gamma$ in G/Γ for the quotient topology. Then there exists $\varepsilon > 0$ such that if $d_G(x, y) < \varepsilon$, then

$y\Gamma \in W$. It follows that if $d(x\Gamma, y\Gamma) < \varepsilon$, then $y\Gamma \in W$. Therefore the topology induced by d on G/Γ is finer than the quotient topology. For any $\varepsilon' > 0$, set $U = \{y \in G | d_G(x, y) < \varepsilon'\}$. Then U is an open neighborhood of x. Thus $U\Gamma/\Gamma$ is an open neighborhood of $x\Gamma$ for the quotient topology. For any $z\Gamma \in U\Gamma/\Gamma$, we can find $z' \in z\Gamma \cap U$ and hence $d(x\Gamma, z\Gamma) \leq d_G(x, z') < \varepsilon'$. Therefore the quotient topology on G/Γ is finer than the topology induced by d. We conclude that d induces the quotient topology. $\square$

Proposition 4.2. *For any* $f \in C^1(\hat{G}/\Gamma, \rho_0) \subseteq C^*(\hat{G}/\Gamma, \rho_0) = C(G/\Gamma)$, *we have*

$$L_{\rho_0}(f) = \sup_{x\Gamma \neq y\Gamma} \frac{|f(x\Gamma) - f(y\Gamma)|}{d(x\Gamma, y\Gamma)}.$$

Proof. The right hand side of the above equation is equal to $\sup_{x\neq y} \frac{|f(x)-f(y)|}{d_G(x,y)}$. So it suffices to show

$$L_{\rho_0}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_G(x, y)}. \tag{8}$$

The proof is similar to that of [31, Proposition 8.6]. Let $\varrho : [0, 1] \to G$ be a continuously differentiable curve. Denote by $\ell(\varrho)$ the length of ϱ. Then $(f \circ \varrho)'(\nu) = (\alpha_{-\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f)(\varrho_\nu)$ for all $\nu \in [0, 1]$, and hence

$$\begin{aligned}
|f(\varrho_1) - f(\varrho_0)| &= |\int_0^1 (f \circ \varrho)'(\nu)\, d\nu| \leq \int_0^1 |(f \circ \varrho)'(\nu)|\, d\nu \\
&= \int_0^1 |(\alpha_{-\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f)(\varrho_\nu)|\, d\nu \leq \int_0^1 \|\alpha_{\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)} f\|\, d\nu \\
&\leq L_{\rho_0}(f) \int_0^1 \|\mathrm{Ad}_{\varrho_\nu}(\varrho'_\nu)\|\, d\nu = L_{\rho_0}(f)\ell(\varrho),
\end{aligned}$$

where in the last equality we use the fact that the Riemannian metric on G is right translation invariant. It follows easily that $|f(\varrho_1) - f(\varrho_0)| \leq L_{\rho_0}(f)\ell(\varrho)$ holds if ϱ is only piecewise continuously differentiable. Considering all piecewise continuously differentiable curves connecting x and y we obtain $|f(x) - f(y)| \leq L_{\rho_0}(f) d_G(x, y)$ for all $x, y \in G$.

Denote by e_G the identity element of G. For any $0 \neq X \in \mathfrak{g}$, we have

$$\begin{aligned}
\sup_{x\neq y} \frac{|f(x)-f(y)|}{d_G(x,y)} &\geq \sup_{x} \sup_{\nu\neq 0} \frac{|f(x)-f(e^{\nu X}x)|}{d_G(x,e^{\nu X}x)} \\
&= \sup_{\nu\neq 0} \sup_{x} \frac{|f(x)-f(e^{\nu X}x)|}{d_G(e_G,e^{\nu X})} \\
&\geq \sup_{\nu\neq 0} \sup_{x} \frac{|f(x)-f(e^{\nu X}x)|}{|\nu|\|X\|} \\
&= \sup_{\nu\neq 0} \frac{\|f-\alpha_{e^{-\nu X}}f\|}{|\nu|\|X\|} \\
&= \sup_{\nu\neq 0} \frac{\|\alpha_{e^{\nu X}}f-f\|}{|\nu|\|X\|} \geq \frac{\|\alpha_X(f)\|}{\|X\|}.
\end{aligned}$$

Therefore $\sup_{x\neq y} \frac{|f(x)-f(y)|}{d_G(x,y)} \geq L_{\rho_0}(f)$. This proves (8). □

5. Lip-norms and compact group actions

In this section we recall the definition of compact quantum metric spaces and prove Theorem 5.2, which enables one to show that certain seminorm defines a quantum metric, via the help of a compact group action.

Rieffel has set up the theory of compact quantum metric spaces in the general framework of order-unit spaces [31, Defintion 2.1]. We shall need it only for C^*-algebras. By a *C^*-algebraic compact quantum metric space* we mean a pair (A, L) consisting of a unital C^*-algebra A and a (possibly $+\infty$-valued) seminorm L on A satisfying the *reality condition*

$$L(a) = L(a^*) \tag{9}$$

for all $a \in A$, such that L vanishes exactly on $\mathbb{C}$ and the metric d_L on the state space $S(A)$ defined by

$$d_L(\psi,\phi) = \sup_{L(a)\leq 1} |\psi(a)-\phi(a)| \tag{10}$$

induces the weak*-topology. The *radius* of (A, L), denote by r_A, is defined to be the radius of $(S(A), d_L)$. We say that L is a *Lip-norm*.

Let A be a unital C^*-algebra and let L be a (possibly $+\infty$-valued) seminorm on A vanishing on $\mathbb{C}$. Then L and $\|\cdot\|$ induce (semi)norms $\tilde{L}$ and $\|\cdot\|^{\sim}$ respectively on the quotient space $\tilde{A} = A/\mathbb{C}$.

Recall that a *character* of a compact group is the trace function of a finite-dimensional complex representation of the group [7, Section II.4].

Lemma 5.1. *Let H be a compact group and H_0 be a closed normal subgroup of H of finite index. Then for any linear combination of finitely many characters of H, its multiplication with the characteristic function of H_0 is also a linear combination of finitely many characters of H.*

Proof. The products and sums of characters of H are still characters [7, Proposition II.4.10]. Thus it suffices to show that the characteristic function of H_0 on H is a linear combination of finitely many characters of H.

Since H/H_0 is finite, every $\mathbb{C}$-valued class function on H/H_0, i.e., functions being constant on conjugate classes, is a linear combination of characters of H/H_0 [16, Proposition 2.30]. Thus the characteristic function of $\{e_{H/H_0}\}$ on H/H_0, where e_{H/H_0} denotes the identity element of H/H_0, is a linear combination of characters of H/H_0. Then the characteristic function H_0 on H is a linear combination of characters of H. ◻

Recall that a *length function* on a topological group H is a continuous $\mathbb{R}_{\geq 0}$-valued function, ℓ, on H such that $\ell(h) = 0$ if and only if h is equal to the identity element e_H of H, that $\ell(h_1h_2) \leq \ell(h_1) + \ell(h_2)$ for all $h_1, h_2 \in H$, and that $\ell(h^{-1}) = \ell(h)$ for all $h \in H$.

Suppose that a compact group H has a strongly continuous action σ on a Banach space A as isometric automorphisms. Endow H with its normalized Haar measure. For any continuous $\mathbb{C}$-valued function φ on H, define a linear map $\sigma_\varphi : A \to A$ by

$$\sigma_\varphi(a) = \int_H \varphi(h)\sigma_h(a)\, dh$$

for $a \in A$. Denote by $\hat{H}$ the set of isomorphism classes of irreducible representations of H. For each $s \in \hat{H}$, denote by A_s the spectral subspace of A corresponding to s. For a finite subset J of $\hat{H}$, set $A_J = \sum_{s\in J} A_s$.

The main tool we use for the proof of Theorem 1.1 will be the following slight generalization of [23, Theorem 4.1].

Theorem 5.2. *Let A be a unital C^*-algebra, let L be a (possibly $+\infty$-valued) seminorm on A satisfying the reality condition (9), and let σ be a strongly continuous action of a compact group H on A by automorphisms. Assume that L takes finite values on a dense subspace of A, and that L vanishes on $\mathbb{C}$. Suppose that the following conditions are satisfied:*

(1) *there are some length function ℓ on a closed normal subgroup H_0 of H of finite index and some constant $C > 0$ such that $L^\ell \leq C \cdot L$ on A, where L^ℓ is the (possibly $+\infty$-valued) seminorm on A defined by*

$$L^\ell(a) = \sup\{\frac{\|\sigma_h(a) - a\|}{\ell(h)} | h \in H_0, h \neq e_H\}. \tag{11}$$

(2) *for any linear combination φ of finitely many characters on H we have $L \circ \sigma_\varphi \leq \|\varphi\|_1 \cdot L$ on A, where $\|\varphi\|_1$ denotes the L^1 norm of φ;*

(3) *for each $s \in \hat{H}$ not being the trivial representation s_0 of H, the set $\{a \in A_s | L(a) \leq 1, \|a\| \leq r\}$ is totally bounded for some $r > 0$, and the only element in A_s vanishing under L is 0;*

(4) *there is a unital C^*-algebra $\mathcal{A}$ containing the fixed-point subalgebra A^σ, with a Lip-norm $L_{\mathcal{A}}$, such that $L_{\mathcal{A}}$ extends the restriction of L to A^σ;*

(5) *for each $s \in \widehat{H/H_0} \subseteq \hat{H}$ not equal to s_0, there exists some constant $C_s > 0$ such that $\| \cdot \| \leq C_s L$ on A_s.*

Then (A, L) is a C^*-algebraic compact quantum metric space with $r_A \leq C \int_{H_0} \ell(h)\, dh + \sum_{s_0 \neq s \in \widehat{H/H_0}} C_s(\dim(s))^2 + r_{\mathcal{A}}$, where H_0 is endowed with its normalized Haar measure.

We need some preparation for the proof of Theorem 5.2. The following lemma generalizes [23, Lemma 3.4].

Lemma 5.3. *Let H be a compact group, and let H_0 be a closed normal subgroup of H of finite index. Let f be a continuous $\mathbb{C}$-valued function on H with $f(e_H) = 0$. Then for any $\varepsilon > 0$ there is a nonnegative function φ on H with support contained in H_0 such that φ is a linear combination of finitely many characters of H, $\|\varphi\|_1 = 1$, and $\|\varphi \cdot f\|_1 < \epsilon$.*

Proof. Denote by χ the characteristic function of H_0 on H. Set $g = f\chi + \varepsilon(1 - \chi)$. Then $g \in C(H)$ and $g(e_H) = 0$. By [23, Lemma 3.4] we can find a nonnegative function ϕ on H such that ϕ is a linear combination of finitely many characters, $\|\phi\|_1 = 1$, and $\|\phi \cdot g\|_1 < \varepsilon/2$. Then $\varepsilon \int_{H \setminus H_0} \phi(h)\, dh \leq \|\phi \cdot g\|_1 < \varepsilon/2$, and hence

$$\|\chi\phi\|_1 = \|\phi\|_1 - \int_{H \setminus H_0} \phi(h)\, dh > 1 - 1/2 = 1/2.$$

Set $\varphi = \chi\phi/\|\chi\phi\|_1$. By Lemma 5.1 φ is a linear combination of finitely many characters of H. One has

$$\|\varphi \cdot f\|_1 = \|\chi\phi f\|_1/\|\chi\phi\|_1 = \|\chi\phi g\|_1/\|\chi\phi\|_1 < (\varepsilon/2)/(1/2) = \varepsilon.$$

□

For a compact group H and a finite subset J of $\hat{H}$, set $\bar{J} = \{\bar{s} | s \in J\}$, where $\bar{s}$ denotes the contragradient representation. Replacing [23, Lemma 3.4] by Lemma 5.3 in the proof of [23, Lemma 4.4], we get:

Lemma 5.4. *Let H be a compact group. For any $\varepsilon > 0$ there is a finite subset $J = \bar{J}$ in $\hat{H}$, containing the trivial representation s_0, depending only on ℓ and ε/C, such that for any strongly continuous isometric action σ of H on a complex Banach space A with a (possibly $+\infty$-valued) seminorm L on A satisfying conditions (1) and (2) in Theorem 5.2, and any $a \in A$, there is some $a' \in A_J$ with*

$$\|a'\| \leq \|a\|, \quad L(a') \leq L(a), \quad \text{and } \|a - a'\| \leq \varepsilon L(a).$$

If A has an isometric involution being invariant under σ, then when a is self-adjoint we can choose a' also to be self-adjoint.

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Most part of the proof of [23, Theorem 4.1] carries over here. In fact, conditions (2)-(4) here are the same as the conditions (2)-(4) in [23, Theorem 4.1]. Since the proof of Lemma 4.5 in [23] does not involve condition (1) there, this lemma still holds in our current situation. Replacing [23, Lemma 4.4] by Lemma 5.4 in the proof of Lemma 4.6 of [23], we see that the latter also holds in our current situation. To finish the proof of Theorem 5.2, we only need to prove the following analogue of Lemma 4.7 of [23]:

Lemma 5.5. *We have*

$$\| \cdot \|^{\sim} \leq (C \int_{H_0} \ell(h)\, dh + \sum_{s_0 \neq s \in \widehat{H/H_0}} C_s (\dim(s))^2 + r_{\mathcal{A}}) L^{\sim}$$

on $(\tilde{A})_{\rm sa}$, where H_0 is endowed with its normalized Haar measure.

Proof. By Lemma 5.1 the characteristic function φ of H_0 on H is a linear combination of characters of H. Set $n = |H/H_0|$. Let $a \in A_{\rm sa}$. Then $\sigma_{n\varphi}(a)$ belongs to $A_{\rm sa}$ and is fixed by $\sigma|_{H_0}$. We have

$$\begin{aligned} \|a - \sigma_{n\varphi}(a)\| &= \| \int_{H_0} a\, dh - \int_{H_0} \sigma_h(a)\, dh \| \leq \int_{H_0} \|a - \sigma_h(a)\|\, dh \\ &\leq L^{\ell}(a) \int_{H_0} \ell(h)\, dh \leq C \cdot L(a) \int_{H_0} \ell(h)\, dh, \end{aligned}$$

where the last inequality comes from the condition (1). By the condition (2) we have

$$L(\sigma_{n\varphi}(a)) \leq \|n\varphi\|_1 \cdot L(a) = L(a).$$

Note that $A^{\sigma|_{H_0}} = \oplus_{s\in\widehat{H/H_0}} A_s$. Say, $\sigma_{n\varphi}(a) = \sum_{s\in\widehat{H/H_0}} a_s$ with $a_s \in A_s$. For each $s \in \widehat{H/H_0}$, denote by χ_s the corresponding character of H/H_0, thought of as a character of H. Then $a_s = \sigma_{\dim(s)\overline{\chi_s}}(\sigma_{n\varphi}(a))$ [23, Lemma 3.2]. Thus

$$L(a_s) = L(\sigma_{\dim(s)\overline{\chi_s}}(\sigma_{n\varphi}(a)) \leq \|\dim(s)\overline{\chi_s}\|_1 L(\sigma_{n\varphi}(a)) \leq (\dim(s))^2 L(a),$$

where the first inequality comes from the condition (2). Note that $a_{s_0} \in A_{\mathrm{sa}}$. By the condition (5) we have

$$\|a_s\| \leq C_s L(a_s) \leq C_s(\dim(s))^2 L(a)$$

for each $s \in \widehat{H/H_0}$ not equal to s_0. By the condition (4), we have

$$\|b\|^{\sim} \leq r_{\mathcal{A}} L^{\sim}(b)$$

for all $b \in (A_{s_0})_{\mathrm{sa}} = (A^{\sigma})_{\mathrm{sa}}$ [30, Proposition 1.6, Theorem 1.9] [23, Proposition 2.11]. Thus

$$\|a_{s_0}\|^{\sim} \leq r_{\mathcal{A}} L^{\sim}(a_{s_0}) = r_{\mathcal{A}} L(a_{s_0}) \leq r_{\mathcal{A}} L(a).$$

Therefore we have

$$\begin{aligned} \|a\|^{\sim} &\leq \|a - \sigma_{n\varphi}(a)\| + \|a_{s_0}\|^{\sim} + \sum_{s_0\neq s\in\widehat{H/H_0}} \|a_s\| \\ &\leq C \cdot L(a) \int_{H_0} \ell(h)\, dh + r_{\mathcal{A}} L(a) + \sum_{s_0\neq s\in\widehat{H/H_0}} C_s(\dim(s))^2 L(a) \end{aligned}$$

as desired. ∎

This finishes the proof of Theorem 5.2. ∎

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Denote by K_0 the connected component of K containing the identity element e_K. Take an inner product on $\mathfrak{k}$ and use it to get a translation invariant Riemannian metric on K in the usual way. For each $x \in K_0$ set $\ell(x)$ to be the geodesic distance form e_K to x. Then ℓ is a length function on K_0.

In order to prove Theorem 1.1, we just need to verify the conditions in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$. Recall that we are given a norm on $\mathfrak{g}$, and

$$L_\rho(a) = \begin{cases} \sup_{0\neq X\in\mathfrak{g}} \frac{\|\alpha_X(a)\|}{\|X\|}, & \text{if } a \in C^1(\hat{G}/\Gamma, \rho); \\ \infty, & \text{otherwise,} \end{cases} \tag{12}$$

for $a \in C^*(\hat{G}/\Gamma, \rho)$.

By Lemma 2.4 the actions α and β on $C^*(\hat{G}/\Gamma, \rho)$ commute with each other. Thus β preserves $C^1(\hat{G}/\Gamma, \rho)$ and L_ρ.

Choose the basis $X_1, \dots, X_{\dim(G)}$ of $\mathfrak{g}$ in Proposition 3.3 to be of norm 1. Denote by C_1 the supremum of $||F_{j,Y}||$ for all $1 \le j \le \dim(G)$ and Y in the unit sphere of $\mathfrak{k}$ (with respect to the inner product on $\mathfrak{k}$ above) in Proposition 3.3.

Lemma 6.1. *We have $L^\ell \le (dim(G)C_1) \cdot L_\rho$ on $C^*(\hat{G}/\Gamma, \rho)$.*

Proof. It suffices to show $L^\ell \le (\dim(G)C_1) \cdot L_\rho$ on $C^1(\hat{G}/\Gamma, \rho)$. By Proposition 3.3 every $a \in C^1(\hat{G}/\Gamma, \rho)$ is once differentiable with respect to the action β. By [31, Proposition 8.6] we have $L^\ell(a) = \sup_{Y \in \mathfrak{k}, ||Y||=1} ||\beta_Y(a)||$. Then from (5) in Proposition 3.3 we get $L^\ell(a) \le (\dim(G)C_1)L_\rho(a)$. □

Lemma 6.2. *For any linear combination φ of finitely many characters of K we have $L_\rho \circ \beta_\varphi \le ||\varphi||_1 \cdot L_\rho$ on $C^*(\hat{G}/\Gamma, \rho)$.*

Proof. We have remarked above that β preserves L_ρ. By Lemma 3.1 one has

$$L_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{e^X}(a) - a\|}{\|X\|} \tag{13}$$

for every $a \in C^1(\hat{G}/\Gamma, \rho)$. It follows that L_ρ is lower semi-continuous on $C^1(\hat{G}/\Gamma, \rho)$ equipped with the relative topology from $C^1(\hat{G}/\Gamma, \rho) \subseteq C^*(\hat{G}/\Gamma, \rho)$. By Lemma 3.2 the action β is also strongly continuous on $C^1(\hat{G}/\Gamma, \rho)$ with respect to the norm defined in Lemma 3.2. Then β_ψ is also well-defined on $C^1(\hat{G}/\Gamma, \rho)$ for any continuous $\mathbb{C}$-valued function ψ on K. By [23, Remark 4.2.(3)] we get Lemma 6.2. □

The conditions (1) and (2) in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ follow from Lemmas 6.1 and 6.2 respectively.

Fix an inner product on $\mathfrak{g}$, and denote by L'_ρ the seminorm on $C^*(\hat{G}/\Gamma, \rho)$ defined by (12) but using this inner product norm instead. Since $\mathfrak{g}$ is finite dimensional, any two norms on $\mathfrak{g}$ are equivalent. Therefore there exists some constant $C_2 > 0$ not depending on ρ such that $L'_\rho \le C_2 L_\rho$.

By Lemma 4.1 and Proposition 4.2 the restriction of L'_{ρ_0} on $C^1(\hat{G}/\Gamma, \rho_0) \subseteq C(G/\Gamma)$ is the Lipschitz seminorm associated to some metric d on G/Γ. The Arzela-Ascoli theorem [12, Theorem VI.3.8] tells us that the set $\{a \in C^*(\hat{G}/\Gamma, \rho_0) | L_{\rho_0}(a) \le r_1, \|a\| \le r_2\}$ is totally bounded for any $r_1, r_2 > 0$. Since for each $s \in \hat{K}$ neither the seminorm L_ρ nor the

C^*-norm on $B_s \subseteq C^*(\hat{G}/\Gamma, \rho)$ depends on ρ, the condition (3) in Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ follows.

From the criterion of Lip-norms in [30, Proposition 1.6, Theorem 1.9] (see also [23, Proposition 2.11]) one sees that the Lipschitz seminorm associated to the metric on any compact metric space is a Lip-norm on the C^*-algebra of continuous functions on this space. Since L'_{ρ_0} on $C^*(\hat{G}/\Gamma, \rho_0) = C(G/\Gamma)$ is no less than the Lipschitz seminorm associated to the metric d on G/Γ, from [30, Proposition 1.6, Theorem 1.9] one concludes that L_{ρ_0} is also a Lip-norm on $C(G/\Gamma)$. Therefore we may take $(\mathcal{A}, L_{\mathcal{A}})$ in condition (4) of Theorem 5.2 to be $(C(G/\Gamma), L_{\rho_0})$ for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$.

Let $s \in \hat{K}$ not being the trivial representation of K, and let $a \in B_s$. Then $L'_{\rho_0}(a) \leq C_2 L_{\rho_0}(a) = C_2 L_\rho(a)$. Thus for any λ in the range of a on G/Γ one has $\|a - \lambda 1_{C(G/\Gamma)}\|_{C(G/\Gamma)} \leq C_2 C_3 L_\rho(a)$, where C_3 denotes the diameter of G/Γ under the metric d. We have

$$\begin{aligned} \|a\|_{C^*(\hat{G}/\Gamma,\rho)} &= \|a\|_{C(G/\Gamma)} = \| \int_K \overline{\langle k, s\rangle} \beta_k(a - \lambda 1_{C(G/\Gamma)})\, dk \|_{C(G/\Gamma)} \\ &\leq \|a - \lambda 1_{C(G/\Gamma)}\|_{C(G/\Gamma)} \leq C_2 C_3 L_\rho(a). \end{aligned}$$

This establishes the condition (5) of Theorem 5.2 for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$.

We have shown that the conditions in Theorem 5.2 hold for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$. Thus Theorem 1.1 follows from Theorem 5.2.

7. Quantum Gromov-Hausdorff distance

In this section we prove Theorem 1.2.

We recall first the definition of the distance $\mathrm{dist}_{\mathrm{nu}}$ from [19, Section 5]. To simplify the notation, for fixed unital C^*-algebras A_1 and A_2, when we take infimum over unital C^*-algebras B containing both A_1 and A_2, we mean to take infimum over all unital isometric $*$-homomorphisms of A_1 and A_2 into some unital C^*-algebra B. Denote by $\mathrm{dist}^B_{\mathrm{H}}$ the Hausdorff distance between subsets of B. For a C^*-algebraic compact quantum metric spaces (A, L_A), set

$$\mathcal{E}(A) = \{a \in A_{\mathrm{sa}} | L_A(a) \leq 1\}.$$

For any C^*-algebraic compact quantum metric spaces (A_1, L_{A_1}) and (A_2, L_{A_2}), the distance $\mathrm{dist}_{\mathrm{nu}}(A_1, A_2)$ is defined as

$$\mathrm{dist}_{\mathrm{nu}}(A_1, A_2) = \inf \mathrm{dist}^B_{\mathrm{H}}(\mathcal{E}(A_1), \mathcal{E}(A_2)),$$

where the infimum is taken over all unital C^*-algebras B containing A_1 and A_2.

Throughout the rest of this section, we fix G, Γ, K such that there exits ρ satisfying the conditions (S1)-(S5). We also fix a norm on $\mathfrak{g}$. Denote by Ω the set of all ρ satisfying the conditions (S1) and (S2), equipped with the weakest topology making the maps $\Omega \to G$ sending ρ to $\rho(s)$ to be continuous for each $s \in \hat{K}$.

Every closed subgroup of a Lie group is also a Lie group [37, Theorem 3.42]. Thus K is a compact abelian Lie group. Then K is the product of a torus and a finite abelian group [7, Corollary 3.7]. Therefore $\hat{K}$ is finitely generated. Let $s_1, \dots, s_n$ be a finite subset of $\hat{K}$ generating $\hat{K}$. Then the map $\varphi : \Omega \to \prod_{j=1}^{n} G$ sending ρ to $(\rho(s_1), \dots, \rho(s_n))$ is injective, and its image is closed. Furthermore, it is easily checked that the topology on Ω is exactly the pullback of the relative topology of $\varphi(\Omega)$ in $\prod_{j=1}^{n} G$. Since G is a Lie group, it is locally compact metrizable. Thus $\prod_{j=1}^{n} G$ and Ω are also locally compact metrizable.

For clarity and convenience, we shall denote the actions α and β on $C^*(\hat{G}/\Gamma, \rho)$ by α_ρ and β_ρ respectively, and denote the C^*-norm on $C^*(\hat{G}/\Gamma, \rho)$ by $\| \cdot \|_\rho$. Consider the (possibly $+\infty$-valued) auxiliary seminorm L''_ρ on $C^*(\hat{G}/\Gamma, \rho)$ defined by

$$L''_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{\rho, e^X}(a) - a\|_\rho}{\|X\|}.$$

Lemma 7.1. *Let W be a locally compact Hausdorff space with a continuous map $W \to \Omega$ sending w to ρ_w. Let f be a continuous section of the continuous field of C^*-algebras over W in Proposition 2.9. Then the function $w \mapsto L''_{\rho_w}(f_w)$ is lower semi-continuous on W.*

Proof. Let $w' \in W$. To show that the above function is lower semi-continuous at w', we consider the case $L''_{\rho_{w'}}(f_{w'}) < \infty$. The case $L''_{\rho_{w'}}(f_{w'}) = \infty$ can be dealt with similarly. Let $\varepsilon > 0$. Take $0 \neq X \in \mathfrak{g}$ such that

$$L''_{\rho_{w'}}(f_{w'})\|X\| < \|\alpha_{\rho_{w'}, e^X}(f_{w'}) - f_{w'}\|_{\rho_{w'}} + \varepsilon\|X\|.$$

It is easily checked that $w \mapsto \alpha_{\rho_w, e^X}(f_w)$ is also a continuous section of the continuous field. Then when w is close enough to w', we have

$$\|\alpha_{\rho_{w'}, e^X}(f_{w'}) - f_{w'}\|_{\rho_{w'}} < \|\alpha_{\rho_w, e^X}(f_w) - f_w\|_{\rho_w} + \varepsilon\|X\|$$

and hence

$$L''_{\rho_{w'}}(f_{w'})\|X\| < \|\alpha_{\rho_w, e^X}(f_w) - f_w\|_{\rho_w} + 2\varepsilon\|X\| \leq (L''_{\rho_w}(f_w) + 2\varepsilon)\|X\|.$$

Therefore $L''_{\rho_{w'}}(f_{w'}) \leq L''_{\rho_w}(f_w) + 2\varepsilon$. $\square$

Note that although the $*$-algebra structure of $C_{0,1}(G/\Gamma,\rho)$ ($C_{\mathrm{b},1}(G,\rho)$ resp.) depends on ρ, the Banach space structure, the left translation action of G and the right translation action of K on $C_{0,1}(G/\Gamma,\rho)$ ($C_{\mathrm{b},1}(G,\rho)$ resp.) do not depend on ρ. Thus we may denote by $C_{0,1}(G/\Gamma)$, α and β this Banach space and these actions respectively. Also denote by $C^1_{0,1}(G/\Gamma)$ the set of once differentiable elements of $C_{0,1}(G/\Gamma)$ with respect to α.

Lemma 7.2. *For any a in $\oplus_{s\in\hat{K}}(B_s\cap C^1_{0,1}(G/\Gamma))$, the function $\rho\mapsto L_\rho(a)$ is continuous on Ω.*

Proof. Say, $a=\sum_{s\in F}a_s$ for some finite subset F of $\hat{K}$ and $a_s\in B_s\cap C^1_{0,1}(G/\Gamma)$ for each $s\in F$. Then $L_\rho(a)=\sup_{X\in\mathfrak{g},\|X\|=1}\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ for each $\rho\in\Omega$. Since α commutes with β, we have $\alpha_X(a_s)\in B_s$. By Proposition 2.9 the function $\rho\mapsto\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on Ω for each $X\in\mathfrak{g}$. Since $\mathfrak{g}$ is a finite-dimensional vector space and $\alpha_X(a_s)$ depends on X linearly, it follows easily that the function $(X,\rho)\mapsto\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on $\mathfrak{g}\times\Omega$. As the unit sphere of $\mathfrak{g}$ is compact, one concludes that the function $\rho\mapsto\sup_{X\in\mathfrak{g},\|X\|=1}\|\sum_{s\in F}\alpha_X(a_s)\|_\rho$ is continuous on Ω. □

Fix $\rho'\in\Omega$. Let Z be a compact neighborhood of ρ' in Ω.

Note that the linear span of $\rho\mapsto f(\rho)a\in C^*(\hat{G}/\Gamma,\rho)$ for a in some B_s and $f\in C(Z)$ is dense in the C^*-algebra of continuous sections of the continuous field over Z in Proposition 2.9. Since Z is a compact metrizable space, $C(Z)$ is separable. As G is a Lie group, it is separable. Then G/Γ is separable, and hence is a compact metrizable space. Thus $C(G/\Gamma)$ is separable, and hence B_s is separable for each $s\in\hat{K}$. On the other hand, since $\hat{K}$ is finitely generated, $\hat{K}$ is countable. Therefore the C^*-algebra of continuous sections of the continuous field over Z in Proposition 2.9 is separable.

By Proposition 2.5 each $C^*(\hat{G}/\Gamma,\rho)$ is nuclear. Every separable continuous field of unital nuclear C^*-algebras over a compact metric space can be subtrivialized [6, Theorem 3.2]. Thus we can find a unital C^*-algebra B and unital embeddings $C^*(\hat{G}/\Gamma,\rho)\to B$ for all $\rho\in Z$ such that, via identifying each $C^*(\hat{G}/\Gamma,\rho)$ with its image in B, the continuous sections of the continuous field over Z in Proposition 2.9 are exactly the continuous maps $Z\to B$ whose images at each ρ are in $C^*(\hat{G}/\Gamma,\rho)$.

For any C^*-algebraic compact quantum metric space (A,L_A) and any constant R no less than the radius of (A,L_A), the set $D_R(A):=\{a\in A_{\mathrm{sa}}|L_A(a)\le 1,\|a\|\le R\}$ is totally bounded and every $a\in\mathcal{E}(A)$ can be written as $x+\lambda$ for some $x\in D_R(A)$ and $\lambda\in\mathbb{R}$ [30, Proposition

1.6, Theorem 1.9]. In Section 6 we have seen that the conditions in Theorem 5.2 hold for $(A, L, H, H_0, \sigma) = (C^*(\hat{G}/\Gamma, \rho), L_\rho, K, K_0, \beta)$ with some C, C_s and $(\mathcal{A}, L_{\mathcal{A}})$ not depending on ρ. Thus, by Theorem 5.2 there is some constant R such that the radius of $(C^*(\hat{G}/\Gamma, \rho_\rho), L_\rho)$ is no bigger than R for all $\rho \in \Omega$. For any $\varepsilon > 0$, by Lemmas 5.4 and 2.4 there is a finite subset $F \subseteq \hat{K}$ satisfying that for any $\rho \in \Omega$ and any $x \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ there is some $y \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho)) \cap \sum_{s\in F} B_s$ with $\|y\|_\rho \leq \|x\|_\rho$ and $\|x - y\|_\rho < \varepsilon$.

Lemma 7.3. *Let $\varepsilon > 0$. Then there is a neighborhood U of ρ' in Z such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ with $\|a - b\|_B < \varepsilon$.*

Proof. According to the discussion above we can find a finite subset Y of $\mathcal{E}(C^*(\hat{G}/\Gamma, \rho')) \cap \sum_{s\in F} B_s$ such that for every $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ there are some $z \in Y$ and $\lambda \in \mathbb{R}$ with $\|a - (z + \lambda)\|_{\rho'} < \varepsilon$. For each $y \in Y$, write y as $\sum_{s\in F} y_s$ with $y_s \in B_s$. Since $L_{\rho'}(y) < \infty$, y is once differentiable with respect to $\alpha_{\rho'}$. It is easy to see that each y_s is once differentiable with respect to $\alpha_{\rho'}$. Thus, by Lemma 7.2 the function $\rho \mapsto L_\rho(y)$ is continuous on Ω. Then we can find a constant $\delta > 0$ and a neighborhood U of ρ' in Z such that $\delta\|y_\rho\|_\rho < \varepsilon$, $\|y_{\rho'} - y_\rho\|_B < \varepsilon$, and $L_\rho(y_\rho) < 1 + \delta$ for all $y \in Y$ and $\rho \in U$, where y_ρ denotes y as an element in $C^*(\hat{G}/\Gamma, \rho)$. Fix $\rho \in U$. Set $b = z_\rho/(1 + \delta)$. Then $L_\rho(b + \lambda) = L_\rho(b) < 1$, and

$$\begin{aligned}\|a - (b + \lambda)\|_B &\leq \|a - (z_{\rho'} + \lambda)\|_{\rho'} + \|z_{\rho'} - z_\rho\|_B + \|z_\rho - b\|_\rho \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

$\square$

Lemma 7.4. *Let $\varepsilon > 0$. Then there is a neighborhood U of ρ' in Z such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho))$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ with $\|a - b\|_B < \varepsilon$.*

Proof. According to the discussion before Lemma 7.3, it suffices to show that there is a neighborhood U of ρ' in Z such that for any $\rho \in U$ and any $a \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho)) \cap \oplus_{s\in F} B_s$ satisfying $\|a\|_\rho \leq R$ there is some $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$ with $\|a - b\|_B < \varepsilon$. Suppose that this fails. Then we can find a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in Z converging to ρ' and an $a_n \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho_n)) \cap \oplus_{s\in F} B_s$ satisfying $\|a_n\|_{\rho_n} \leq R$ for each $n \in \mathbb{N}$ such that $\|a_n - b\|_B \geq \varepsilon$ for all $n \in \mathbb{N}$ and $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$. Write a_n as $\sum_{s\in F} a_{n,s}$ with $a_{n,s} \in B_s$. Then $a_{n,s} = \int_K \overline{\langle k, s\rangle} \beta_{\rho_n,k}(a_n)\, dk$. Thus $\|a_{n,s}\|_{\rho_n} \leq \|a_n\|_{\rho_n} \leq R$ and $L_{\rho_n}(a_{n,s}) \leq L_{\rho_n}(a_n) \leq 1$ by Lemma 6.2. Since the restriction of L_ρ on B_s does not depend on ρ, and the set $\{a \in$

$B_s | L_\rho(a) \leq 1, \|a\| \leq R\}$ is totally bounded, passing to a subsequence if necessary, we may assume that $a_{n,s}$ converges to some a_s in B_s when $n \to \infty$ for each $s \in F$. Set $a = \sum_{s\in F} a_s$. Then $(a_n)_{\rho_n}$ converges to $a_{\rho'}$ in B as $n \to \infty$, where $(a_n)_{\rho_n}$ and $a_{\rho'}$ denote a_n and a as elements in $C^*(\hat{G}/\Gamma, \rho_n)$ and $C^*(\hat{G}/\Gamma, \rho')$ respectively. In particular, a is self-adjoint and $\|a\|_{\rho'} \leq \lim_{n\to\infty} \|a_n\|_{\rho_n} \leq R$.

By Lemma 3.1 we have $L''_{\rho_n}(a_n) = L_{\rho_n}(a_n) \leq 1$ for all $n \in \mathbb{N}$. On the one-point compactification $W = \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$, consider the continuous map $W \to \Omega$ sending $n \in \mathbb{N}$ to ρ_n and ∞ to ρ'. Then the section f defined as $f_n = a_n \in C^*(\hat{G}/\Gamma, \rho_n)$ for $n \in \mathbb{N}$ and $f_\infty = a \in C^*(\hat{G}/\Gamma, \rho')$ is a continuous section of the continuous field on W in Proposition 2.9. Thus, by Lemma 7.1 we have $L''_{\rho'}(a) \leq \liminf_{n\to\infty} L''_{\rho_n}(a_n) \leq 1$. By Lemma 3.4 we can find some self-adjoint $b \in C^1(\hat{G}/\Gamma, \rho')$ with $\|b\|_{\rho'} \leq \|a\|_{\rho'} \leq R$, $\|b - a\|_{\rho'} \leq \varepsilon/2$, and $L_{\rho'}(b) \leq L''_{\rho'}(a) \leq 1$. Then $b \in \mathcal{E}(C^*(\hat{G}/\Gamma, \rho'))$, and

$$\|b - a_n\|_B \to \|b - a\|_{\rho'} \leq \varepsilon/2$$

as $n \to \infty$. Therefore, when n is large enough, we have $\|b - a_n\|_B < \varepsilon$, contradicting our assumption. This finishes the proof of the lemma. $\square$

From Lemmas 7.3 and 7.4 we conclude that Theorem 1.2 holds.

Appendix A. Comparison of $\mathrm{dist}_{\mathrm{nu}}$ and prox

In this appendix we compare the distance $\mathrm{dist}_{\mathrm{nu}}$ and the proximity Rieffel introduced in [35].

A (possibly $+\infty$-valued) seminorm L on a unital (possibly incomplete) C^*-norm algebra A is called a *C^*-metric* [35, Definition 4.1] if

(1) L is lower semi-continuous, satisfies the reality condition (9), and is *strongly-Leibniz* in the sense that $L(ab) \leq L(a)\|b\| + \|a\|L(b)$ for all $a, b \in A$, $L(1_A) = 0$, and $L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$ for all a being invertible in A,
(2) L extended to the completion $\bar{A}$ of A by $L(a) = \infty$ for $a \in \bar{A} \setminus A$ is a Lip-norm on $\bar{A}$,
(3) the algebra $\{a \in A | L(a) < \infty\}$ is spectrally stable in $\bar{A}$.

In such case, the pair (A, L) is called a *compact C^*-metric space.*

The seminorm L_ρ in Theorem 1.1 may fail to be a C^*-metric since it may fail to be lower semi-continuous. However, it is lower semi-continuous on $C^1(\hat{G}/\Gamma, \rho)$ by Lemma 3.1. Thus its restriction on the algebra of smooth elements in $C^*(\hat{G}/\Gamma, \rho)$ with respect to α is a C^*-metric. By [35, Proposition 3.2] its closure $\bar{L}_\rho$ is a C^*-metric on

$C^*(\hat{G}/\Gamma, \rho)$. Lemma 3.4 tells us that

$$\bar{L}_\rho(a) = \sup_{0 \neq X \in \mathfrak{g}} \frac{\|\alpha_{e^X}(a) - a\|}{\|X\|}$$

for all $a \in C^*(\hat{G}/\Gamma, \rho)$.

In [35, Definition 5.6, Section 14] Rieffel introduced the notions of *proximity* $\mathrm{prox}(A, B)$ and *complete proximity* $\mathrm{prox}_s(A, B)$ between two compact C^*-metric spaces (A, L_A) and (B, L_B). In general, one has $\mathrm{prox}_s(A, B) \geq \mathrm{prox}(A, B)$. For each $q \in \mathbb{N}$, denote by $\mathrm{UCP}_q(A)$ the set of unital completely positive linear maps from the completion $\bar{A}$ of A to $M_q(\mathbb{C})$. Define $\mathrm{prox}^q(A, B)$ as the infimum of the Hausdorff distance of $\mathrm{UCP}_q(A)$ and $\mathrm{UCP}_q(B)$ in $\mathrm{UCP}_q(A \oplus B)$ under the metric d^q_L, for L running through C^*-metrics L on $A \oplus B$ whose quotients on A and B agree with L_A and L_B on A_{sa} and B_{sa} respectively. Here the metric d^q_L is defined as

$$d^q_L(\varphi, \psi) = \sup_{L(a,b) \leq 1} \|\varphi(a, b) - \psi(a, b)\|.$$

Then $\mathrm{prox}_s(A, B)$ is defined as $\sup_q \mathrm{prox}^q(A, B)$.

Note that the definition of $\mathrm{dist}_{\mathrm{nu}}$ extends to compact C^*-metric spaces (A, L_A) and (B, L_B) directly.

Theorem A.1. *For any compact C^*-metric spaces (A, L_A) and (B, L_B), one has*

$$\mathrm{dist}_{\mathrm{nu}}(A, B) \geq \mathrm{prox}_s(A, B).$$

Proof. The proof is similar to those of [24, Proposition 4.7] and [19, Theorem 3.7]. Let $\mathcal{A}$ be a unital C^*-algebra containing $\bar{A}$ and $\bar{B}$. Set $c = \mathrm{dist}^{\mathcal{A}}_{\mathrm{H}}(\mathcal{E}(A), \mathcal{E}(B))$. Let $\varepsilon > 0$. Define a seminorm L on $A \oplus B$ by

$$L(a, b) = \max(L_A(a), L_B(b), \frac{\|a - b\|}{c + \varepsilon}).$$

It was pointed in the proof of [24, Proposition 4.7] that L extended to $\overline{A \oplus B} = \bar{A} \oplus \bar{B}$ as in the condition (2) of the definition of C^*-metrics above is a Lip-norm, and that the quotients of L on A and B agree with L_A and L_B on A_{sa} and B_{sa} respectively. It is readily checked that L satisfies the conditions (1) and (3) in the definition of C^*-metrics. Thus L is a C^*-metric on $A \oplus B$. For any $q \in \mathbb{N}$ and $\varphi \in \mathrm{UCP}_q(A)$, by Arveson's extension theorem [8, Theorem 1.6.1] extend φ to a ϕ in $\mathrm{UCP}_q(\mathcal{A})$. Set ψ to be the restriction of ϕ on $\bar{B}$. For any $(a, b) \in \mathcal{E}(A \oplus B)$ one has

$$\|\varphi(a, b) - \psi(a, b)\| = \|\varphi(a) - \psi(b)\| = \|\phi(a - b)\| \leq \|a - b\| \leq c + \varepsilon.$$

Thus $d^q_L(\varphi,\psi) \le c+\varepsilon$. Similarly, for any $\psi' \in \mathrm{UCP}_q(B)$, we can find some $\varphi' \in \mathrm{UCP}_q(A)$ with $d^q_L(\varphi',\psi') \le c+\varepsilon$. Therefore $\mathrm{prox}^q(A,B) \le c+\varepsilon$. It follows that $\mathrm{prox}^q(A,B) \le \mathrm{dist}_{\mathrm{nu}}(A,B)$, and hence $\mathrm{prox}_s(A,B) \le \mathrm{dist}_{\mathrm{nu}}(A,B)$ as desired. $\square$

It was pointed out in Section 5 of [19] that one has continuity of quantum tori and θ-deformation, convergence of matrix algebras to integral coadjoint orbits of compact connected semisimple Lie groups, and approximation of quantum tori by finite quantum tori with respect to $\mathrm{dist}_{\mathrm{nu}}$. It follows from Theorem A.1 that we also have such continuity, convergence and approximation with respect to prox_s and prox. In particular, this yields a new proof for [35, Theorem 14.1].

References

[1] B. Abadie. Generalized fixed-point algebras of certain actions on crossed products. *Pacific J. Math.* **171** (1995), no. 1, 1–21. arXiv:funct-an/9301005.

[2] B. Abadie. The range of traces on quantum Heisenberg manifolds. *Trans. Amer. Math. Soc.* **352** (2000), no. 12, 5767–5780 (electronic).

[3] B. Abadie. Morita equivalence for quantum Heisenberg manifolds. *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3515–3523 (electronic). arXiv:math.OA/0503466.

[4] B. Abadie and R. Exel. Hilbert C^*-bimodules over commutative C^*-algebras and an isomorphism condition for quantum Heisenberg manifolds. *Rev. Math. Phys.* **9** (1997), no. 4, 411–423. arXiv:funct-an/9609001.

[5] B. Abadie and R. Exel. Deformation quantization via Fell bundles. *Math. Scand.* **89** (2001), no. 1, 135–160. arXiv:funct-an/9706001.

[6] É. Blanchard. Subtriviality of continuous fields of nuclear C^*-algebras. *J. Reine Angew. Math.* **489** (1997), 133–149. math.OA/0012128.

[7] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*. Graduate Texts in Mathematics, 98. Springer-Verlag, New York, 1995.

[8] N. P. Brown and N. Ozawa. *C^*-algebras and Finite-dimensional Approximations*. Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[9] P. S. Chakraborty. Metrics on the quantum Heisenberg manifold. *J. Operator Theory* **54** (2005), no. 1, 93–100. arXiv:math.OA/0112309.

[10] A. Connes and M. Dubois-Violette. Moduli space and structure of noncommutative 3-spheres. *Lett. Math. Phys.* **66** (2003), no. 1-2, 91–121. arXiv:math.QA/0308275.

[11] A. Connes and M. Dubois-Violette. Noncommutative finite dimensional manifolds. II. Moduli space and structure of noncommutative 3-spheres. *Comm. Math. Phys.* **281** (2008), no. 1, 23–127. arXiv:math.QA/0511337.

[12] J. B. Conway. *A Course in Functional Analysis*. Second edition. Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.

[13] J. Dixmier. *C^*-Algebras*. Translated from the French by Francis Jellett. North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[14] S. Doplicher, R. Longo, J. E. Roberts, and L. Zsidó. A remark on quantum group actions and nuclearity. Dedicated to Professor Huzihiro Araki on the occasion of his 70th birthday. *Rev. Math. Phys.* **14** (2002), no. 7-8, 787–796. arXiv:math.OA/0204029.

[15] R. S. Doran and J. M. G. Fell. *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles.* Pure and Applied Mathematics, 125 and 126. Academic Press, Inc., Boston, MA, 1988.

[16] W. Fulton and J. Harris. *Representation Theory. A First Course.* Graduate Texts in Mathematics, 129. Springer-Verlag, New York, 1991.

[17] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), no. 1, 65–222.

[18] D. Kerr. Matricial quantum Gromov-Hausdorff distance. *J. Funct. Anal.* **205** (2003), no. 1, 132–167. arXiv:math.OA/0207282.

[19] D. Kerr and H. Li. On Gromov-Hausdorff convergence for operator metric spaces. *J. Operator Theory* to appear. arXiv:math.OA/0411157.

[20] M. B. Landstad. Traces on noncommutative homogeneous spaces. *J. Funct. Anal.* **191** (2002), no. 2, 211–223. arXiv:math.OA/0104067.

[21] M. B. Landstad, and I. Raeburn. Twisted dual-group algebras: equivariant deformations of $C_0(G)$. *J. Funct. Anal.* **132** (1995), no. 1, 43–85.

[22] M. B. Landstad, and I. Raeburn. Equivariant deformations of homogeneous spaces. *J. Funct. Anal.* **148** (1997), no. 2, 480–507.

[23] H. Li. θ-deformations as quantum compact metric spaces. *Comm. Math. Phys.* **256** (2005), no. 1, 213–238. arXiv:math.OA/0311500.

[24] H. Li. Order-unit quantum Gromov-Hausdorff distance. *J. Funct. Anal.* **231** (2006), no. 2, 312–360. arXiv:math.OA/0312001.

[25] H. Li. Compact quantum metric spaces and ergodic actions of compact quantum groups. *J. Funct. Anal.* **256** (2009), no. 10, 3368–3408. arXiv:math.OA/0411178.

[26] H. Li. C^*-algebraic quantum Gromov-Hausdorff distance. arXiv:math.OA/0312003.

[27] A. L. T. Paterson. *Amenability.* Mathematical Surveys and Monographs, 29. American Mathematical Society, Providence, RI, 1988.

[28] M. A. Rieffel. Deformation quantization of Heisenberg manifolds. *Comm. Math. Phys.* **122** (1989), no. 4, 531–562.

[29] M. A. Rieffel. Continuous fields of C^*-algebras coming from group cocycles and actions. *Math. Ann.* **283** (1989), no. 4, 631–643.

[30] M. A. Rieffel. Metrics on states from actions of compact groups. *Doc. Math.* **3** (1998), 215–229 (electronic). arXiv:math.OA/9807084.

[31] M. A. Rieffel. Gromov-Hausdorff distance for quantum metric spaces. *Mem. Amer. Math. Soc.* **168** (2004), no. 796, 1–65. arXiv:math.QA/0011063.

[32] M. A. Rieffel. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. *Mem. Amer. Math. Soc.* **168** (2004), no. 796, 67–91. arXiv:math.QA/0108005.

[33] M. A. Rieffel. Compact quantum metric spaces. In: *Operator Algebras, Quantization, and Noncommutative Geometry*, 315–330, Contemp. Math., 365, Amer. Math. Soc., Providence, RI, 2004. arXiv:math.QA/0308207.

[34] M. A. Rieffel. Vector bundles and Gromov-Hausdorff distance. *J. K-theory* to appear. arXiv:math/0608266.

[35] M. A. Rieffel. Leibniz seminorms for "Matrix algebras converge to the sphere". arXiv:0707.3229.

[36] M. Takesaki. *Theory of Operator Algebras. I.* Encyclopaedia of Mathematical Sciences, 124. Springer-Verlag, Berlin, 2002.

[37] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Corrected reprint of the 1971 edition. Graduate Texts in Mathematics, 94. Springer-Verlag, New York-Berlin, 1983.

[38] N. Weaver. Sub-Riemannian metrics for quantum Heisenberg manifolds. *J. Operator Theory* **43** (2000), no. 2, 223–242. arXiv:math.OA/9801014.

[39] W. Wu. Quantized Gromov-Hausdorff distance. *J. Funct. Anal.* **238** (2006), no. 1, 58–98.

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