HOMOCLINICALLY EXPANSIVE ACTIONS AND A GARDEN OF EDEN THEOREM FOR HARMONIC MODELS

TULLIO CECCHERINI-SILBERSTEIN, MICHEL COORNAERT, AND HANFENG LI

Abstract. Let Γ be a countable Abelian group and $f \in \mathbb{Z}[\Gamma]$, where $\mathbb{Z}[\Gamma]$ denotes the integral group ring of Γ. Consider the Pontryagin dual X_f of the cyclic $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ and suppose that f is weakly expansive (e.g., f is invertible in $\ell^1(\Gamma)$, or, when Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, f is well-balanced) and that X_f is connected. We prove that if $\tau \colon X_f \to X_f$ is a Γ-equivariant continuous map, then τ is surjective if and only if the restriction of τ to each Γ-homoclinicity class is injective. We also show that this equivalence remains valid in the case when $\Gamma = \mathbb{Z}^d$ and $f \in \mathbb{Z}[\Gamma] = \mathbb{Z}[u_1, u_1^{-1}, \dots, u_d, u_d^{-1}]$ is an irreducible atoral polynomial whose zero-set $Z(f)$ satisfies some suitable finiteness conditions (e.g., when $d \geq 2$ such that $Z(f)$ is finite). These two results are analogues of the classical Garden of Eden theorem of Moore and Myhill for cellular automata with finite alphabet over Γ.

Contents

Date: August 28, 2018.

2010 *Mathematics Subject Classification.* 37D20, 37C29, 54H20, 37A45, 22D32, 22D45.

Key words and phrases. Garden of Eden theorem, finitely presented algebraic action, principal algebraic action, harmonic model, homoclinic group, Pontryagin duality, topological rigidity, Moore property, Myhill property, expansive action, homoclinic expansive action, weakly expansive polynomial.

1. Introduction

Consider a dynamical system (X, α), consisting of a compact metrizable space X, called the *phase space*, equipped with a continuous action α of a countable group Γ. Let d be a metric on X that is compatible with the topology. Two points $x, y \in X$ are said to be *homoclinic* if $\lim_{\gamma \to \infty} d(\gamma x, \gamma y) = 0$, i.e., for every $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ such that $d(\gamma x, \gamma y) < \varepsilon$ for all $\gamma \in \Gamma \setminus F$. Homoclinicity is an equivalence relation on X. By compactness of X, this relation does not depend on the choice of the compatible metric d. A map with source set X is called *pre-injective* (with respect to α) if its restriction to each homoclinicity class is injective.

An *endomorphism* of the dynamical system (X, α) is a continuous map $\tau\colon X \to X$ that is Γ-equivariant (i.e., $\tau(\gamma x) = \gamma \tau(x)$ for all $\gamma \in \Gamma$ and $x \in X$).

The original Garden of Eden theorem is a statement in symbolic dynamics that characterizes surjective endomorphisms of shift systems with finite alphabet. To be more specific, let us fix a compact metrizable space A, called the *alphabet*. Given a countable group Γ, the *shift* over the group Γ with alphabet A is the dynamical system (A^Γ, σ), where $A^\Gamma = \{x\colon \Gamma \to A\}$ is equipped with the product topological group structure and σ is the action defined by $\gamma x(\gamma') := x(\gamma^{-1}\gamma')$ for all $x \in A^\Gamma$ and $\gamma, \gamma' \in \Gamma$. The *Garden of Eden theorem* states that, under the hypotheses that the group Γ is amenable and the alphabet A is finite, an endomorphism of (A^Γ, σ) is surjective if and only if it is pre-injective. It was first proved for $\Gamma = \mathbb{Z}^d$ by Moore and Myhill in the early 1960s. Actually, the implication surjective $\implies$ pre-injective was first proved by Moore in [27] while the converse implication was established shortly after by Myhill in [29]. The Garden of Eden theorem was subsequently extended to finitely generated groups of subexponential growth by Machì and Mignosi [25] and finally to all countable amenable groups by Machì, Scarabotti, and the first author in [8].

Let us say that the dynamical system (X, α) has the *Moore property* if every surjective endomorphism of (X, α) is pre-injective and that it has the *Myhill property* if every pre-injective endomorphism of (X, α) is surjective. We say that the dynamical system (X, α) has the *Moore-Myhill property*, or that it satisfies the *Garden of Eden theorem*, if it has both the Moore and the Myhill properties.

The goal of the present paper is to establish a version of the Garden of Eden theorem for principal algebraic dynamical systems associated with weakly expansive polynomials over countable Abelian groups and with connected phase space. By an *algebraic dynamical system*, we mean a dynamical system of the form (X, α), where X is a compact metrizable Abelian group and α is an action of a countable group Γ on X by continuous group automorphisms. Note that, in this case, the set $\Delta(X, \alpha) = \{x \in X : x \text{ is homoclinic to } 0_X\} \subset X$, where 0_X is the identity element of X, is a subgroup of X, called the *homoclinic group* and two points $x, y \in X$ are homoclinic if and only if $x - y \in \Delta(X, \alpha)$, that is, they belong to the same coset of $\Delta(X, \alpha)$ in X. By Pontryagin duality, algebraic dynamical systems with acting group Γ are in one-to-one correspondence with countable left $\mathbb{Z}[\Gamma]$-modules. Here $\mathbb{Z}[\Gamma]$ denotes the integral group ring of Γ. This correspondence has been intensively studied in the last decades and revealed fascinating connections between commutative algebra, number theory, harmonic analysis, ergodic theory, and dynamical systems (see in particular the monograph [31] and the survey [20]).

Let $f \in \mathbb{Z}[\Gamma]$ and consider the cyclic left $\mathbb{Z}[\Gamma]$-module $M_f \coloneqq \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ obtained by quotienting the ring $\mathbb{Z}[\Gamma]$ by the principal left ideal generated by f. The algebraic dynamical system associated by Pontryagin duality with M_f is denoted by (X_f, α_f) and is called the *principal algebraic dynamical system* associated with f.

We denote by $\mathcal{C}_0(\Gamma)$ the real vector space of all functions $g \colon \Gamma \to \mathbb{R}$ vanishing at infinity (i.e., for every $\varepsilon > 0$ there exists a finite subset $\Omega \subset \Gamma$ such that $|g_\gamma| < \varepsilon$ for all $\gamma \in \Gamma \setminus \Omega$). Moreover, for $f \in \mathbb{Z}[\Gamma]$ and $g \in \mathcal{C}_0(\Gamma)$ we denote by $fg \in \mathcal{C}_0(\Gamma)$ their convolution product (see Subsection 2.3).

Definition 1.1. A polynomial $f \in \mathbb{Z}[\Gamma]$ is said to be *weakly expansive* provided:
(we-1) $\forall g \in \mathcal{C}_0(\Gamma)$, $fg = 0 \Rightarrow g = 0$;
(we-2) $\exists \omega \in \mathcal{C}_0(\Gamma)$ such that $f\omega = 1_\Gamma$.

Our first result is the following.

Theorem 1.2 (Garden of Eden theorem for algebraic actions associated with weakly expansive polynomials)**.** *Let Γ be a countable Abelian group and $f \in \mathbb{Z}[\Gamma]$. Suppose that f is weakly expansive and that X_f is connected. Then the dynamical system (X_f, α_f) has the Moore-Myhill property.*

There are two main ingredients in our proof of Theorem 1.2. The first one, Corollary 4.5, is a rigidity result (a generalization of [1, Corollary 1]) for algebraic dynamical systems associated with weakly expansive polynomials and with connected phase space. We use it to prove that, under the above conditions, every endomorphism of (X_f, α_f) is affine with linear part of the form $x \mapsto rx$ for some $r \in \mathbb{Z}[\Gamma]$. The second one, Theorem 3.9, a generalization of [19, Lemma 4.5]), asserts that, if f is weakly expansive, then the homoclinic group $\Delta(X_f, \alpha_f)$, equipped with the induced action of Γ, is dense in X_f and isomorphic, as a $\mathbb{Z}[\Gamma]$-module, to $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f^*$, where $f^* \in \mathbb{Z}[\Gamma]$ is defined by $(f^*)_\gamma \coloneqq f_{\gamma^{-1}}$ for all $\gamma \in \Gamma$.

Recall that a dynamical system (X, α) is called *expansive* if there exists a constant $\varepsilon_0 > 0$ such that, for every pair of distinct points $x, y \in X$, there exists an element $\gamma \in \Gamma$ such that $d(\gamma x, \gamma y) > \varepsilon_0$. Such a constant ε_0 is called an *expansivity constant* for (X, α, d). The fact that (X, α) is expansive or not does not depend on the choice of the metric d. For instance, the shift system (A^Γ, σ) is expansive for every countable group Γ whenever the alphabet A is finite.

Let $f \in \mathbb{Z}[\Gamma]$ and suppose that the associated principal algebraic dynamical system (X_f, α_f) is expansive. In Corollary 3.12 we show that f is weakly expansive, and from Theorem 1.2 we thus deduce the following:

Corollary 1.3. *Let Γ be a countable Abelian group and $f \in \mathbb{Z}[\Gamma]$. Suppose that the principal algebraic dynamical system (X_f, α_f) associated with f is expansive and that X_f is connected. Then the dynamical system (X_f, α_f) has the Moore-Myhill property.* □

This result was obtained by the first two named authors in [7] and, shortly after, as a particular case of a much more general Garden of Eden theorem for expansive actions (where Γ is amenable and no connectedness of the phase space is assumed) proved by the third author in [18].

Recall that a polynomial $f \in \mathbb{R}[\Gamma]$ is said to be *well-balanced* (cf. [4, Definition 1.2]) if:

(wb-1) $\sum_{\gamma \in \Gamma} f_\gamma = 0$,
(wb-2) $f_\gamma \leq 0$ for all $\gamma \in \Gamma \setminus \{1_\Gamma\}$,
(wb-3) $f_\gamma = f_{\gamma^{-1}}$ for all $\gamma \in \Gamma$ (i.e., f is *self-adjoint*),
(wb-4) and $\operatorname{supp}(f) \coloneqq \{\gamma \in \Gamma : f_\gamma \neq 0\}$, the *support* of f, generates Γ.

If $f \in \mathbb{Z}[\Gamma]$ is well-balanced, the associated dynamical system (X_f, Γ) is called a *harmonic model*. As an example, for $\Gamma = \mathbb{Z}^d$, the polynomial $f = 2d - \sum_{i=1}^d (u_i + u_i^{-1}) \in \mathbb{Z}[u_1, u_1^{-1}, \ldots, u_d, u_d^{-1}] = \mathbb{Z}[\mathbb{Z}^d]$ is well-balanced. The corresponding dynamical system is called the *Laplace harmonic model* and shares interesting measure theoretic and entropic properties with other different models in mathematical physics, probability theory, and dynamical systems such as the Abelian sandpile model, spanning trees,and the dimer models [32, 4].

Since a well-balanced polynomial $f \in \mathbb{Z}[\Gamma]$, with Γ infinite countable not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, is weakly expansive (cf. Proposition 3.14), from Theorem 1.2 we deduce:

Corollary 1.4 (Garden of Eden theorem for harmonic models)**.** *Let Γ be an infinite countable Abelian group which is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ (e.g. $\Gamma = \mathbb{Z}^d$, with $d \geq 3$). Suppose that $f \in \mathbb{Z}[\Gamma]$ is well-balanced and that X_f is connected. Then the dynamical system (X_f, α_f) has the Moore-Myhill property.* □

Let (X, α) be an algebraic dynamical system and $1 \leq p \leq \infty$. In [10] (see also Subsection 3.1), the p-homoclinic group $\Delta^p(X, \alpha) \subset X$ of (X, α) was introduced. We say that a map $\tau \colon X \to X$ is p-pre-injective if the restriction of τ to each coset of the p-homoclinic group $\Delta^p(X, \alpha)$ is injective. Note that $\Delta^\infty(X, \alpha) = \Delta(X, \alpha)$ so that ∞-pre-injectivity is the same thing as pre-injectivity.

After identifying $\mathbb{Z}[\mathbb{Z}^d]$ with $\mathbb{Z}[u_1, u_1^{-1}, \ldots, u_d, u_d^{-1}]$, the integral ring of Laurent polynomials in d commuting indeterminates, every polynomial $f \in \mathbb{Z}[\mathbb{Z}^d]$ yields a function $\mathbb{S}^d \to \mathbb{C}$

given by $(s_1, \dots, s_d) \mapsto f(s_1, \dots, s_d)$, where $\mathbb{S} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$. We denote by

$$Z(f) \coloneqq \{(s_1, \dots, s_d) \in \mathbb{S}^d : f(s_1, \dots, s_d) = 0\}$$

its *zero-set*. Recall that an irreducible polynomial $f \in \mathbb{Z}[\Gamma]$ is *atoral* [22, Definition 2.1] if there is some $g \in \mathbb{Z}[\Gamma]$ such that $g \notin \mathbb{Z}[\Gamma]f$ and $Z(f) \subset Z(g)$.

We are now in position to state the following:

Theorem 1.5 (A Garden of Eden theorem for irreducible atoral polynomials)**.** *Let $f \in \mathbb{Z}[\mathbb{Z}^d]$ be an irreducible atoral polynomial such that $Z(f)$ is contained in the image of the intersection of $[0,1]^d$ and a finite union of hyperplanes in $\mathbb{R}^d$ under the composition of the quotient map $\mathbb{R}^d \to \mathbb{T}^d \coloneqq \mathbb{R}^d/\mathbb{Z}^d$ and the standard homeomorphism $\mathbb{T}^d \to \mathbb{S}^d$ (e.g., when $d \geq 2$ such that $Z(f)$ is finite). Let $\tau \colon X_f \to X_f$ be a Γ-equivariant continuous map. Then the following conditions are equivalent:*

(a) *τ is surjective,*
(b) *τ is pre-injective,*
(c) *τ is p-pre-injective for all $1 \leq p \leq \infty$,*
(d) *τ is p-pre-injective for some $1 \leq p \leq \infty$,*
(e) *τ is 1-pre-injective.*

In particular, (X_f, α_f) satisfies the Moore-Myhill property.

Our last result is the following:

Corollary 1.6 (Garden of Eden theorem for Laplace harmonic models)**.** *The Laplace harmonic model (i.e. the principal algebraic dynamical system (X_f, α_f) associated with the polynomial $f = 2d - \sum_{i=1}^{d}(u_i + u_i^{-1}) \in \mathbb{Z}[u_1, u_1^{-1}, \dots, u_d, u_d^{-1}] = \mathbb{Z}[\mathbb{Z}^d]$) satisfies the Moore-Myhill property if and only if $d \geq 2$.*

Our motivation for the present work originated from a sentence of Gromov [13, p. 195] suggesting that the Garden of Eden theorem could be extended to dynamical systems with a suitable hyperbolic flavor other than shifts and subshifts. A first step in that direction was made in [6], where the two first named authors proved that all Anosov diffeomorphisms on tori generate $\mathbb{Z}$-actions with the Moore-Myhill property, and another one in [5], where sufficient conditions for expansive actions of countable amenable groups to have the Myhill property were presented. Finally, in [18] the third named author proved the very general Garden of Eden theorem for expansive actions of amenable groups we alluded to above.

The paper is organized as follows. Section 2 introduces notation and collects background material on algebraic dynamical systems.

In Section 3 we study several weak forms of expansivity, namely p-expansivity and p-homoclinicity (from [10]), and the new notions of homoclinical expansive action and of principal algebraic action associated with a weakly expansive polynomial. In Subsection 3.4 we then show that polynomials yielding principal algebraic expansive actions are weakly expansive and, in Subsection 3.5, we prove that well-balanced polynomials, with not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ infinite countable group, are weakly expansive as well.

In Section 4, we discuss topological rigidity of algebraic dynamical systems associated with weakly expansive polynomials. The proof of Theorem 1.2 is then given in Section 5.

In the following section we discuss the notion of atorality for irreducible polynomials in $\mathbb{Z}[\mathbb{Z}^d]$, we present a few examples, and give the proofs of Theorem 1.5 and of Corollary 1.6. In the last section, we collect some final remarks. In particular, we exhibit some examples showing that Theorem 1.2 becomes false if weak expansivity of $f \in \mathbb{Z}[\Gamma]$ is replaced by the weaker hypothesis that the homoclinic group $\Delta(X_f, \alpha_f)$ is dense in X_f, or that the dynamical system (X_f, α_f) is mixing. We also introduce and discuss the notions of p-pre-injectivity, p-Moore, and p-Myhill properties for algebraic actions, and prove some variations on the Garden of Eden theorem in this framework.

Acknowledgments. We express our gratitude to the referees for their careful reading of the paper and for providing useful and interesting comments that helped us improving our presentation. Hanfeng Li was partially supported by NSF and NSFC grants.

2. Background material and preliminaries

2.1. **Notation.** We denote by $\mathbb{N} \coloneqq \{0, 1, 2 \ldots\}$ the set of all natural numbers, by $\mathbb{S} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$ the multiplicative group of all complex numbers of modulus one, and by $\mathbb{T} \coloneqq \mathbb{R}/\mathbb{Z}$ the additive group of real numbers mod $\mathbb{Z}$. For any integer $d \geq 1$ we denote by $\Theta \colon \mathbb{T}^d \to \mathbb{S}^d$ the isomorphism of topological groups given by $\Theta(t_1, \ldots, t_d) = (\exp(2\pi t_1 i), \ldots, \exp(2\pi t_d i))$.

2.2. **Group actions.** Let Γ be a countable group. We use multiplicative notation for the group operation in Γ and denote by 1_Γ its identity element.

An *action* of Γ on a set X is a map $\alpha \colon \Gamma \times X \to X$ such that $\alpha(1_\Gamma, x) = x$ and $\alpha(\gamma_1, \alpha(\gamma_2, x)) = \alpha(\gamma_1\gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$. In the sequel, to simplify, we shall write γx instead of $\alpha(\gamma, x)$, if there is no risk of confusion.

If α is an action of Γ on a set X, we denote by $\mathrm{Fix}(X, \alpha)$ the set of points of X that are *fixed* by α, i.e., the set of points $x \in X$ such that $\gamma x = x$ for all $\gamma \in \Gamma$.

If Γ acts on two sets X and Y, a map $\tau \colon X \to Y$ is said to be Γ-*equivariant* if one has $\tau(\gamma x) = \gamma\tau(x)$ for all $\gamma \in \Gamma$ and $x \in X$.

2.3. **Convolution.** Let Γ be a countable group. We denote by $\ell^\infty(\Gamma)$ the vector space consisting of all formal series

$$f = \sum_{\gamma \in \Gamma} f_\gamma \gamma,$$

with coefficients $f_\gamma \in \mathbb{R}$ for all $\gamma \in \Gamma$ and

$$\|f\|_\infty \coloneqq \sup_{\gamma \in \Gamma} |f_\gamma| < \infty.$$

For $1 \leq p < \infty$ we denote by $\ell^p(\Gamma)$ the vector subspace of $\ell^\infty(\Gamma)$ consisting of all $f \in \ell^\infty(\Gamma)$ such that

$$\|f\|_p \coloneqq \left(\sum_{\gamma \in \Gamma} |f_\gamma|^p \right)^{\frac{1}{p}} < \infty.$$

Note that $\ell^1(\Gamma) \subset \ell^p(\Gamma) \subset \ell^q(\Gamma) \subset \ell^\infty(\Gamma)$ for all $1 < p < q < \infty$. When $f \in \ell^\infty(\Gamma)$ and $g \in \ell^1(\Gamma)$ (resp. $f \in \ell^1(\Gamma)$ and $g \in \ell^\infty(\Gamma)$) we define the *convolution product* $fg \in \ell^\infty(\Gamma)$ by setting

$$(fg)_\gamma := \sum_{\substack{\gamma_1,\gamma_2\in\Gamma:\\ \gamma_1\gamma_2=\gamma}} f_{\gamma_1} g_{\gamma_2} = \sum_{\delta\in\Gamma} f_{\gamma\delta^{-1}} g_\delta \tag{2.1}$$

for all $\gamma \in \Gamma$. Note that $\|fg\|_\infty \leq \|f\|_\infty \cdot \|g\|_1$ (resp. $\|fg\|_\infty \leq \|f\|_1 \cdot \|g\|_\infty$). We have the associativity rule

$$(fg)h = f(gh) \text{ for all } f \in \ell^\infty(\Gamma),\ g,h \in \ell^1(\Gamma) \text{ (resp. } f,g \in \ell^1(\Gamma),\ h \in \ell^\infty(\Gamma)). \tag{2.2}$$

We denote by $\mathbb{R}[\Gamma] = \{f \in \ell^\infty(\Gamma) : f_\gamma = 0 \text{ for all but finitely many } \gamma \in \Gamma\}$ and by $\mathbb{Z}[\Gamma] = \{f \in \mathbb{R}[\Gamma] : f_\gamma \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}$ the *real* and, respectively, the *integral group ring* of Γ. Observe that the convolution product extends the group operation on $\Gamma \subset \mathbb{Z}[\Gamma] \subset \mathbb{R}[\Gamma]$.

Note also that, as a $\mathbb{Z}$-module, $\mathbb{Z}[\Gamma]$ is free with base Γ.

If we take $\Gamma = \mathbb{Z}^d$, then $\mathbb{Z}[\Gamma]$ is the Laurent polynomial ring $R_d := \mathbb{Z}[u_1^{\pm1}, \ldots, u_d^{\pm1}]$ on d commuting indeterminates $u_1, \ldots, u_d$.

Recall that $\mathcal{C}_0(\Gamma)$ denotes the vector space consisting of all functions $f \colon \Gamma \to \mathbb{R}$ vanishing at infinity: we express this condition by writing $\lim_{\gamma\to\infty} f(\gamma) = 0$. We then have the inclusions

$$\Gamma \subset \mathbb{Z}[\Gamma] \subset \mathbb{R}[\Gamma] \subset \ell^1(\Gamma) \subset \ell^p(\Gamma) \subset \mathcal{C}_0(\Gamma) \subset \ell^\infty(\Gamma), \tag{2.3}$$

for all $1 \leq p < \infty$. Moreover, there is a natural involution $f \mapsto f^*$ on $\ell^\infty(\Gamma)$ defined by

$$(f^*)_\gamma := f_{\gamma^{-1}} \tag{2.4}$$

for all $f \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$. Observe that every set in (2.3) is $*$-invariant and that

$$(fg)^* = g^* f^* \text{ for all } f \in \ell^\infty(\Gamma) \text{ and } g \in \ell^1(\Gamma) \text{ (resp. } f \in \ell^1(\Gamma) \text{ and } g \in \ell^\infty(\Gamma)). \tag{2.5}$$

The normed space $(\ell^1(\Gamma), \|\cdot\|_1)$ is a unital Banach *-algebra for the convolution product and the involution. The unity element of $\ell^1(\Gamma)$ is 1_Γ. From the associative rule in Γ and linearity one easily shows that if $f, h \in \mathbb{R}[\Gamma]$ and $g \in \ell^\infty(\Gamma)$ then

$$(fg)h = f(gh). \tag{2.6}$$

Let now $k, n \in \mathbb{N}$ and denote by $\mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma]) := \{(a^{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq k}} : a^{ij} \in \mathbb{Z}[\Gamma]\}$ the space of all n-by-k matrices with coefficients in the group ring $\mathbb{Z}[\Gamma]$. We identify $\mathbb{Z}[\Gamma]^k$ (resp. $\mathbb{Z}[\Gamma]^n$) and $\mathrm{Mat}_{1,k}(\mathbb{Z}[\Gamma])$ (resp. $\mathrm{Mat}_{1,n}(\mathbb{Z}[\Gamma])$) so that if $g = (g^1, g^2, \ldots, g^n) \in \mathbb{Z}[\Gamma]^n$ and $A = (a^{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq k}} \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$, the element $gA \in \mathbb{Z}[\Gamma]^k$ is defined by $(gA)^j := \sum_{i=1}^n g^i a^{ij} \in \mathbb{Z}[\Gamma]$, that is,

$$(gA)^j_\gamma = \sum_{i=1}^n \sum_{\eta\in\Gamma} g^i_{\gamma\eta} a^{ij}_{\eta^{-1}} \tag{2.7}$$

for all $j = 1, 2, \dots, k$ and $\gamma \in \Gamma$. Given $A = (a^{ij})_{\substack{1 \le i \le n \\ 1 \le j \le k}} \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$, we define

$$A^* = \left((a^{ji})^*\right)_{\substack{1 \le i \le k \\ 1 \le j \le n}} \in \mathrm{Mat}_{k,n}(\mathbb{Z}[\Gamma]). \tag{2.8}$$

Note that when $k = n = 1$ and $A = f \in \mathbb{Z}[\Gamma]$, then (2.8) reduces to (2.4). Also, we set

$$\|A\|_\infty = \sup_{\substack{1 \le i \le n \\ 1 \le j \le k}} \sup_{\gamma \in \Gamma} |a^{ij}_\gamma|$$

and

$$\|A\|_1 = \sum_{\substack{1 \le i \le n \\ 1 \le j \le k}} \|a^{ij}\|_1 = \sum_{\substack{1 \le i \le n \\ 1 \le j \le k}} \sum_{\gamma \in \Gamma} |a^{ij}_\gamma|.$$

2.4. **Algebraic dynamical systems.** An *algebraic dynamical system* is a pair (X, α), where X is a compact metrizable Abelian topological group and α is an action of a countable group Γ on X by continuous group automorphisms.

As an example, if A is a compact metrizable Abelian topological group (e.g. $A = \mathbb{T}$) and Γ a countable group, then the system (A^Γ, σ), where $A^\Gamma = \{x \colon \Gamma \to A\}$ is equipped with the product topology, and σ is the *shift action*, defined by

$$(\sigma(\gamma, x))(\gamma') \coloneqq x(\gamma^{-1}\gamma') \quad \text{for all } \gamma, \gamma' \in \Gamma \text{ and } x \in A^\Gamma,$$

is an algebraic dynamical system.

Let (X, α) be an algebraic dynamical system with acting group Γ. As X is compact and metrizable, its Pontryagin dual $\widehat{X}$ is a discrete countable Abelian group.

There is a dual left $\mathbb{Z}[\Gamma]$-module structure on $\widehat{X}$ induced by the action of Γ on X. Conversely, if M is a countable left $\mathbb{Z}[\Gamma]$-module and we equip M with its discrete topology, then its Pontryagin dual $\widehat{M}$ is a compact metrizable Abelian group and there is, by duality, an action α_M of Γ on $\widehat{M}$ by continuous group automorphisms, so that $(\widehat{M}, \alpha_M)$ is an algebraic dynamical system. In this way, algebraic dynamical systems with acting group Γ are in one-to-one correspondence with countable left $\mathbb{Z}[\Gamma]$-modules (see [31], [20] for more details).

2.5. **Finitely presented algebraic dynamical systems.** Let Γ be a countable group. One says that an algebraic dynamical system (X, α) with acting group Γ is *finitely generated* (resp. *finitely presented*) if its Pontryagin dual $\widehat{X}$ is finitely generated (resp. finitely presented) as a left $\mathbb{Z}[\Gamma]$-module.

Let now $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$. Then $\mathbb{Z}[\Gamma]^n A$ is a finitely generated $\mathbb{Z}[\Gamma]$-submodule of $\mathbb{Z}[\Gamma]^k$ and the quotient $M_A \coloneqq \mathbb{Z}[\Gamma]^k / \mathbb{Z}[\Gamma]^n A$ is a finitely presented left $\mathbb{Z}[\Gamma]$-module. Note that every finitely presented left $\mathbb{Z}[\Gamma]$-module is isomorphic to some M_A for $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$ suitably chosen.

To simplify notation, let us write X_A instead of $\widehat{M_A}$ and α_A instead of α_{M_A}. The algebraic dynamical system (X_A, α_A) is called the *finitely presented algebraic dynamical system* associated with A.

If $k = n = 1$ and $A = f \in \mathbb{Z}[\Gamma]$, then $M_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$, where $\mathbb{Z}[\Gamma]f$ is the principal left ideal of $\mathbb{Z}[\Gamma]$ generated by f, and the algebraic dynamical system (X_f, α_f) is called the *principal algebraic dynamical system* associated with f.

One can regard (X_A, α_A) as a *group subshift* of $((\mathbb{T}^k)^\Gamma, \sigma)$, i.e., as a closed subgroup of $(\mathbb{T}^k)^\Gamma$ that is invariant under the shift action σ of Γ on $(\mathbb{T}^k)^\Gamma$, in the following way. The Pontryagin dual of $(\mathbb{T}^k)^\Gamma$ is $\mathbb{Z}[\Gamma]^k$ with pairing $\langle \cdot, \cdot \rangle \colon \mathbb{Z}[\Gamma]^k \times (\mathbb{T}^k)^\Gamma \to \mathbb{T}$ given by

$$\langle g, x \rangle = \sum_{j=1}^{k} \sum_{\eta \in \Gamma} g^j_\eta x_j(\eta)$$

for all $g = (g^1, g^2, \dots, g^k) \in \mathbb{Z}[\Gamma]^k$ and $x = (x_1, x_2, \dots, x_k) \in (\mathbb{T}^\Gamma)^k = (\mathbb{T}^k)^\Gamma$. Then,

$$X_A = \{x \in (\mathbb{T}^k)^\Gamma : xA^* = 0_{(\mathbb{T}^n)^\Gamma}\}, \tag{2.9}$$

and the action α_A of Γ on $X_A \subset (\mathbb{T}^k)^\Gamma$ is the restriction to X_A of the shift action σ.

In particular, if $A = f \in \mathbb{Z}[\Gamma]$, then (2.9) becomes

$$X_f = \{x \in \mathbb{T}^\Gamma : xf^* = 0_{\mathbb{T}^\Gamma}\}. \tag{2.10}$$

Consider the surjective map $\pi \colon \ell^\infty(\Gamma)^k \to (\mathbb{T}^k)^\Gamma$ defined by $\pi(g)(\gamma)^i = g^i_\gamma \mod 1$ for all $g = (g^1, g^2, \dots, g^k) \in \ell^\infty(\Gamma)^k$, $\gamma \in \Gamma$, and $i = 1, 2, \dots, k$. We then denote by $\ell^\infty(\Gamma, \mathbb{Z})^k$ the set consisting of all $g \in \ell^\infty(\Gamma)^k$ such that $g^i_\gamma \in \mathbb{Z}$ for all $\gamma \in \Gamma$ and $i = 1, 2, \dots, k$. From (2.9) one easily deduces that if $x \in (\mathbb{T}^k)^\Gamma$ and $g \in \ell^\infty(\Gamma)^k$ satisfies that $\pi(g) = x$, then $x \in X_A$ if and only if $gA^* \in \ell^\infty(\Gamma, \mathbb{Z})^n$.

2.6. **The homoclinic group.** Let (X, α) be an algebraic dynamical system with acting group Γ. The set of points in X that are homoclinic to 0_X with respect to α is a $\mathbb{Z}[\Gamma]$-submodule $\Delta(X, \alpha) \subset X$, which is called the *homoclinic group* of (X, α) (cf. [19], [20]). Note that $x \in \Delta(X, \alpha)$ if and only if $\lim_{\gamma \to \infty} \gamma x = 0_X$. We can choose a compatible metric d on X that is translation-invariant so that

$$d(\gamma x, \gamma y) = d(\gamma x - \gamma y, 0_X) = d(\gamma(x - y), 0_X)$$

for all $x, y \in X$ and $\gamma \in \Gamma$. We deduce that x and y are homoclinic if and only if $x - y \in \Delta(X, \alpha)$. Moreover, a straightforward argument shows that for $k \in \mathbb{N}$ and $x \in (\mathbb{T}^k)^\Gamma$ the following conditions are equivalent:

(a) $x \in \Delta((\mathbb{T}^k)^\Gamma, \sigma)$;
(b) $\lim_{\gamma \to \infty} x(\gamma) = 0_{\mathbb{T}^k}$;
(c) there exists $g \in \mathcal{C}_0(\Gamma)^k$ such that $x = \pi(g)$.

2.7. **Connectedness of the phase space.** A non-zero element $f \in \mathbb{Z}[\Gamma]$ is called *primitive* if there is no integer $n \geq 2$ that divides all coefficients of f. Every nonzero element $f \in \mathbb{Z}[\Gamma]$ can be uniquely written in the form $f = mf_0$ with m a positive integer and f_0 primitive. The integer m is called the *content* of f. For principal algebraic dynamical systems with elementary amenable acting group we have the following criterion for connectedness of the phase space. Recall (cf. for instance [9]) that the class of *elementary amenable groups* is the smallest class of groups containing all finite groups and all Abelian groups that is closed under the operations of taking subgroups, quotiens, extensions, and direct limits.

Proposition 2.1. *Let Γ be a countable torsion-free elementary amenable group (e.g. $\Gamma = \mathbb{Z}^d$). Let $f \in \mathbb{Z}[\Gamma]$ with $f \neq 0$. Then the following conditions are equivalent:*

(a) *X_f is connected;*

(b) *f is primitive.*

Proof. (a) $\Rightarrow$ (b): Suppose that f is not primitive. Then $f = mg$ for some integer $m \geq 2$ and $g \in \mathbb{Z}[\Gamma]$. By [15, Theorem 1.4] the ring $\mathbb{Q}[\Gamma]$ is a domain. If $g = hf$ for some $h \in \mathbb{Z}[\Gamma]$, then $\frac{1}{m}f = hf$ in $\mathbb{Q}[\Gamma]$, and hence $h = \frac{1}{m}1_\Gamma$, which is a contradiction. Thus $g + \mathbb{Z}[\Gamma]f$ is a nonzero element of $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$, while $m(g + \mathbb{Z}[\Gamma]f) = 0$. Therefore $g + \mathbb{Z}[\Gamma]f$ is a nonzero torsion element of M_f, and hence X_f is not connected.

(b) $\Rightarrow$ (a): Suppose that f is primitive and that X_f is not connected. Then M_f has torsion, so that there exists $a \in M_f$ with finite order $n \geq 2$. Replacing a by some integral multiple of a, we may assume that n is a prime number p. Write $a = g + \mathbb{Z}[\Gamma]f$ for some $g \in \mathbb{Z}[\Gamma]$. Then $pg = hf$ for some $h \in \mathbb{Z}[\Gamma]$. Denote by ψ the natural ring morphism $\mathbb{Z}[\Gamma] \to (\mathbb{Z}/p\mathbb{Z})[\Gamma]$ obtained by reducing coefficients modulo p. Then $\psi(h)\psi(f) = 0$. By [15, Theorem 1.4], the ring $(\mathbb{Z}/p\mathbb{Z})[\Gamma]$ is a domain. Since f is primitive, $\psi(f) \neq 0$. Thus $\psi(h) = 0$, i.e. $h = pw$ for some $w \in \mathbb{Z}[\Gamma]$. Then $pg = hf = pwf$, and hence $g = wf \in \mathbb{Z}[\Gamma]$. This means that $a = 0$, which is a contradiction. ☐

3. Weak forms of expansivity for algebraic actions

In this section we present and study weak forms of expansivity for algebraic actions. This applies in particular to the harmonic models introduced in [32] (see also [4]).

3.1. p-expansive algebraic actions and p-homoclinic groups.

In this section we review the notions of p-expansive algebraic actions and of p-homoclinic groups introduced by Chung and the third named author in [10, Sections 4 and 5].

Let Γ be a countable group acting by automorphisms of a compact metrizable Abelian group X. Let also $1 \leq p \leq \infty$.

For $x \in X$ and $\chi \in \widehat{X}$, define the function $\Psi'_{x,\chi}$ on Γ by setting

$$\Psi'_{x,\chi}(\gamma) = e^{2\pi i \langle \gamma x, \chi \rangle} - 1, \tag{3.1}$$

for all $\gamma \in \Gamma$.

One then says that the algebraic dynamical system (X, α) is *p-expansive* provided there exists a finite subset $W \subset \widehat{X}$ and $\varepsilon > 0$ such that 0_X is the only point $x \in X$ satisfying

$$\sum_{\chi \in W} \|\Psi'_{x,\chi}\|_p < \varepsilon. \tag{3.2}$$

The following collects the main properties of p-expansivity.

Theorem 3.1 ([10, Proposition 4.3 and Theorem 4.11])**.** *Let (X, Γ, α) be an algebraic dynamical system. Let $1 \leq p \leq \infty$. Then the following hold:*

(1) *If α is p-expansive, then it is q-expansive for all $1 \leq q \leq p$.*
(2) *If α is p-expansive, then $\widehat{X}$ is a finitely generated left $\mathbb{Z}[\Gamma]$-module.*
(3) *If α is p-expansive, then for any finite subset $W \subset \widehat{X}$ generating $\widehat{X}$ as a left $\mathbb{Z}[\Gamma]$-module, there exists $\varepsilon > 0$ such that 0_X is the only point $x \in X$ satisfying* (3.2).
(4) *α is ∞-expansive if and only if it is expansive.*
(5) *Let $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$. Then α_A is p-expansive if an only if the $\mathbb{R}[\Gamma]$-morphism $\ell^p(\Gamma)^k \to \ell^p(\Gamma)^n$ sending g to gA^* is injective. Moreover, if in addition, Γ is amenable, then the following conditions are equivalent:*
   (a) *the topological entropy of (X_A, α_A) is finite;*
   (b) *α_A is 1-expansive;*
   (c) *α_A is 2-expansive;*
   (d) *the $\mathbb{Z}[\Gamma]$-morphism $\mathbb{Z}[\Gamma]^k \to \mathbb{Z}[\Gamma]^n$ sending g to gA^* is injective;*
   (e) *the $\mathbb{R}[\Gamma]$-morphism $\mathbb{R}[\Gamma]^k \to \mathbb{R}[\Gamma]^n$ sending g to gA^* is injective.*

We now recall the definitions of a p-homoclinic point and of the p-homoclinic group (cf. [10, Section 5]). Let Γ be a countable group acting by automorphisms of the compact metrizable Abelian group X and let $1 \leq p < \infty$. One says that a point $x \in X$ is *p-homoclinic* provided that $\Psi'_{x,\chi} \in \ell^p(\Gamma)$ for all $\chi \in \widehat{X}$. Let then $\Delta^p(X, \alpha)$ denote the set of all p-homoclinic points of X. This is called the *p-homoclinic group* (cf. Theorem 3.2.(2)) of the algebraic dynamical system (X, α). Also one sets $\Delta^\infty(X, \alpha) \coloneqq \Delta(X, \alpha)$. Note that for $p = 1$, the set $\Delta^1(X, \alpha)$ was studied in [32] and [21]. Here below we collect some basic properties of the p-homoclinic groups.

Theorem 3.2 ([10, Proposition 5.2, Lemma 5.3, and Lemma 5.4])**.** *Let Γ be a countable group acting by automorphisms of the compact metrizable Abelian group X and let $1 \leq p \leq \infty$. Then the following hold:*
(1) *One has $\Delta^p(X, \alpha) \subset \Delta^q(X, \alpha)$ for all $p \leq q \leq \infty$.*
(2) *$\Delta^p(X, \alpha)$ is a Γ-invariant subgroup of X.*
(3) *If α is p-expansive, then $\Delta^p(X, \alpha)$ is countable.*
(4) *If $\mathbb{Z}[\Gamma]$ is left Noetherian and α is p-expansive, then $\Delta^p(X, \alpha)$ is a finitely generated left $\mathbb{Z}[\Gamma]$-module.*
(5) *Assume that $p < \infty$ and let $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$. If α_A is p-expansive, then $\Delta^p(X_A, \alpha_A)$ is isomorphic to a $\mathbb{Z}[\Gamma]$-submodule of $\mathbb{Z}[\Gamma]^n/\mathbb{Z}[\Gamma]^k A^*$. If, in addition, the $\mathbb{R}[\Gamma]$-morphism $\ell^p(\Gamma)^k \to \ell^p(\Gamma)^n$ sending g to gA^* is invertible, then $\Delta^p(X_A, \alpha_A)$ is isomorphic to $\mathbb{Z}[\Gamma]^n/\mathbb{Z}[\Gamma]^k A^*$.*

3.2. **Homoclinically expansive actions.** In this section we introduce and study a new form of weak expansivity for dynamical systems.

Definition 3.3. Let (X, α) be a dynamical system with acting group Γ. One says that the action is *homoclinically expansive* if there exists a constant $\varepsilon_0 > 0$ such that, for each pair of distinct homoclinic points $x, y \in X$, there exists an element $\gamma \in \Gamma$ such that $d(\gamma x, \gamma y) > \varepsilon_0$, where d is any compatible metric on X. Such a constant ε_0 is then called a *homoclinic-expansivity constant* for (X, α, d).

Note that the fact that (X, α) is homoclinically expansive is in fact independent of the choice of the metric d by compactness of X. A pseudometric $\tilde{d}$ on X is said to be *dynamically-generating* if for all distinct $x, y \in X$ there is $\gamma \in \Gamma$ such that $\tilde{d}(\gamma x, \gamma y) > 0$ (cf. [14, Definition 9.35]). Now, given a dynamically-generating continuous pseudometric $\tilde{d}$ on X, we can define a compatible metric d on X by setting

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \tilde{d}(\gamma_n x, \gamma_n y),$$

where $\gamma_0 = 1_\Gamma, \gamma_1, \dots$ is an enumeration of the elements of Γ. Then

$$\sup_{\gamma \in \Gamma} \tilde{d}(\gamma x, \gamma y) \leq \sup_{\gamma \in \Gamma} d(\gamma x, \gamma y) \leq 2 \sup_{\gamma \in \Gamma} \tilde{d}(\gamma x, \gamma y)$$

for all $x, y \in X$. Thus in Definition 3.3 we may take d to be any dynamically-generating continuous pseudometric on X.

In the following, we study homoclinic expansivity for algebraic actions. Let (X, α) be an algebraic dynamical system with acting group Γ.

For any $t \in \mathbb{R}$, we set $|t + \mathbb{Z}| := \min_{m \in \mathbb{Z}} |t + m|$. More generally, for $k \in \mathbb{N}$ and $t = (t_1, t_2, \dots, t_k) \in \mathbb{R}^k$ we set

$$|t + \mathbb{Z}^k| := \max_{1 \leq j \leq k} |t_j + \mathbb{Z}|. \tag{3.3}$$

Given $x \in X$ and $\chi \in \widehat{X}$, define a function $\Psi_{x,\chi}$ on Γ by setting

$$\Psi_{x,\chi}(\gamma) = |\langle \gamma x, \chi \rangle|, \tag{3.4}$$

for all $\gamma \in \Gamma$.

As a comparison between (3.1) and (3.4), note that there is some constant $C > 0$ such that

$$C|t| \leq |e^{2\pi i t} - 1| \leq C^{-1}|t|$$

for all $t \in [-1/2, 1/2]$. It follows that, for all $1 \leq p \leq \infty$,

$$C\|\Psi_{x,\chi}\|_p \leq \|\Psi'_{x,\chi}\|_p \leq C^{-1}\|\Psi_{x,\chi}\|_p.$$

It is easy to see that for any $x \in X$, one has that $x \in \Delta(X, \alpha)$ if and only if $\Psi_{x,\chi} \in \mathcal{C}_0(\Gamma)$ for all $\chi \in \widehat{X}$.

Proposition 3.4. *Let (X, α) be an algebraic dynamical system with acting group Γ. Then the following conditions are equivalent:*

(a) *the action α is homoclinically expansive;*

(b) *there exist a finite subset W of $\widehat{X}$ and $\varepsilon > 0$ such that 0_X is the only point x in $\Delta(X, \alpha)$ satisfying*

$$\sum_{\chi \in W} \|\Psi_{x,\chi}\|_\infty < \varepsilon. \tag{3.5}$$

Proof. Take a function $h \in \ell^1(\widehat{X})$ such that $h(\chi) > 0$ for every $\chi \in \widehat{X}$. Consider the compatible metric ρ on X defined by

$$\rho(x,y) = \sum_{\chi \in \widehat{X}} h(\chi)|\langle x, \chi\rangle - \langle y, \chi\rangle| = \sum_{\chi \in \widehat{X}} h(\chi)|\langle x - y, \chi\rangle|.$$

Assume that (b) holds with W and ε and let us set $\varepsilon_0 := \varepsilon \min_{\chi \in W} h(\chi)/|W|$. Let (x,y) be a homoclinic pair of X with $\sup_{\gamma \in \Gamma} \rho(\gamma x, \gamma y) < \varepsilon_0$. Note that

$$\begin{aligned}
\sup_{\gamma \in \Gamma} \rho(\gamma x, \gamma y) &= \sup_{\gamma \in \Gamma} \sum_{\chi \in \widehat{X}} h(\chi)|\langle \gamma x - \gamma y, \chi\rangle| \\
&= \sup_{\gamma \in \Gamma} \sum_{\chi \in \widehat{X}} h(\chi)\Psi_{x-y,\chi}(\gamma) \\
&\geq \sup_{\gamma \in \Gamma} (\min_{\chi \in W} h(\chi)) \sum_{\chi \in W} \Psi_{x-y,\chi}(\gamma) \\
&\geq \frac{\min_{\chi \in W} h(\chi)}{|W|} \sum_{\chi \in W} \|\Psi_{x-y,\chi}\|_\infty.
\end{aligned}$$

Thus $\sum_{\chi \in W} \|\Psi_{x-y,\chi}\|_\infty < \varepsilon$, and hence $x = y$. This shows that ε_0 is a homoclinic-expansivity constant for (X, α), and (b)$\Rightarrow$(a) follows.

Now assume that (a) holds and let $\varepsilon_0 > 0$ be a homoclinic-expansivity constant for (X, α). Take a finite subset W of $\widehat{X}$ such that $\sum_{\chi \in \widehat{X} \setminus W} h(\chi) < \varepsilon_0/2$ and set $\varepsilon := \varepsilon_0/(2 \max_{\chi \in W} h(\chi))$. Let $x \in \Delta(X, \alpha)$ with $\sum_{\chi \in W} \|\Psi_{x,\chi}\|_\infty < \varepsilon$. Then

$$\begin{aligned}
\sup_{\gamma \in \Gamma} \rho(\gamma x, \gamma 0_X) &= \sup_{\gamma \in \Gamma} \sum_{\chi \in \widehat{X}} h(\chi)\Psi_{x,\chi}(\gamma) \\
&\leq \varepsilon_0/2 + \sup_{\gamma \in \Gamma} \sum_{\chi \in W} h(\chi)\Psi_{x,\chi}(\gamma) \\
&\leq \varepsilon_0/2 + (\max_{\chi \in W} h(\chi)) \sum_{\chi \in W} \|\Psi_{x,\chi}\|_\infty \\
&< \varepsilon_0,
\end{aligned}$$

and hence $x = 0_X$. This shows that (a)$\Rightarrow$(b). $\square$

Proposition 3.5. *Let (X, Γ, α) be an algebraic dynamical system. Suppose that α is homoclinically expansive and $\widehat{X}$ is finitely generated (as a left $\mathbb{Z}[\Gamma]$-module). Then the following hold:*

(1) *α is p-expansive for all $1 \leq p < \infty$;*
(2) *for any finite subset W of $\widehat{X}$ generating $\widehat{X}$ as a left $\mathbb{Z}[\Gamma]$-module, there exists $\varepsilon > 0$ such that 0_X is the only point x in $\Delta(X, \alpha)$ satisfying $\sum_{\chi \in W} \|\Psi_{x,\chi}\|_\infty < \varepsilon$.*

Proof. Note that for any $x \in X$, $\chi, \chi' \in \widehat{X}$, and $u, v \in \mathbb{Z}[\Gamma]$ we have

$$\|\Psi_{x,u\chi+v\chi'}\|_q \leq \|u\|_1\|\Psi_{x,\chi}\|_q + \|v\|_1\|\Psi_{x,\chi'}\|_q \tag{3.6}$$

for all $1 \leq q \leq \infty$.

(1). Since α is homoclinically expansive, by virtue of Proposition 3.4 we can find a finite subset W of $\widehat{X}$ and $\varepsilon > 0$ satisfying (3.5). Enlarging W if necessary, we may assume that W generates $\widehat{X}$ as a left $\mathbb{Z}[\Gamma]$-module. Let $1 \leq p < \infty$. If $x \in X$ and $\sum_{\chi \in W} \|\Psi_{x,\chi}\|_p < \varepsilon$, then from (3.6) we know that $x \in \Delta^p(X, \alpha) \subset \Delta(X, \alpha)$, and using $\sum_{\chi\in W} \|\Psi_{x,\chi}\|_\infty \leq \sum_{\chi\in W} \|\Psi_{x,\chi}\|_p < \varepsilon$, we conclude that $x = 0_X$. Therefore α is p-expansive.

(2) follows from Proposition 3.4 and (3.6). $\square$

For finitely presented (e.g. principal) algebraic actions we have the following characterization of homoclinic expansivity (compare with Theorem 3.1.(5)).

Theorem 3.6. *Let Γ be a countable group. Let $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$. Then the following conditions are equivalent:*

(a) *(X_A, α_A) is homoclinically expansive;*
(b) *the linear map $\mathcal{C}_0(\Gamma)^k \to \mathcal{C}_0(\Gamma)^n$ sending g to gA^* is injective.*

Proof. For $x \in X_A$ consider the function $\Phi_x \in \ell^\infty(\Gamma)$ defined by $\Phi_x(\gamma) := |x_\gamma|$ for all $\gamma \in \Gamma$, where $|\cdot|$ is as in (3.3).

Assume that (b) fails. Then $gA^* = 0$ for some nonzero $g \in \mathcal{C}_0(\Gamma)^k$. As a consequence, for all $\lambda \in \mathbb{R}$ one also has $\lambda g A^* = 0$ and hence $\pi(\lambda g) \in \Delta(X_A, \alpha_A)$. Let $W \subset \widehat{X_A}$ denote the image of the canonical basis of $\mathbb{Z}[\Gamma]^k$ under the quotient map $\mathbb{Z}[\Gamma]^k \to \mathbb{Z}[\Gamma]^k/\mathbb{Z}[\Gamma]^n A$. Note that W then generates $\widehat{X_A}$ as a left $\mathbb{Z}[\Gamma]$-module. When $\lambda \to 0$, one has $\|\Phi_{\pi(\lambda g)}\|_\infty \to 0$, and hence $\sum_{\chi\in W} \|\Psi_{\pi(\lambda g),\chi}\|_\infty \to 0$. Since $g \neq 0$, when $|\lambda|$ is sufficiently small and nonzero, $\pi(\lambda g) \neq 0_{X_A}$. From Proposition 3.5.(2), we deduce that (X_A, α_A) is not homoclinically expansive. This shows (a)$\Rightarrow$(b).

Now assume that (b) holds so that, in particular, $A \neq 0$. Let d be a translation-invariant compatible metric on X_A. Then there is some $\varepsilon_0 > 0$ such that for any $x \in X_A$ with $d(x, 0_{X_A}) \leq \varepsilon_0$, one has $|x_{1_\Gamma}| < 1/(2\|A\|_1)$. Let $x, y \in X_A$ be two homoclinic points with $\max_{\gamma\in\Gamma} d(\gamma x, \gamma y) \leq \varepsilon_0$. Then $x - y \in \Delta(X_A, \alpha_A)$, and for any $\gamma \in \Gamma$ we have $d(\gamma(x - y), 0_{X_A}) \leq \varepsilon_0$, and hence $|(x - y)_{\gamma^{-1}}| = |(\gamma(x - y))_{1_\Gamma}| < 1/(2\|A\|_1)$. Let g be the unique element of $\ell^\infty(\Gamma)^k$ satisfying $\|g\|_\infty \leq 1/(2\|A\|_1)$ and $\pi(g) = x - y$. Since $x - y$ is in $\Delta(X_A, \alpha_A)$, we have $g \in \mathcal{C}_0(\Gamma)^k$. It follows that $\|gA^*\|_\infty \leq \|g\|_\infty\|A\|_1 \leq 1/2$ and therefore $gA^* \in \ell^\infty(\Gamma, \mathbb{Z})$. Thus, $gA^* = 0$. By (b) we have $g = 0$, and hence $x = y$. This shows that ε_0 is a homoclinic expansivity constant for (X_A, α_A), and (b)$\Rightarrow$(a) follows as well. $\square$

Note that when $n = k = 1$ and $A = f \in \mathbb{Z}[\Gamma]$, condition (b) in Theorem 3.6 (and therefore homoclinic expansivity of α_f) is equivalent to (we-1) in Definition 1.1.

Proposition 3.7. *Let Γ be a countable group. Let $k, n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,k}(\mathbb{Z}[\Gamma])$ and suppose that (X_A, α_A) is homoclinically expansive. Then $\Delta(X_A, \alpha_A)$ is isomorphic to a left $\mathbb{Z}[\Gamma]$-submodule of $\mathbb{Z}[\Gamma]^n/\mathbb{Z}[\Gamma]^k A^*$.*

Proof. For each $x \in \Delta(X_A, \alpha_A)$, take $\tilde{x} \in \mathcal{C}_0(\Gamma)^k$ with $\pi(\tilde{x}) = x$. Then, $\tilde{x}A^* \in \ell^\infty(\Gamma, \mathbb{Z})^n \cap \mathcal{C}_0(\Gamma)^n = \mathbb{Z}[\Gamma]^n$. If we choose another $\tilde{x}' \in \mathcal{C}_0(\Gamma)^k$ with $\pi(\tilde{x}') = x$, then $\tilde{x} - \tilde{x}' \in \ell^\infty(\Gamma, \mathbb{Z})^k \cap \mathcal{C}_0(\Gamma)^k = \mathbb{Z}[\Gamma]^k$, and hence $\tilde{x}A^* - \tilde{x}'A^* \in \mathbb{Z}[\Gamma]^k A^*$. Thus the map $\varphi\colon \Delta(X_A, \alpha_A) \to \mathbb{Z}[\Gamma]^n/\mathbb{Z}[\Gamma]^k A^*$ sending x to $\tilde{x}A^* + \mathbb{Z}[\Gamma]^k A^*$ is well defined. Clearly, φ is a left $\mathbb{Z}[\Gamma]$-module morphism. Let $x \in \ker(\varphi)$. Then $\tilde{x}A^* = gA^*$, for some $g \in \mathbb{Z}[\Gamma]^k$. By virtue of Theorem 3.6, we have $\tilde{x} = g$ and hence $x = 0_{X_A}$. Thus φ is injective. □

Examples 3.8. Here below we describe some examples of principal algebraic actions and discuss their homoclinic expansivity.

(1) Suppose that $\gamma \in \Gamma$ has infinite order and denote by $\Gamma' \cong \mathbb{Z}$ the subgroup of Γ it generates. It follows from [23, Theorem 5.1] that for any nonzero $f \in \mathbb{C}[\Gamma']$ and any nonzero $g \in \mathcal{C}_0(\Gamma')$, one has $fg \neq 0$. Using the right-coset decomposition of Γ, it follows that for any nonzero $f \in \mathbb{Z}[\Gamma']$ and any nonzero $g \in \mathcal{C}_0(\Gamma)$, one has $fg \neq 0$. This shows that the associated principal algebraic action α_f is homoclinically expansive. Note that if $f \coloneqq 1_\Gamma - \gamma \in \mathbb{Z}[\Gamma']$, then f is not invertible in $\ell^1(\Gamma)$ and hence, by [11, Theorem 3.2] (cf. Theorem 3.11 below), α_f is not expansive. (cf. [10, Example 4.6].)

(2) Let $\Gamma = \mathbb{Z}^d$ and let $f \in \mathbb{R}[\Gamma]$. Let us denote by $P\colon \mathbb{R}^d \to \mathbb{S}^d$ the composition of the quotient map $\mathbb{R}^d \to \mathbb{T}^d$ and the homeomorphism $\Theta\colon \mathbb{T}^d \to \mathbb{S}^d$. It follows from [24, Theorem 2.2] that its zero-set $Z(f)$ is contained in the P-image of the intersection of $[0,1]^d$ and a finite union of hyperplanes in $\mathbb{R}^d$ if and only if $fg \neq 0$ for all nonzero $g \in \mathcal{C}_0(\Gamma)$. From Theorem 3.6 we thus deduce that the principal algebraic action associated with $f \in \mathbb{Z}[\Gamma]$ is homoclinically expansive if and only if $Z(f)$ is contained in the image of the intersection of $[0,1]^d$ and a finite union of hyperplanes in $\mathbb{R}^d$ under P. This is the case, for instance, if $Z(f)$ is finite. (cf. [10, Example 4.9].)

(3) Suppose that Γ contains two elements $\gamma, \gamma' \in \Gamma$ that generate a non-Abelian free subsemigroup. Consider the polynomial $f \coloneqq \pm 3 \cdot 1_\Gamma - (1_\Gamma + \gamma - \gamma^2)\gamma' \in \mathbb{Z}[\Gamma]$. It follows from an argument similar to that in [19, Example 7.2] that the associated principal algebraic action is homoclinically expansive (though not expansive by [17, Example A.1] and [11, Theorem 3.2]). (cf. [10, Example 4.10].)

(4) In [10, Example 4.7] it is shown that for $\Gamma = \mathbb{Z}^d$, $d \geq 2$, the element $h = 2d - 1 - \sum_{j=1}^d (u_j + u_j^{-1}) \in \mathbb{Z}[u_1, u_1^{-1}, \cdots, u_d, u_d^{-1}] = \mathbb{Z}[\mathbb{Z}^d]$ satisfies that, for any $\frac{2d}{d-1} < p \leq +\infty$, the corresponding principal algebraic action α_h is not p-expansive. It follows from Proposition 3.5.(1) that α_h is not homoclinically expansive either.

Let us remark that, if $d = 2$, so that $h = 3 - u_1 - u_1^{-1} - u_2 - u_2^{-1} \in \mathbb{Z}[\mathbb{Z}^2]$, the zero-set $Z(h)$ is a 1-dimensional curve. As shown in [19, Example 7.3], any measure supported on $Z(h)$ with a smooth density yields, via its Fourier coefficients, a homoclinic point. As a consequence, the homoclinic group $\Delta(X_h, \alpha_h)$ is uncountable and therefore there are no nontrivial summable homoclinic points ($\Delta^1(X_h, \alpha_h) = \{0_{X_h}\}$). Now, it is the curvature of $Z(f)$ which makes this possible. This example emphasizes why the condition in Theorem 1.5 on the zero-set, namely, being contained in (the P-image of) a finite union of hyperplanes, is at least reasonable, since it prevents the curvature of the zero-set to lead to the above mentioned phenomenon. We thank one of the referees for pointing this out to us.

3.3. **Principal algebraic actions associated with weakly expansive polynomials.**

Theorem 3.9. *Let Γ be a countable group and suppose that $f \in \mathbb{Z}[\Gamma]$ is weakly expansive. Then the following hold:*

(1) *The element $\omega \in \mathcal{C}_0(\Gamma)$ satisfying* (we-2) *in Definition 1.1 is unique and, moreover, $\omega f = 1_\Gamma$.*
(2) *$\Delta(X_f, \alpha_f)$ is dense in X_f.*
(3) *$\Delta(X_f, \alpha_f)$ is isomorphic to $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f^*$ as a left $\mathbb{Z}[\Gamma]$-module.*

Proof. Let $\omega, \omega' \in \mathcal{C}_0(\Gamma)$ satisfying (we-2). Then $f\omega = 1_\Gamma = f\omega'$ yields $f(\omega - \omega') = 0$ and condition (we-1) in Definition 1.1 infers $\omega = \omega'$. This proves uniqueness of ω. Moreover, from (2.6) and (we-2) we deduce

$$f(\omega f) = (f\omega)f = 1_\Gamma f = f = f 1_\Gamma.$$

Thus, $f(\omega f - 1_\Gamma) = 0$, and, again by (we-1), we get $\omega f = 1_\Gamma$. This shows (1).

In order to prove (2), let now $a \in \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ with $\langle a, \pi(\gamma\omega^*)\rangle = 0$ for all $\gamma \in \Gamma$ (where $\pi \colon \ell^\infty(\Gamma) \to \mathbb{T}^\Gamma$ is as in Subsection 2.5). Write $a = g + \mathbb{Z}[\Gamma]f$ for some $g \in \mathbb{Z}[\Gamma]$. Then

$$\begin{aligned} 0 &= \langle a, \pi(\gamma\omega^*)\rangle \\ &= \sum_{\delta \in \Gamma} (\gamma\omega^*)_\delta g_\delta + \mathbb{Z} \\ &= \sum_{\delta \in \Gamma} (\gamma\omega^*)_\delta (g^*)_{\delta^{-1}} + \mathbb{Z} \\ &= (\gamma\omega^* g^*)_{1_\Gamma} + \mathbb{Z} \\ &= (\omega^* g^*)_{\gamma^{-1}} + \mathbb{Z} \\ &= (g\omega)_\gamma + \mathbb{Z} \end{aligned}$$

for all $\gamma \in \Gamma$. Thus $h := g\omega$ lies in $\ell^\infty(\Gamma, \mathbb{Z})$. Since $\omega \in \mathcal{C}_0(\Gamma)$ and $g \in \mathbb{Z}[\Gamma]$, one has $g\omega \in \mathcal{C}_0(\Gamma)$. Therefore $h \in \ell^\infty(\Gamma, \mathbb{Z}) \cap \mathcal{C}_0(\Gamma) = \mathbb{Z}[\Gamma]$. Using (2.6) and (1) it follows that

$$hf = (g\omega)f = g(\omega f) = g1_\Gamma = g,$$

and hence $g = hf \in \mathbb{Z}[\Gamma]f$, which means that $a = 0$. We have $\{\pi(\gamma\omega^*) : \gamma \in \Gamma\} \subset \Delta(X_f, \alpha_f)$ (cf. the end of Subsection 2.6) and by Pontryagin duality we conclude that $\Delta(X_f, \alpha_f)$ is dense in X_f.

We are only left to prove (3). In the proof of Proposition (3.7) (here we take $n = k = 1$ and $A = f$) we have defined an injective left $\mathbb{Z}[\Gamma]$-module morphism $\varphi \colon \Delta(X_f, \alpha_f) \to \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f^*$ sending x to $\tilde{x}f^* + \mathbb{Z}[\Gamma]f^*$. Let us show that φ is surjective. Let $h \in \mathbb{Z}[\Gamma]$. Using (we-2) and (2.6) we deduce that

$$\begin{aligned} \varphi(h\pi(\omega^*)) = \varphi(\pi(h\omega^*)) = (h\omega^*)f^* + \mathbb{Z}[\Gamma]f^* &= h(\omega^* f^*) + \mathbb{Z}[\Gamma]f^* \\ &= h(f\omega)^* + \mathbb{Z}[\Gamma]f^* = h1_\Gamma + \mathbb{Z}[\Gamma]f^* = h + \mathbb{Z}[\Gamma]f^*. \end{aligned}$$

This shows that φ is surjective. Therefore φ is indeed an isomorphism. □

It follows from the proof of Theorem 3.9.(3) and the notation therein that the cyclic $\mathbb{Z}[\Gamma]$-module $\Delta(X_f, \alpha_f)$ is generated by the element $x^\Delta := \pi(\omega^*) \in \Delta(X_f, \alpha_f)$, called the *fundamental homoclinic point* of (X_f, α_f) (cf. [19]).

Let (X, α) be an algebraic dynamical system with an infinite acting group Γ and denote by μ the Haar probability measure on X. Let $r \in \mathbb{N}$ with $r \geq 2$. One says that (X, α) is *mixing of order* r if, for all measurable subsets $B_1, B_2, \ldots, B_r \subset X$, one has

$$\mu(\gamma_1 B_1 \cap \gamma_2 B_2 \cap \cdots \cap \gamma_r B_r) \longrightarrow \mu(B_1)\mu(B_2)\cdots\mu(B_r) \tag{3.7}$$

as $\gamma_i^{-1}\gamma_j \to \infty$ in Γ for all $1 \leq i < j \leq r$. If (X, α) is mixing of order $r = 2$, one simply says that (X, α) is *mixing*. Note that every mixing algebraic dynamical system is ergodic. If (X, α) is mixing of order r for all $r \geq 2$, one then says that (X, α) is *mixing of all orders*.

Observe that if A is a compact metrizable Abelian group and Γ is any infinite countable group, then the Γ-shift (A^Γ, σ) is mixing of all orders since (3.7) is trivially satisfied when the B_is are cylinders, for all $r \geq 2$.

It follows from [4, Proposition 4.6] that an algebraic dynamical system (X, α) admitting a dense homoclinic group is mixing of all orders. From Theorem 3.9.(2) we then deduce the following:

Corollary 3.10. *Let Γ be an infinite countable group and suppose that $f \in \mathbb{Z}[\Gamma]$ is weakly expansive. Then the associated algebraic dynamical system (X_f, α_f) is mixing of all orders.*

3.4. **Expansive principal algebraic actions.** The following result is due to Deninger and Schmidt [11, Theorem 3.2] (see also [20, Theorem 5.1]).

Theorem 3.11. *Let Γ be a countable group and $f \in \mathbb{Z}[\Gamma]$. Then the following conditions are equivalent:*

(a) *the dynamical system (X_f, α_f) is expansive;*
(b) *f is invertible in $\ell^1(\Gamma)$.*

For other characterizations of expansivity for algebraic dynamical systems we refer to [30, 31] (for $\Gamma = \mathbb{Z}^d$, $d \in \mathbb{N}$), [26] (for Γ Abelian), [12] (for (X, α) finitely presented), [2] (for X connected and finite-dimensional), and [10, Theorem 3.1].

As observed in [20], if f is *lopsided*, i.e., there exists an element $\gamma_0 \in \Gamma$ such that $|f_{\gamma_0}| > \sum_{\gamma \neq \gamma_0} |f_\gamma|$, then f is invertible in $\ell^1(\Gamma)$. On the other hand, there are $f \in \mathbb{Z}[\Gamma]$ invertible in $\ell^1(\Gamma)$ that are not lopsided. For instance, if we take $\Gamma = \mathbb{Z}$, then the polynomial $u^2 - u - 1 \in \mathbb{Z}[\Gamma] = \mathbb{Z}[u, u^{-1}]$ is not lopsided although it is invertible in $\ell^1(\Gamma)$ (the associated principal algebraic dynamical system is conjugate to the $\mathbb{Z}$-system generated by Arnold's cat map $(x_1, x_2) \mapsto (x_2, x_1 + x_2)$ on the 2-dimensional torus $\mathbb{T}^2$, see e.g. [31, Example 2.18.(2)]).

The following result justifies our terminology for weakly expansive polynomials.

Corollary 3.12. *Let Γ be a countable group and $f \in \mathbb{Z}[\Gamma]$. Suppose that the dynamical system (X_f, α_f) is expansive. Then f is weakly expansive.*

Proof. Expansivity of (X_f, α_f) implies, by Theorem 3.11, that f is invertible in $\ell^1(\Gamma)$. Then $\omega := f^{-1} \in \ell^1(\Gamma) \subset \mathcal{C}_0(\Gamma)$ yields (we-2) in Definition 1.1.

Let now $g \in \mathcal{C}_0(\Gamma)$ and suppose that $fg = 0$. Then, recalling (2.2), we deduce that

$$0 = \omega 0 = \omega(fg) = (\omega f)g = 1_\Gamma g = g,$$

and (we-1) follows as well. □

3.5. **Harmonic models.** Let $f \in \mathbb{Z}[\Gamma]$ be well-balanced. It follows from (2.10) that $x \in \mathbb{T}^\Gamma$ belongs to X_f if and only if x satisfies the *harmonicity equation* (mod 1)

$$\sum_{\eta \in \Gamma} f_\eta x(\gamma\eta) = 0,$$

for all $\gamma \in \Gamma$. This explains the terminology. Note that for $\Gamma = \mathbb{Z}^d$, the polynomial $f \in \mathbb{Z}[\Gamma] = \mathbb{Z}[u_1, u_1^{-1}, \dots, u_d, u_d^{-1}]$ given by

$$f = 2d - \sum_{i=1}^{d}(u_i + u_i^{-1})$$

is well-balanced and the corresponding harmonicity equation is the discrete analogue of the *Laplace equation* (cf. Eq. (4.5) in [32]).

Lemma 3.13. *Let Γ be a countable infinite group and $f \in \mathbb{R}[\Gamma]$. Suppose that f is well-balanced. Then the map $g \mapsto fg$ from $\mathcal{C}_0(\Gamma)$ to $\mathcal{C}_0(\Gamma)$ is injective. In particular, harmonic models are homoclinically expansive.*

Proof. Set

$$\mu \coloneqq 1_\Gamma - \frac{1}{f_{1_\Gamma}} f \in \mathbb{R}[\Gamma]. \tag{3.8}$$

Then μ is a probability measure on Γ which is symmetric and its support $S \coloneqq \operatorname{supp}(\mu)$ generates Γ as a semigroup, by (wb-1), (wb-3), and (wb-4), respectively.

In order to show (we-1) we apply the maximum principle. Let $g \in \mathcal{C}_0(\Gamma)$ and suppose that $fg = 0$, equivalently, $\mu g = g$. Set $M \coloneqq \max_{\delta \in \Gamma} |g_\delta|$ and observe that $A \coloneqq \{\gamma \in \Gamma : |g_\gamma| = M\}$ is non-empty. Moreover, if $\gamma \in A$ one has, using the triangle inequality and the properties of μ we alluded to above,

$$M = |g_\gamma| = |(\mu g)_\gamma| \le \sum_{\delta \in S} \mu_{\delta^{-1}} |g_{\delta\gamma}| \le \sum_{\delta \in S} M \mu_\delta = M,$$

forcing $|g_{\delta\gamma}| = M$ for all $\delta \in S$. This shows that $SA \subset A$. A recursive argument immediately shows that $S^n A \subset A$ for all $n \in \mathbb{N}$. Since S generates Γ as a semigroup, we get that $A = \Gamma$. In other words, $|g|$ is a constant function. As $g \in \mathcal{C}_0(\Gamma)$, we conclude that $g = 0$.

The last statement follows immediately after Theorem 3.6. □

In the arguments preceding Lemma 4.8 in [4] it is shown that if Γ is a countable infinite group which is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ and $f \in \mathbb{Z}[\Gamma]$ is well-balanced, then

$$\omega \coloneqq \frac{1}{f_{1_\Gamma}} \sum_{k=0}^{\infty} \mu^k \in \mathcal{C}_0(\Gamma), \tag{3.9}$$

where μ is as in (3.8), satisfies that $f\omega = 1_\Gamma$, so that (we-2) holds. Combining this with Lemma 3.13, we deduce:

Proposition 3.14. *Let Γ be a countable infinite group Γ that is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Then every balanced polynomial $f \in \mathbb{Z}[\Gamma]$ is weakly expansive.*

Remark 3.15. It is a well known fact in the theory of Markov chains (cf. for instance [36, Definition 1.14]) that the sum in (3.9) expressing ω is (pointwise) convergent if and only if the random walk on Γ associated with μ is *transient* (i.e., given any finite subset $\Omega \subset \Gamma$, there exists $t(\Omega) \in \mathbb{N}$ such that, with probability one, the position $x(t) \in \Gamma$ of the random walker on Γ at time $t \geq t(\Omega)$ satisfies that $x(t) \in \Gamma \setminus \Omega$) and it is a deep result of Varopoulos (cf. [33, 34] and [36, Theorem 3.24]) that this is the case exactly if Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$.

4. Topological rigidity

4.1. **Affine maps.** Let X and Y be two topological Abelian groups. A map $\tau\colon Y \to X$ is called *affine* if there is a continuous group morphism $\lambda\colon Y \to X$ and an element $t \in X$ such that $\tau(y) = \lambda(y) + t$ for all $y \in Y$. Note that λ and t are then uniquely determined by τ since they must satisfy $t = \lambda(0_Y)$ and $\lambda(y) = \tau(y) - t$ for all $y \in Y$. One says that λ and t are respectively the *linear part* and the *translational part* of the affine map τ.

The following two obvious criteria will be useful in the sequel.

Proposition 4.1. *Let (X, α) be an algebraic dynamical system and let $\tau\colon X \to X$ be an affine map with linear part $\lambda\colon X \to X$. Then the following conditions are equivalent:*
(a) *τ is pre-injective;*
(b) *λ is pre-injective;*
(c) $\mathrm{Ker}(\lambda) \cap \Delta(X, \alpha) = \{0_X\}$.

Proposition 4.2. *Let (X, α) be an algebraic dynamical system and let $\tau\colon X \to X$ be an affine map with linear part $\lambda\colon X \to X$ and translational part $t \in X$. Then the following conditions are equivalent:*
(a) *τ is Γ-equivariant;*
(b) *λ is Γ-equivariant and $t \in \mathrm{Fix}(X, \alpha)$.*

4.2. **Topological rigidity.** Let (X, α) be an algebraic dynamical system with acting group Γ. One says that (X, α) is *topologically rigid* if every endomorphism $\tau\colon X \to X$ of (X, α) is affine.

Before stating our rigidity results, let us introduce some notation. Let $L(X)$ denote the real vector space of all group homomorphisms $\widehat{X} \to \mathbb{R}$ equipped with the topology of pointwise convergence. Note that Γ acts on $L(X)$ by setting $[\gamma\psi](\chi) \coloneqq \psi(\gamma^{-1}\chi)$ for all $\psi \in L(X)$ and $\chi \in \widehat{X}$. Moreover, the map $E\colon L(X) \to X$ defined by $E(\psi)(\chi) \coloneqq \psi(\chi) + \mathbb{Z}$ for all $\psi \in L(X)$ and $\chi \in \widehat{X}$ is a continuous (Γ-equivariant) group homomorphism.

Theorem 4.3. *Let (X, α) and (Y, β) be algebraic dynamical systems with acting group Γ. Suppose that Y is connected and that the homoclinic group $\Delta(Y, \beta)$ is dense in Y. Also* *suppose that* (X, α) *is homoclinically expansive. Then every* Γ*-equivariant continuous map* $Y \to X$ *is affine.*

Proof. Let $\tau\colon Y \to X$ be a Γ-equivariant continuous map. By Bhattacharya's extension of van Kampen theorem [1, Theorem 1], there are a Γ-equivariant affine map $\lambda\colon Y \to X$ and a Γ-equivariant continuous map $\Phi\colon Y \to L(X)$ such that $\Phi(0_Y) = 0$ and $\tau = \lambda + E \circ \Phi$. (Note that in the statement of [1, Theorem 1], X is assumed to be connected as well; however, in its proof, this condition is never used.) Thus it suffices to show that $\Phi = 0$. Let $y \in \Delta(Y, \beta)$ and $\chi \in \widehat{X}$. For any $\gamma \in \Gamma$ we have

$$[\Phi(\gamma y)](\chi) = [\gamma\Phi(y)](\chi) = [\Phi(y)](\gamma^{-1}\chi).$$

When $\gamma \to \infty$, we have $\Phi(\gamma y) \to \Phi(0_Y) = 0$, and hence $[\Phi(y)](\gamma^{-1}\chi) \to 0$. For any $t \in \mathbb{R}$, we have

$$\begin{aligned}\Psi_{E(t\Phi(y)),\chi}(\gamma) = |\langle \gamma E(t\Phi(y)), \chi\rangle| = |\langle E(t\Phi(y)), \gamma^{-1}\chi\rangle| \\ \leq |[t\Phi(y)](\gamma^{-1}\chi)| = |t| \cdot |[\Phi(y)](\gamma^{-1}\chi)|\end{aligned}$$

for all $\gamma \in \Gamma$, and hence $\Psi_{E(t\Phi(y)),\chi} \in \mathcal{C}_0(\Gamma)$. Therefore $E(t\Psi(y)) \in \Delta(X, \alpha)$. Since α is homoclinically expansive, by Proposition 3.4 there exist a finite subset W of $\widehat{X}$ and $\varepsilon > 0$ such that 0_X is the only point x in $\Delta(X, \alpha)$ satisfying

$$\sum_{\chi \in W} \|\Psi_{x,\chi}\|_\infty < \varepsilon.$$

Set $C := \sum_{\chi \in W} \sup_{\gamma \in \Gamma} |[\Phi(y)](\gamma^{-1}\chi)| < \infty$. Then

$$\sum_{\chi \in W} \|\Psi_{E(t\Phi(y)),\chi}\|_\infty \leq |t| C.$$

Thus for all $t \in \mathbb{R}$ with $|t| < \varepsilon/C$ we have $E(t\Phi(y)) = 0_X$, which means that $t\Phi(y)$ takes integer values. It follows that $\Phi(y) = 0$. Since $\Delta(Y, \beta)$ is dense in Y and Φ is continuous, we conclude that $\Phi = 0$ as desired. ☐

Corollary 4.4. *Let* (X, α) *be an algebraic dynamical system. Suppose that* X *is connected, that the homoclinic group* $\Delta(X, \alpha)$ *is dense in* X*, and that* (X, α) *is homoclinically expansive. Then* (X, α) *is topologically rigid.*

Corollary 4.5. *Let* Γ *be a countable group. Suppose that* $f \in \mathbb{Z}[\Gamma]$ *satisfies that* (X_f, α_f) *is homoclinically expansive and* $\Delta(X_f, \alpha_f)$ *is dense in* X_f *(e.g., that* f *is weakly expansive), and the phase space* X_f *is connected. Then* (X_f, α_f) *is topologically rigid.*

If, in addition, Γ *is Abelian, then for a map* $\tau\colon X_f \to X_f$ *the following conditions are equivalent:*

(a) τ *is an endomorphism of the dynamical system* (X_f, α_f)*;*
(b) *there exist* $r \in \mathbb{Z}[\Gamma]$ *and* $t \in \mathrm{Fix}(X_f, \alpha_f)$ *such that* $\tau(x) = rx + t$ *for all* $x \in X_f$*.*

Proof. The first statement follows immediately from Theorem 4.3 after taking $X = Y = X_f$ (if f is weakly expansive, recall Theorem 3.6 and Theorem 3.9.(2)).

Suppose now that Γ is Abelian and let $\tau\colon X_f \to X_f$ be a map. For each $r \in \mathbb{Z}[\Gamma]$, the self-mapping of X_f given by $x \mapsto rx$ is Γ-equivariant since $\mathbb{Z}[\Gamma]$ is commutative. Therefore, the fact that (b) implies (a) follows from Proposition 4.2.

To prove the converse, suppose that τ is an endomorphism of the dynamical system (X_f, α_f). It follows from the first statement that τ is affine. Therefore, by using Proposition 4.2, there exist a continuous $\mathbb{Z}[\Gamma]$-module morphism $\lambda\colon X_f \to X_f$ and $t \in \mathrm{Fix}(X_f, \alpha_f)$ such that $\tau(x) = \lambda(x) + t$ for all $x \in X_f$. As the ring $\mathbb{Z}[\Gamma]$ is commutative and $\widehat{X_f} = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ is a cyclic $\mathbb{Z}[\Gamma]$-module, there is $s \in \mathbb{Z}[\Gamma]$ such that $\widehat{\lambda}(\chi) = s\chi$ for all $\chi \in \widehat{X_f}$. Since $\lambda = \widehat{\widehat{\lambda}}$, setting $r \coloneqq s^* \in \mathbb{Z}[\Gamma]$ it follows that $\lambda(x) = s^*x = rx$, and hence $\tau(x) = rx+t$, for all $x \in X_f$. $\square$

5. Proof of Theorem 1.2

Proof of Theorem 1.2. Let Γ be a countable Abelian group and let $f \in \mathbb{Z}[\Gamma]$ be a weakly expansive polynomial such that X_f is connected. Also let τ be an endomorphism of (X_f, α_f), i.e., a Γ-equivariant continuous map $\tau\colon X_f \to X_f$. We want to show that τ is surjective if and only if it is pre-injective.

By Corollary 4.5, there exists $r \in \mathbb{Z}[\Gamma]$ such that τ is affine with linear part $\lambda\colon X_f \to X_f$ given by $\lambda(x) = rx$ for all $x \in X_f$. Clearly τ is surjective if and only if λ is. As X_f is compact, we know that surjectivity of λ is equivalent to injectivity of its Pontryagin dual $\widehat{\lambda}\colon \widehat{X_f} \to \widehat{X_f}$ (cf. [28, Proposition 30]). Now we observe that $\widehat{\lambda}(\chi) = r^*\chi$ for all $\chi \in \widehat{X_f}$.

As Γ is Abelian, the natural mapping $\Phi\colon \widehat{X_f} = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f \to \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f^*$ defined by

$$\Phi(g + \mathbb{Z}[\Gamma]f) = g^* + \mathbb{Z}[\Gamma]f^*$$

for all $g \in \mathbb{Z}[\Gamma]$ is a group isomorphism. Note that

$$\Phi(g_1(g_2+\mathbb{Z}[\Gamma]f)) = \Phi(g_1g_2+\mathbb{Z}[\Gamma]f) = (g_1g_2)^*+\mathbb{Z}[\Gamma]f^* = g_1^*g_2^*+\mathbb{Z}[\Gamma]f^* = g_1^*\Phi(g_2+\mathbb{Z}[\Gamma]f)$$

for all $g_1, g_2 \in \mathbb{Z}[\Gamma]$. As a consequence, denoting by $\kappa\colon \widehat{X_f} \to \Delta(X_f, \alpha_f)$ the composition of Φ with the $\mathbb{Z}[\Gamma]$-module isomorphism $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f^* \to \Delta(X_f, \alpha_f)$ in Theorem 3.9.(3), we have the commuting diagram

$$\begin{array}{ccc} \widehat{X_f} & \xrightarrow{\chi \mapsto r^*\chi} & \widehat{X_f} \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle\kappa} \\ \Delta(X_f, \alpha_f) & \xrightarrow{x \mapsto rx} & \Delta(X_f, \alpha_f). \end{array}$$

We deduce that injectivity of $\widehat{\lambda}$ is equivalent to injectivity of the group endomorphism μ of $\Delta(X_f, \alpha_f)$ defined by $\mu(x) \coloneqq rx$ for all $x \in \Delta(X_f, \alpha_f)$. As μ is the restriction of λ to $\Delta(X_f, \alpha_f)$, we conclude that surjectivity of τ is equivalent to pre-injectivity of τ, by using Proposition 4.1. $\square$

From the proof of Theorem 1.2 and Proposition 3.7 we immediately deduce the following:

Corollary 5.1. *Let Γ be a countable Abelian group and let $f \in \mathbb{Z}[\Gamma]$. Suppose that (X_f, α_f) is homoclinically expansive, $\Delta(X_f, \alpha_f)$ is dense in X_f, and that X_f is connected. Then (X_f, α_f) has the Moore property.*

6. Atoral polynomials and proof of Theorem 1.5

Lind, Schmidt, and Verbistkiy [22, Theorem 3.2] gave the following geometric-dynamical characterization of atorality for irreducible polynomials in $\mathbb{Z}[\mathbb{Z}^d]$.

Theorem 6.1. *Let $\Gamma = \mathbb{Z}^d$ and suppose that $f \in \mathbb{Z}[\Gamma]$ is irreducible. Then the following conditions are equivalent:*
(a) $\Delta^1(X_f, \alpha_f) \neq \{0_{X_f}\}$*;*
(b) $\Delta^1(X_f, \alpha_f)$ *is dense in* X_f*;*
(c) f *is* atoral *in the sense that there is some* $g \in \mathbb{Z}[\Gamma]$ *such that* $g \notin \mathbb{Z}[\Gamma]f$ *and* $Z(f) \subset Z(g)$*;*
(d) $\dim Z(f) \leq d - 2$.

The meaning of $\dim(\cdot)$ in Theorem 6.1.(d) is explained in [22, page 1063]; in particular, one has $\dim(\varnothing) := -\infty$. Also remark that, if $d = 1$, an irreducible polynomial $f \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[u_1, u_1^{-1}]$ is atoral if and only if $Z(f) = \varnothing$ and this, in turn, is equivalent to (X_f, α_f) being expansive (cf. [19, Lemmma 2.1.(1)]).

Examples 6.2. Here below, we present some examples of irreducible atoral polynomials $f \in \mathbb{Z}[\mathbb{Z}^d]$, mainly from [21, Section 3] and [22, Section 4]. We can then apply Theorem 1.5 and deduce that the corresponding algebraic dynamical systems (X_f, α_f) satisfy the Garden of Eden theorem.

(1) Let $d = 1$ and $f(u) = u^2 - u - 1 \in \mathbb{Z}[u, u^{-1}] = \mathbb{Z}[\mathbb{Z}]$. Then f is irreducible and, since $Z(f) = \varnothing$, atoral. The associated principal algebraic dynamical system (X_f, α_f) is conjugated to the hyperbolic dynamical system $(\mathbb{T}^2, \varphi_A)$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ is *Arnold's cat* and $\varphi_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ is the associated hyperbolic automorphism of the two-dimensional torus. Note that (X_f, α_f) is expansive so that we can deduce the Moore-Myhill property also from Corollary 1.3 (this is a particular case of the Garden of Eden theorem for Anosov diffeomorphisms on tori [5], we alluded to in the Introduction).
(2) Let $d = 2$ and $f(u_1, u_2) = 2 - u_1 - u_2 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$. Then $Z(f) = \{(1, 1)\}$, and so f is atoral. Thus, by Example 3.8.(2), (X_f, α_f) is homoclinically expansive. Also, it follows from [21, Section 5] and [22, Example 4.3]) that there is some ω in $\mathcal{C}_0(\mathbb{Z}^2)$ with $f\omega = \omega f = 1_\Gamma$. As a consequence, f is weakly expansive (though not well-balanced). Moreover, f is also primitive, so that, by Proposition 2.1, X_f is connected. Applying Theorem 1.2, we obtain an alternative proof of the fact that (X_f, α_f) has the Moore-Myhill property.
(3) Let $d = 2$, and consider the Laplace harmonic model $f(u_1, u_2) = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1} \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$. One has $Z(f) = \{(1, 1)\}$. Thus f is atoral and (X_f, α_f)

satisfies the Garden of Eden theorem, by virtue of Theorem 1.5. (Note that we cannot apply Theorem 1.2.)

(4) Let $d = 2$, and $f(u_1, u_2) = 1 + u_1 + u_2 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$. Then $Z(f) = \{(\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = \exp(2\pi i/3)$. The algebraic dynamical system (X_f, α_f) is called the *connected Ledrappier shift*.

(5) Let $d = 2$, and $f(u_1, u_2) = 2 - u_1^2 + u_2 - u_1 u_2 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$. One has $Z(f) = \{(\xi, \eta), (\overline{\xi}, \overline{\eta})\}$, where ξ, η are algebraic numbers.

(6) Let $d = 2$ and $f(u_1, u_2) = 2 - u_1^3 + u_2 - u_1 u_2 - u_1^2 u_2 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$. Here $Z(f) = \{(1, 1), (i, \zeta), (-i, \overline{\zeta}), (\xi, \eta), (\overline{\xi}, \overline{\eta})\}$, where ζ, ξ, η are algebraic numbers.

(7) Let $d = 3$ and $f(u_1, u_2, u_3) = 1 + u_1 + u_2 + u_3 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}, u_3, u_3^{-1}] = \mathbb{Z}[\mathbb{Z}^3]$. The corresponding zero-set $Z(f)$ is the union of three circles, namely, $\{(-1, s, -s) : s \in \mathbb{S}\}$, $\{(s, -1, -s) : s \in \mathbb{S}\}$, and $\{(s, -s, -1) : s \in \mathbb{S}\}$. Hence, f is atoral.

(8) Let $d = 3$ and $f(u_1, u_2, u_3) = 3 + 3u_1 - 3u_1^3 + u_1^4 - u_2 - u_3 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}, u_3, u_3^{-1}] = \mathbb{Z}[\mathbb{Z}^3]$. One has has $Z(f) = \{(\eta, \overline{\eta}, \overline{\eta}), (\overline{\eta}, \eta, \eta)\}$, where η is an algebraic integer.

Remark 6.3. (i) Let $d = 1$ and $f = 2 - u - u^{-1} \in \mathbb{Z}[u, u^{-1}] = \mathbb{Z}[\mathbb{Z}]$. Then the associated dynamical system $X_f = \{x \in \mathbb{T}^{\mathbb{Z}} : x(n-1) + x(n+1) = 2x(n) \text{ for all } n \in \mathbb{Z}\}$ is the one-dimensional Laplace harmonic model. It is easy to see that $\Delta(X_f, \alpha_f) = \{0_{\mathbb{T}^{\mathbb{Z}}}\}$. Then (X_f, α_f) satisfies the Moore property but not the Myhill property (the constant map $x \mapsto 0_{\mathbb{T}^{\mathbb{Z}}}$ (which is a pre-injective endomorphism of (X_f, α_f)) is clearly not surjective).

(ii) It follows from Examples 6.2.(4) that the connected Ledrappier shift $X = \{x \in \mathbb{T}^{\mathbb{Z}^2} : x(m, n) + x(m+1, n) + x(m, n+1) = 0 \text{ for all } m, n \in \mathbb{Z}\}$ satisfies the Garden of Eden theorem. On the other hand, the *(disconnected) Ledrappier shift* $X' \coloneqq \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x(m, n) + x(m+1, n) + x(m, n+1) = 0 \text{ for all } m, n \in \mathbb{Z}\}$ (which may be regarded as an algebraic dynamical system with phase space $\widehat{\mathbb{Z}[\mathbb{Z}^2]/I}$, where $I = 2\mathbb{Z}[\mathbb{Z}^2] + f\mathbb{Z}[\mathbb{Z}^2]$ is the ideal generated by 2 and $f(u_1, u_2) = 1 + u_1 + u_2 \in \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}] = \mathbb{Z}[\mathbb{Z}^2]$) does not satisfy the Garden of Eden theorem. Indeed, one has $\Delta(X', \alpha') = \{0_{(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}}\}$ so that every map $\tau \colon X' \to X'$ is pre-injective. This ensures the Moore property for (X', α'). However, the constant map $x \mapsto 0_{(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}}$ (which is a pre-injective endomorphism of (X', α')) is clearly not surjective, showing that (X', α') does not satisfy the Myhill property. This example shows, in particular, that the Garden of Eden theorem fails to hold for general (not connected) algebraic dynamical systems. Note that, however, in the classical Garden of Eden Theorem of Moore and Myhill [27, 29], as well as in its generalizations [25, 8, 13, 18], the phase space is the full shift A^G (A a finite alphabet set and G an amenable group) or a subshift $X \subset A^G$ (satisfying suitable irreducibility conditions), and therefore is totally disconnected. Now, the method used in the present paper (rigidity) is totally different from the method used in these previous papers (entropy), and we don't know how to combine the two methods together.

We are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. We first note that, by Example 3.8.(2), α_f is homoclinically expansive. Moreover, by Theorem 6.1, $\Delta^1(X_f, \alpha_f)$ and therefore $\Delta(X_f, \alpha_f)$ are dense in X_f. Finally, since f is irreducible it is primitive and hence (cf. Proposition 2.1) X_f is connected.

Thus we may apply Corollary 4.5 and deduce that (X_f, α_f) is topologically rigid.
(a)$\Rightarrow$(b) is given by Corollary 5.1.
(b)$\Rightarrow$(c) follows from $\Delta^p(X_f, \alpha_f) \subset \Delta(X_f, \alpha_f)$ for all $1 \leq p \leq \infty$ (cf. Theorem 3.2.(1)).
(c)$\Rightarrow$(d) this is trivial. (d)$\Rightarrow$(e) follows from $\Delta^p(X) \subseteq \Delta^q(X)$ for all $1 \leq p \leq q \leq +\infty$ (cf. [10, Proposition 5.2.(1)]). (e)$\Rightarrow$(a). Let $\tau \colon X_f \to X_f$ be a Γ-equivariant 1-pre-injective continuous map. By topological rigidity, there exist $r \in \mathbb{Z}[\Gamma]$ and $t \in X_f$ such that the group endomorphism λ of X_f defined by setting $\lambda(x) := rx$ for all $x \in X_f$ satisfies that $\tau(x) = \lambda(x) + t$ for all $x \in X_f$. Then the dual map $\widehat{\lambda} \colon \widehat{X_f} \to \widehat{X_f}$ is given by $\widehat{\lambda}(\chi) = r^*\chi$ for all $\chi \in \widehat{X_f}$. Set $\mathbf{m}_f = \mathbb{Z}[\Gamma] \cap (f\ell^1(\Gamma))$, which is an ideal of $\mathbb{Z}[\Gamma]$. For each $g \in \mathbf{m}_f$, one has $g = fv_g$ for a unique $v_g \in \ell^1(\Gamma)$. Then $x^{(g)} := \pi(v_g^*)$ lies in $\Delta^1(X_f, \alpha_f)$, where π is, as usual, the projection map $\ell^\infty(\Gamma) \to \mathbb{T}^\Gamma$. The map $g + \mathbb{Z}[\Gamma]f \mapsto x^{(g)}$ is clearly a group isomorphism from $\mathbf{m}_f/\mathbb{Z}[\Gamma]f$ onto $\Delta^1(X_f, \alpha_f)$ (cf. [22, Corollary 3.3]). Take $g \in \mathbf{m}_f \setminus \mathbb{Z}[\Gamma]f$. Since f is irreducible, the group homomorphism $h + \mathbb{Z}[\Gamma]f \mapsto hg + \mathbb{Z}[\Gamma]f$ from $\widehat{X_f}$ to $\mathbf{m}_f/\mathbb{Z}[\Gamma]f$ is injective. Thus the group homomorphism $\overline{\kappa} \colon \widehat{X_f} \to \Delta^1(X_f, \alpha_f)$ defined by $\overline{\kappa}(h + \mathbb{Z}[\Gamma]f) := x^{(hg)}$ is injective. We have the commutative diagram:

$$\begin{array}{ccc}
\widehat{X_f} & \xrightarrow{\chi \mapsto r^*\chi} & \widehat{X_f} \\
\scriptstyle{\overline{\kappa}}\downarrow & & \downarrow\scriptstyle{\overline{\kappa}} \\
\Delta^1(X_f, \alpha_f) & \xrightarrow{x \mapsto rx} & \Delta^1(X_f, \alpha_f).
\end{array}$$

Since τ is 1-pre-injective, so is λ, that is, λ is injective on $\Delta^1(X_f, \alpha_f)$. Hence the dual map $\widehat{\lambda}$ is injective. By Pontryagin duality, this is equivalent to λ being surjective. It follows that τ is surjective as well. $\square$

We are now in position to present a proof of the Garden of Eden theorem for Laplace harmonic models:

Proof of Corollary 1.6. For $d = 1$ it follows from Remark 6.3.(i) that the Laplace harmonic model fails to satisfy the Myhill property. On the other hand, it follows from Examples 6.2.(3) (resp. Corollary 1.4) that the Laplace harmonic model satisfies the Moore-Myhill property for $d = 2$ (resp. $d \geq 3$). $\square$

7. Concluding remarks

7.1. **Surjunctivity.** A dynamical system (X, α) is called *surjunctive* if every injective endomorphism of (X, α) is surjective (and hence a homeomorphism). As injectivity implies pre-injectivity, we deduce from Theorem 1.2 that if Γ is a countable Abelian group, $f \in \mathbb{Z}[\Gamma]$ is weakly expansive and X_f is connected, then the dynamical system (X_f, α_f) is surjunctive. Actually, in the case $\Gamma = \mathbb{Z}^d$, this last result is a particular case of Theorem 1.5 in [3] which asserts that if $\Gamma = \mathbb{Z}^d$ and M is a finitely generated $\mathbb{Z}[\Gamma]$-module, then $(\widehat{M}, \alpha_M)$ is surjunctive.

7.2. **Counterexamples for mixing algebraic dynamical systems.** Recall (cf. Corollary 3.10) that if Γ is infinite and $f \in \mathbb{Z}[\Gamma]$ is weakly expansive, then the associated algebraic dynamical system (X_f, α_f) is mixing of all orders.

The examples below show that Theorem 1.2 becomes false if the hypothesis that $f \in \mathbb{Z}[\Gamma]$ is weakly expansive is replaced by the weaker hypotheses that the homoclinic group $\Delta(X_f, \alpha_f)$ is dense and that (X_f, α_f) is mixing.

Example 7.1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\Gamma = \mathbb{Z}^d$, $d \geq 1$, and consider the Γ-shift $(\mathbb{T}^\Gamma, \sigma)$ (this is (X_f, α_f) for $f = 0 \in \mathbb{Z}[\Gamma]$). Then the endomorphism τ of $(\mathbb{T}^\Gamma, \sigma)$ defined by $\tau(x)(\gamma) = 2x(\gamma)$ for all $x \in \mathbb{T}^\Gamma$ and $\gamma \in \Gamma$, is clearly surjective. However, τ is not pre-injective since the configuration $y \in \mathbb{T}^\Gamma$, defined by $y(\gamma) = 1/2 \mod 1$ if $\gamma = 1_\Gamma$ and 0 otherwise, is a non-trivial element in the homoclinic group of $(\mathbb{T}^\Gamma, \sigma)$ and satisfies $\tau(y) = \tau(0_{\mathbb{T}^\Gamma}) = 0_{\mathbb{T}^\Gamma}$. It follows that $(\mathbb{T}^\Gamma, \sigma)$ does not have the Moore property.

Example 7.2. Let $\Gamma = \mathbb{Z}$ and consider the polynomial

$$f = 1 - 2u_1 + u_1^2 - 2u_1^3 + u_1^4 \in \mathbb{Z}[u_1, u_1^{-1}] = \mathbb{Z}[\Gamma].$$

The associated algebraic dynamical system (X_f, α_f) is conjugate to the system $(\mathbb{T}^4, \beta)$, where β is the action of $\mathbb{Z}$ on $\mathbb{T}^4$ generated by the companion matrix of f. It is mixing since f is not divisible by a cyclotomic polynomial (cf. [31, Theorem 6.5.(2)]). On the other hand, f has four distinct roots in $\mathbb{C}$, two on the unit circle, one inside and one outside. As f is irreducible over $\mathbb{Q}$, it follows that the homoclinic group $\Delta(X_f, \alpha_f)$ is reduced to 0 (cf. [19, Example 3.4]). The trivial endomorphism of (X_f, α_f), that maps each $x \in X_f$ to 0, is pre-injective but not surjective. Consequently, (X_f, α_f) does not have the Myhill property. However, (X_f, α_f) has the Moore property since each homoclinicity class of (X_f, α_f) is reduced to a single point, so that every map with source set X_f is pre-injective. Note that $(X_f, \alpha_f) = (\mathbb{T}^4, \beta)$ is topologically rigid since every mixing toral automorphism is topologically rigid by a result of Walters [35].

7.3. **p-pre-injectivity and the p-Moore and p-Myhill properties.** In Subsection 3.1 we have recalled from [10] the notions and the main properties of p-expansivity and of p-homoclinic group (denoted $\Delta^p(X, \alpha)$) for an algebraic dynamical system (X, α). In the Introduction we have also defined the notion of a p-pre-injective map $\tau \colon X \to X$.

Note that, if τ is a group endomorphism of X, then (cf. Proposition 4.1):

$$\tau \text{ is } p\text{-pre-injective if and only if } \ker(\tau) \cap \Delta^p(X, \alpha) = \{0_X\}. \tag{7.1}$$

We shall then say that the algebraic dynamical system (X, α) has the *p-Moore property* if every surjective endomorphism of (X, α) is p-pre-injective and that it has the *p-Myhill property* if every p-pre-injective endomorphism of (X, α) is surjective. Note that the ∞-Moore property (resp. ∞-Myhill property) is nothing but the Moore property (resp. Myhill property).

It follows immediately from Theorem 3.2.(1) that if $1 \leq p \leq q \leq \infty$ the every q-pre-injective map $\tau \colon X \to X$ is p-pre-injective, so that every algebraic dynamical system satisfying the q-Moore property (resp. the p-Myhill property) satisfies the p-Moore property (resp. the q-Myhill property).

From the proof of Theorem 1.2 and from Theorem 3.2.(5) we immediately deduce the following:

Corollary 7.3. *Let Γ be a countable Abelian group and let $f \in \mathbb{Z}[\Gamma]$. Let also $1 \leq p < \infty$. Suppose that (X_f, α_f) is p-expansive and topologically rigid. Then (X_f, α_f) has the p-Moore property.*

In the following result we relax the commutativity condition for the acting group Γ.

Theorem 7.4. *Let (X, α) be a finitely generated algebraic dynamical system. Suppose that Γ is amenable and that (X, α) is topologically rigid and has finite entropy. Then (X, α) has the 1-Moore property.*

Proof. Let $\tau \colon X \to X$ be a continuous Γ-equivariant surjective map. By topological rigidity, we can find a continuous group endomorphism $\lambda \colon X \to X$ and $t \in X$ such that $\tau(x) = \lambda(x) + t$ for all $x \in X$. Since τ is surjective (resp. injective) if and only if λ is surjective (resp. injective), it is not restrictive to suppose that $t = 0$, that is, τ is a group endomorphism of X. Then the *entropy addition formula* (cf. [17, Corollary 6.3], see also [14, Theorem 13.48]) yields (we denote by $h(\cdot)$ topological entropy)

$$h(\ker(\tau)) = h(X) - h(X/\ker(\tau)) = h(X) - h(\tau(X)) = 0,$$

since τ is surjective and X has finite entropy. For any finitely generated algebraic dynamical system (Y, β) with acting group Γ, if $\Delta^1(Y, \beta)$ is nontrivial, then $h(Y) > 0$ (cf. [10, Theorem 7.3 and Propositions 5.7 and 7.10], see also [14, Theorems 13.23 and 13.35]). Since $\widehat{X}$ is a finitely generated left $\mathbb{Z}[\Gamma]$-module and $\widehat{\ker(\tau)}$ is a quotient $\mathbb{Z}[\Gamma]$-module of $\widehat{X}$, $\widehat{\ker(\tau)}$ is also a finitely generated left $\mathbb{Z}[\Gamma]$-module. It follows that

$$\Delta^1(X, \alpha) \cap \ker(\tau) = \Delta^1(\ker(\tau), \alpha|_{\ker(\tau)}) = \{0_X\}.$$

This, together with (7.1), shows that τ is 1-pre-injective. □

References

[1] S. Bhattacharya, *Orbit equivalence and topological conjugacy of affine actions on compact Abelian groups*, Monatsh. Math. 129 (2000), pp. 89–96.

[2] S. Bhattacharya, *Expansiveness of algebraic actions on connected groups*, Trans. Am. Math. Soc. 356 (12) (2004), pp. 4687–4700.

[3] S. Bhattacharya, T. Ceccherini-Silberstein, and M. Coornaert, *Surjunctivity and topological rigidity of algebraic dynamical systems*, arXiv:1702.06201, to appear in Ergodic Theory and Dynamical Systems.

[4] L. Bowen and H. Li, *Harmonic models and spanning forsets of residually finite groups*, J. Funct. Anal. 263 (2012), pp. 1769–1808.

[5] T. Ceccherini-Silberstein and M. Coornaert, *Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property*, Illinois J. Math. 59 (2015), pp. 597–621.

[6] T. Ceccherini-Silberstein and M. Coornaert, *A Garden of Eden theorem for Anosov diffeomorphisms on tori*, Topology Appl. 212 (2016), pp. 49–56.

[7] T. Ceccherini-Silberstein and M. Coornaert, *A Garden of Eden theorem for principal algebraic actions*, arXiv:1706.06548.

[8] T. Ceccherini-Silberstein, A. Machì, and F. Scarabotti, *Amenable groups and cellular automata*, Ann. Inst. Fourier (Grenoble) 49 (1999), pp. 673–685.
[9] C. Chou, *Elementary amenable groups*, Ill. J. Math. 24(3) (1980), pp. 396–407.
[10] N.-Ph. Chung and H. Li, *Homoclinic group, IE groups, and expansive algebraic actions*, Invent. math. 199 (2015), pp. 805–858.
[11] C. Deninger and K. Schmidt, *Expansive algebraic actions of discrete residually finite amenable groups and their entropy*, Ergodic Theory Dynam. Systems 27 (2007), pp. 769–786.
[12] M. Einsiedler and H. Rindler, *Algebraic actions of the discrete Heisenberg group and other non-abelian groups*, Aequ. Math. 62(1-2) (2001), pp. 117–135.
[13] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) 1 (1999), pp. 109–197.
[14] D. Kerr and H. Li, *Ergodic theory. Independence and dichotomies.* Springer Monographs in Mathematics. Springer, Cham, 2016.
[15] P. H. Kropholler, P. A. Linnell, and J. A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. 104 (1988), pp. 675–684.
[16] T. Y. Lam, *Lectures on modules and rings.* Graduate Texts in Mathematics. Springer, New York (1999).
[17] H. Li, *Compact group automorphisms, addition formulas and Fuglede-Kadison determinants*, Ann. of Math. (2) 176 (2012), pp. 303–347.
[18] H. Li, *Garden of Eden and specification*, Ergodic Theory Dynam. Systems (to appear). arXiv:1708.09012.
[19] D. Lind and K. Schmidt, *Homoclinic points of algebraic $\mathbf{Z}^d$-actions*, J. Amer. Math. Soc. 12 (1999), pp. 953–980.
[20] D. Lind and K. Schmidt, *A survey of algebraic actions of the discrete Heisenberg group*, Uspekhi Mat. Nauk 70 (2015), pp. 77–142.
[21] D. Lind, K. Schmidt and E. Verbitskiy, *Entropy and growth rate of periodic points of algebraic $\mathbf{Z}^d$-actions.* In: Dynamical numbers: interplay between dynamical systems and number theory, ed. S. Kolyada, Yu. Manin, M. Möller, P. Moree and T. Ward, pp. 195–211, Contemporary Mathematics, vol. 523. American Mathematical Society, Providence (RI) (2010).
[22] D. Lind, K. Schmidt, and E. Verbitskiy, *Homoclinic points, atoral polynomials, and periodic points of algebraic $\mathbb{Z}^d$-actions*, Ergodic Theory Dynam. Systems 33 (2013), pp. 1060–1081.
[23] P.A. Linnell, *Analytic versions of the zero divisor conjecture.* In: Geometry and cohomology in group thoery (Durham, 1994), Lecture Notes Series, vol. 252, pp. 209–248. London Mathematical Society, Cambridge University Press, Cambridge (1998).
[24] P.A. Linnell and M.J. Puls, *Zero divisors and $L^p(G)$, II*, New York J. Math. 7 (2001), pp. 49–58.
[25] A. Machì and F. Mignosi, *Garden of Eden configurations for cellular automata on Cayley graphs of groups*, SIAM J. Discrete Math. 6 (1993), pp. 44–56.
[26] R. Miles, *Expansive algebraic actions of countable abelian groups*, Mon. Math. 147(2) (2006), pp. 155–164.
[27] E. F. Moore, *Machine models of self-reproduction*, vol. 14 of Proc. Symp. Appl. Math., American Mathematical Society, Providence, 1963, pp. 17–34.
[28] S. A. Morris, *Pontryagin duality and the structure of locally compact Abelian groups*, Cambridge University Press, Cambridge-New York-Melbourne, 1977. London Mathematical Society Lecture Note Series, No. 29.
[29] J. Myhill, *The converse of Moore's Garden-of-Eden theorem*, Proc. Amer. Math. Soc. 14 (1963), pp. 685–686.
[30] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. (3) 61 (1990), pp. 480–496.
[31] K. Schmidt, *Dynamical systems of algebraic origin*, vol. 128 of Progress in Mathematics, Birkhäuser Verlag, Basel, 1995.

[32] K. Schmidt and E. Verbitskiy, *Abelian sandpiles and the harmonic model*, Comm. Math. Phys. 292 (2009), pp. 721–759.

[33] N.Th. Varopoulos, *Long range estimates for Markov chains*, Bull. Sci. Math. (2) 109 (1985), pp. 225–252.

[34] N.Th. Varopoulos, L. Saloff-Coste, and Th. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Math., vol. 100, Cambridge University Press, Cambridge, 1992.

[35] P. Walters, *Topological conjugacy of affine transformations of tori*, Trans. Amer. Math. Soc. 131 (1968), pp. 40–50.

[36] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Math., vol. 138, Cambridge University Press, Cambridge, 2000.

Dipartimento di Ingegneria, Università del Sannio, 82100 Benevento, Italy
E-mail address: tullio.cs@sbai.uniroma1.it

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
E-mail address: michel.coornaert@math.unistra.fr

Department of Mathematics, Chongqing University, Chongqing 401331, China

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900, USA
E-mail address: hfli@math.buffalo.edu