FLABBY STRICT DEFORMATION QUANTIZATIONS AND K-GROUPS

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ABSTRACT. We construct examples of flabby strict deformation quantizations not preserving K-groups. This answers a question of Rieffel negatively.

1. INTRODUCTION

In the passage from classical mechanics to quantum mechanics, one replaces smooth functions on symplectic manifolds (more generally, Poisson manifolds) by operators on Hilbert spaces, and replaces the Poisson bracket of smooth functions by commutators of operators. Thinking of classical mechanics as limits of quantum mechanics, one requires that the Poisson brackets becomes limits of commutators.

There is an algebraic way of studying such process using formal power series, called *deformation quantization* [1, 13]. In order to study it in a stricter way, Rieffel introduced [6] *strict deformation quantization* of Poisson manifolds, within the framework of C^*-algebras. He showed that noncommutative tori arise naturally as strict deformation quantizations of the ordinary torus in the direction of certain Poisson bracket. After that, a lot of interesting examples of strict deformation quantizations have been constructed. See [8, 9] and the references therein.

We refer the reader to [2, Sections 10.1–10.3] for the basic information about continuous fields of C^*-algebras. Recall the definition of strict deformation quantization [6, 9]:

Definition 1.1. [9, Definition 1] *Let M be a Poisson manifold, and let $C_\infty(M)$ be the algebra of $\mathbb{C}$-valued continuous functions on M vanishing at ∞. By a* strict deformation quantization *of M we mean a dense $*$-subalgebra A of $C_\infty(M)$ closed under the Poisson bracket, together with a continuous field of C^*-algebras $\mathcal{A}_\hbar$ over a closed subset I of the real line containing 0 as a non-isolated point, and linear maps $\pi_\hbar : A \to \mathcal{A}_\hbar$ for each $\hbar \in I$, such that*

(1) $\mathcal{A}_0 = C_\infty(M)$ and π_0 is the canonical inclusion of A into $C_\infty(M)$,

(2) the section $(\pi_\hbar(f))$ is continuous for every $f \in A$,

(3) for all $f, g \in A$ we have

$$\lim_{\hbar \to 0} \| [\pi_\hbar(f), \pi_\hbar(g)]/(i\hbar) - \pi_\hbar(\{f, g\}) \| = 0,$$

(4) $\pi_\hbar$ is injective and $\pi_\hbar(A)$ is a dense $*$-subalgebra of $\mathcal{A}_\hbar$ for every $\hbar \in I$.

If $A \supseteq C_c^\infty(M)$, the space of compactly supported $\mathbb{C}$-valued smooth functions on M, we say that the strict deformation quantization is flabby.

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Condition (4) above enables us to define a new $*$-algebra structure and a new C^*-norm on A at each $\hbar$ by pulling back the $*$-algebra structure and norm of $\pi_\hbar(A) \subseteq \mathcal{A}_\hbar$ to A via $\pi_\hbar$. Condition (2) means that this deformation of the $*$-algebra structure and norm on A is continuous.

Given a strict deformation quantization, a natural question is whether the deformed C^*-algebras $\mathcal{A}_\hbar$ have the same "algebraic topology", in particular, whether they have isomorphic K-groups. Rieffel's quantization of Poisson manifolds induced from actions of $\mathbb{R}^d$ [7] and many other examples [5] are known to preserve K-groups. Rieffel showed examples of non-flabby strict deformation quantizations not preserving K-groups, and asked [9, Question 18]: *Are the K-groups of the deformed C^*-algebras of any flabby strict deformation quantization all isomorphic?* A nice survey of various positive results on related problems may be found in [10].

Shim [11] showed that above question has a negative answer if one allows orbifolds. But it is not clear whether one can adapt the method there to get smooth examples.

Rieffel also pointed out [9, page 321] that in any strict deformation quantization of a non-zero Poisson bracket if one reparametrizes by replacing $\hbar$ by $\hbar^2$ one obtains a strict deformation quantization of the 0 Poisson bracket. Thus to answer Rieffel's question it suffices to consider strict deformation quantizations of the 0 Poisson bracket.

The main purpose of this paper is to answer above question. In Section 2 we give a general method of constructing flabby strict deformation quantization for the 0 Poisson bracket. In particular, we prove

Theorem 1.2. *Let M be a smooth manifold with* $\dim M \geq 2$*, equipped with the* 0 *Poisson bracket. If* $\dim M$ *is even (odd, resp.), then for any integers $n_0 \geq n_1 \geq 0$ ($n_1 \geq n_0 \geq 0$ resp.) there is a flabby strict deformation quantization $\{\mathcal{A}_\hbar, \pi_\hbar\}_{\hbar \in I}$ of M over $I = [0,1]$ with $A = C_c^\infty(M)$ such that $K_i(\mathcal{A}_\hbar) \cong K_i(C_\infty(M)) \oplus \mathbb{Z}^{n_i}$ for all $0 < \hbar \leq 1$ and $i = 0, 1$.*

Theorem 1.2 is far from being the most general result one can obtain using our construction in Section 2. However, it illustrates clearly that a lot of manifolds equipped with the 0 Poisson bracket have flabby strict deformation quantizations not preserving K-groups.

In order to accommodate some other interesting examples such as Berezin-Toeplitz quantization of Kähler manifolds, Landsman introduced a weaker notion *strict quantization* [3, Definition II.1.1.1] [9, Definition 23]. This is defined in a way similar to a strict deformation quantization, but without requiring the condition (4) in Definition 1.1. If $\pi_\hbar$ is injective for each $\hbar \in I$ we say that the strict quantization is *faithful*. It is natural to ask for the precise relation between strict quantizations and strict deformation quantizations. Rieffel also raised the question [9, Question 25]: *Is there an example of a faithful strict quantization such that it is impossible to restrict $\pi_\hbar$ to a dense $*$-subalgebra $B \subseteq A$ to get a strict deformation quantization of M?* Adapting our method in Section 2 we also give such an example for every manifold M equipped with the 0 Poisson bracket. In [4] strict quantizations are constructed for every Poisson manifold, and it is impossible to restrict the strict quantizations constructed there to dense $*$-subalgebras to get strict deformation quantizations unless the Poisson bracket is 0 [4, Corollary 5.6]. Thus we get a complete answer to Rieffel's question.

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2. Strict deformation quantizations for the 0 Poisson bracket

We start with a general method of deforming a C^*-algebra. Let $\mathcal{A}$ be a C^*-algebra and $A \subseteq \mathcal{A}$ a dense $*$-subalgebra. Let $I(A) = \{b \in \mathcal{M}(\mathcal{A}) : bA, Ab \subseteq A\}$ be the idealizer of A in the multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$. Then $I(A)$ is a $*$-algebra containing A as an ideal, and for every $b \in (I(A))_{sa}$ clearly bAb is a $*$-subalgebra of $\mathcal{A}$. If furthermore the multiplication by b is injective on A, that is, $b \notin Ann := \{b' \in \mathcal{M}(\mathcal{A}) : b'a = 0$ for some $0 \neq a \in A\}$, then we can pull back the multiplication and norm on bAb to define a new multiplication $\times_b$ and a new norm $\| \cdot \|_b$ on A via the bijection $A \to bAb$. Explicitly, $a \times_b a' = ab^2a'$ and $\| a \|_b = \| bab \|$. The completion of $(A, \times_b, \| \cdot \|_b)$ is isomorphic to $\overline{bAb}$ naturally.

Let X be a topological space, and consider a bounded map $x \mapsto b_x$ from X to $(I(A))_{sa} \subseteq \mathcal{M}(\mathcal{A})$ continuous with respect to the strict topology [12, Definition 2.3.1] on $\mathcal{M}(\mathcal{A})$, i.e. the $\mathcal{A}$-valued functions $x \mapsto \bar{a} \cdot b_x$ and $x \mapsto b_x \cdot \bar{a}$ on X are norm-continuous for every $\bar{a} \in \mathcal{A}$. Then it follows easily that the $\mathcal{A}$-valued function $x \mapsto b_x a b_x$ on X is norm-continuous for every $a \in A$. Thus we get a continuous field of C^*-algebras over X with fibre algebra $\overline{b_x A b_x}$ at $x \in X$, as a subfield of the trivial continuous field of C^*-algebras over X with fibres $\mathcal{A}$, and it contains $(b_x a b_x)$ as a continuous section for every $a \in A$.

Now we specialize to the commutative case. Let M be a smooth manifold, and let $\mathcal{A} = C_\infty(M)$, $A = C_c^\infty(M)$. Then $\mathcal{M}(\mathcal{A})$ is the space $C_b(M)$ consisting of all $\mathbb{C}$-valued bounded continuous functions on M, and the strict topology on $C_b(M)$ is determined by uniform convergence on every compact subset of M. The idealizer $I(A)$ is the space $C_b^\infty(M)$ consisting of all $\mathbb{C}$-valued bounded smooth functions on M. Given $b \in I(A)$, it is not in Ann exactly if the zero set Z_b of b is nowhere dense. Clearly $C_\infty(M \setminus Z_b) \supseteq bAb \supseteq C_c^\infty(M \setminus Z_b)$, and hence $\overline{bAb} = C_\infty(M \setminus Z_b)$. Let $X = I = [0,1]$. If $\hbar \mapsto b_\hbar$ is a bounded map from I to $C_b^\infty(M)$ continuous with respect to the strict topology on $C_b^\infty(M)$, then we get a continuous field of C^*-algebras over I with fibre $C_\infty(M \setminus Z_b)$ at $\hbar$ and $(\pi_\hbar(a))$ is a continuous section for each $a \in A$, where $\pi_\hbar(a) = b_\hbar a b_\hbar$. If furthermore $b_0 = 1$ then the condition (1) of Definition 1.1 is satisfied. Notice that when M is equipped with the 0 Poisson bracket, the condition (3) of Definition 1.1 holds trivially in our construction. Summarizing above discussion we have reached:

Proposition 2.1. *Let M be a smooth manifold equipped with the* 0 *Poisson bracket. For any bounded map $\hbar \mapsto b_\hbar$ from $I = [0,1]$ to $(C_b^\infty(M))_{sa}$ continuous respect to the strict topology on $C_b^\infty(M)$, if $b_0 = 1$ and the zero set $Z_{b_\hbar}$ of $b_\hbar$ is nowhere dense for every $\hbar \in I$, then there is a flabby strict deformation quantization $\{A_\hbar, \pi_\hbar\}_{\hbar \in I}$ of M over I with $A = C_c^\infty(M)$ and $\mathcal{A}_\hbar = C_\infty(M \setminus Z_{b_\hbar})$ for every $\hbar \in I$.*

Example 2.2. *Let $M = \mathbb{R}^n$. Take a bounded smooth real-valued function F on $\mathbb{R}^n$ such that $F = 1$ in a neighborhood of the origin and F vanishes exactly at one point P. Set $b_\hbar(x) = F(\hbar x)$ for all $0 \leq \hbar \leq 1$ and $x \in \mathbb{R}^n$. Then for each $0 < \hbar \leq 1$ the space $\mathbb{R}^n \setminus Z_{b_\hbar} = \mathbb{R}^n \setminus \{P/\hbar\}$ is homeomorphic to $\mathbb{R} \times S^{n-1}$. Now by Proposition 2.1 there is a flabby strict deformation quantization $\{A_\hbar, \pi_\hbar\}_{\hbar \in I}$ of $\mathbb{R}^n$ over $I = [0,1]$ with $A = C_c^\infty(\mathbb{R}^n)$ and $\mathcal{A}_\hbar = C_\infty(M \setminus Z_{b_\hbar}) \cong C_\infty(\mathbb{R} \times S^{n-1})$ for every $0 < \hbar \leq 1$. Then by the Bott periodicity $K_i(\mathcal{A}_\hbar) \cong K_{i+1}(C(S^{n-1}))$* [12, Theorem 7.2.5, page

158] *for every* $0 < \hbar \leq 1$ *and* $i = 0, 1$. *Thus when* n *is odd* $K_i(\mathcal{A}_\hbar) \ncong K_i(C_\infty(\mathbb{R}^n))$ *for all* $0 < \hbar \leq 1$ *and* $i = 0, 1$ *(see for instance* [12, page 123] *for the* K*-groups of* $\mathbb{R}^n$ *and* S^n*). When* n *is even,* $K_1(\mathcal{A}_\hbar) \ncong K_1(C_\infty(\mathbb{R}^n))$ *for all* $0 < \hbar \leq 1$.

When M is compact, in Proposition 2.1 the element $b_\hbar$ has to be invertible in $C(M)$ for small $\hbar$ and consequently $\mathcal{A}_\hbar = C(M)$. Thus in order to construct strict deformation quantizations for compact M such that the K-groups of $\mathcal{A}_\hbar$ are not isomorphic to those of $C(M)$ for any $\hbar \neq 0$, we have to modify the construction in Proposition 2.1. Notice that if we set $\pi'_\hbar(a + \lambda) = b_\hbar a b_\hbar + \lambda$ for $a \in C_c^\infty(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ in Example 2.2, then we get a strict deformation quantization of S^n equipped with the 0 Poisson bracket. This leads to Proposition 2.4 below.

Notation 2.3. *We denote by* $\mathcal{F}_m$ *the space of smooth real-valued functions* F *on* $\mathbb{R}^m$ *such that* F *is equal to* 1 *outside a compact subset of* $\mathbb{R}^m$ *and the zero set* Z_F *of* F *is nowhere dense.*

Proposition 2.4. *Let* M *be a smooth manifold equipped with the* 0 *Poisson bracket. Let* U *be an open subset of* M *with a diffeomorphism* $\varphi : U \to \mathbb{R}^m$. *For any* $F \in \mathcal{F}_m$ *there is a flabby strict deformation quantization* $\{A_\hbar, \pi_\hbar\}_{\hbar \in I}$ *of* M *over* $I = [0, 1]$ *with* $A = C_c^\infty(M)$ *such that* $\mathcal{A}_\hbar \cong C_\infty(M/Y)$ *for every* $0 < \hbar \leq 1$, *where* $Y = \varphi^{-1}(Z_F \cup \{0\})$.

Proof. Set $F_0 = 1$ and $F_\hbar(x) = F(x/\hbar)$ for all $0 < \hbar \leq 1$ and $x \in \mathbb{R}^m$. Then $F_\hbar \in \mathcal{F}_m$ for each $\hbar \in I$ and we can extend the pull-back $F_\hbar \circ \varphi \in C^\infty(U)$ to a smooth function $b_\hbar$ on M by setting it to be 1 outside U. Clearly $b_\hbar A' b_\hbar$ is a $*$-subalgebra of A'. Notice that there is a compact set $W \subset U$ such that $b_\hbar = 1$ on $M \setminus W$ for all $\hbar \in I$, and W contains $\varphi^{-1}(0)$. Take an $H \in (C_c^\infty(M))_\mathbb{R}$ such that $H = 1$ on W. Denote by A' the space of functions in $C_c^\infty(M)$ vanishing at $\varphi^{-1}(0)$. Then $C_c^\infty(M) = A' \oplus \mathbb{C}H$ as complex vector spaces, and $H^2 - H = b_\hbar(H^2 - H)b_\hbar \in b_\hbar A' b_\hbar$. It is easy to see that $b_\hbar A' b_\hbar + \mathbb{C}H$ is a $*$-subalgebra of $C_c^\infty(M)$ and the linear map $\pi_\hbar : C_c^\infty(M) \to b_\hbar A' b_\hbar + \mathbb{C}H$ defined by $\pi_\hbar(a' + \lambda H) = b_\hbar a' b_\hbar + \lambda H$ for $a' \in A'$ and $\lambda \in \mathbb{C}$ is bijective. For each $a' \in A'$ clearly the map $\hbar \mapsto b_\hbar a' b_\hbar \in C_\infty(M)$ is continuous on $I = [0, 1]$. Thus for each $a \in A = C_c^\infty(M)$, $(\pi_\hbar(a))$ is a continuous section in the continuous subfield $\{\mathcal{A}_\hbar = \overline{b_\hbar A' b_\hbar + \mathbb{C}H}\}_{\hbar \in I}$ of the trivial field of C^*-algebras over I with fibres $C_\infty(M)$. Therefore $\{\mathcal{A}_\hbar, \pi_\hbar\}_{\hbar \in I}$ is a flabby strict deformation quantization of M.

Set $Y_\hbar = \varphi^{-1}((\hbar Z_F) \cup \{0\})$. Clearly $C_\infty(M \setminus Y_\hbar) \supseteq \overline{b_\hbar A' b_\hbar} \supseteq C_c^\infty(M \setminus Y_\hbar)$. Thus $\overline{b_\hbar A' b_\hbar + \mathbb{C}H} = \overline{b_\hbar A' b_\hbar} + \mathbb{C}H$ is exactly the space of functions in $C_\infty(M)$ taking the same value on $Y_\hbar$, which is just $C_\infty(M/Y_\hbar)$. When $0 < \hbar \leq 1$, the space $M/Y_\hbar$ is homeomorphic to M/Y, and hence $\mathcal{A}_\hbar = \overline{b_\hbar A' b_\hbar + \mathbb{C}H} \cong C_\infty(M/Y)$ as desired. ∎

Next we describe a case in which we can relate the K-groups of $C_\infty(M/Y)$ to those of $C_\infty(M)$ easily:

Lemma 2.5. *Let* D *be the subset of* $\mathbb{R}^m$ *consisting of points* $(x_1, \cdots, x_m)$ *with* $0 < x_1, \cdots, x_m < 1$. *Let* M, φ, F *and* Y *be as in Proposition 2.4. Suppose that* $\partial D \subseteq Z_F \subseteq \bar{D}$. *Then*

$$K_i(C_\infty(M \setminus Y)) \cong K_i(C_\infty(M)) \oplus K_i(C_\infty(D \setminus Z_F))$$

for $i = 0, 1$.

Proof. Let $\phi : M \to M/Y$ be the quotient map, and let $W = \phi(M \setminus \varphi^{-1}(D))$. Then W is a closed subset of M/Y, and the complement is homeomorphic to $D \setminus Z_F$. Define a map $\psi : M/Y \to W$ as the identity map on W and $\psi((M/F)\setminus W) = \phi(Y)$. Then ψ is continuous and proper, i.e. the inverse image of every compact subset of W is compact. Thus the exact sequence

$$0 \to C_\infty(D \setminus Z_F) \to C_\infty(M/Y) \to C_\infty(W) \to 0$$

splits. Therefore $K_i(C_\infty(M \setminus Y)) \cong K_i(C_\infty(W)) \oplus K_i(C_\infty(D \setminus Z_F))$ for $i = 0, 1$. Now Lemma 2.5 follows from the fact that W is homeomorphic to M. $\square$

Notice that if a compact set $Z \subseteq \mathbb{R}^m$ is the zero set of some non-negative $f \in C^\infty(M)$, then it is also the zero set of some $F \in \mathcal{F}_m$ (for instance, take a non-negative $g \in C_c^\infty(M)$ with $g|_Z = 1$ and set $F(x) = \frac{f(x)}{f(x)+g(x)}$ for all $x \in \mathbb{R}^m$). Also notice that if closed subsets Z_1 and Z_2 of $\mathbb{R}^m$ are the zero sets of non-negative smooth functions on $\mathbb{R}^m$, then so are $Z_1 \cap Z_2$ and $Z_1 \cup Z_2$. From these observation we get easily

Lemma 2.6. *Let $m \geq 2$, and let D be as in Lemma 2.5. For any $k, j \geq 0$ and $1 \leq s \leq 2$ there exits an $F \in \mathcal{F}_m$ satisfying $\partial D \subseteq Z_F \subseteq \bar{D}$ such that $D \setminus Z_F$ is homeomorphic to the disjoint union of k many $\mathbb{R}^m$ and j many $\mathbb{R}^s \times S^{m-s}$.*

Now we are ready to prove Theorem 1.2:

Proof of Theorem 1.2. The case in which $\dim M$ is even follows from Proposition 2.4 and Lemma 2.5 by taking $k = n_0 - n_1$, $j = n_1$, $s = 1$ in Lemma 2.6. Similarly the case in which $\dim M$ is odd follows by taking $k = n_1 - n_0$, $j = n_0$, $s = 2$. $\square$

Finally we discuss how to adapt our method to construct strict quantizations which can't be restricted to dense $*$-subalgebras to yield strict deformation quantizations. Notice that if we relax the condition $b_\hbar \in (C_b^\infty(M))_{sa}$ in Proposition 2.1 to $b_\hbar \in (C_b(M))_{sa}$ and set $\mathcal{A}_\hbar$ to be the C^*-subalgebra of $_\infty(M)$ generated by $b_\hbar A b_\hbar$, then we get a faithful strict quantization of M over $I = [0, 1]$ with $A = C_c^\infty(M)$ and $\pi_\hbar(a) = b_\hbar a b_\hbar$. Take a nonnegative $F \in C_b(M)$ such that F is not smooth at some point P. Set $b_\hbar = ((1-\hbar) + \hbar F)^{1/2}$ for every $\hbar \in I$. Let $B \subseteq A$ be a dense $*$-subalgebra. Then we can find $f \in B$ such that $f(P) \neq 0$. Clearly $f^2 b_\hbar^2$ is not smooth at P for $0 < \hbar \leq 1$. It follows that $(\pi_\hbar(f))^2$ is not in $\pi_\hbar(B)$ for $0 < \hbar \leq 1$. Thus we get:

Proposition 2.7. *Let M be a smooth manifold equipped with the 0 Poisson bracket. Then there is a faithful strict quantization $\{\mathcal{A}_\hbar, \pi_\hbar\}_{\hbar \in I}$ of M over $I = [0, 1]$ with $A = C_c^\infty(M)$ such that it is impossible to restrict $\pi_\hbar$ to a dense $*$-subalgebra $B \subseteq A$ to get a strict deformation quantization of M.*

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