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Ergodicity of principal algebraic group actions

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Dedicated to Shrikrishna Gopalrao Dani on the occasion of his 65th birthday

Abstract. An *algebraic* action of a discrete group Γ is a homomorphism from Γ to the group of continuous automorphisms of a compact abelian group X. By duality, such an action of Γ is determined by a module $M = \widehat{X}$ over the integer group ring $\mathbb{Z}\Gamma$ of Γ. The simplest examples of such modules are of the form $M = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ with $f \in \mathbb{Z}\Gamma$; the corresponding algebraic action is the *principal algebraic* Γ*-action* α_f defined by f.

In this note we prove the following extensions of results by Hayes [**2**] on ergodicity of principal algebraic actions: If Γ is a countably infinite discrete group which is not virtually cyclic, and if $f \in \mathbb{Z}\Gamma$ satisfies that right multiplication by f on $\ell^2(\Gamma, \mathbb{R})$ is injective, then the principal Γ-action α_f is ergodic (Theorem 1.3). If Γ contains a finitely generated subgroup with a single end (e.g. a finitely generated amenable subgroup which is not virtually cyclic), or an infinite nonamenable subgroup with vanishing first ℓ^2-Betti number (e.g., an infinite property T subgroup), the injectivity condition on f can be replaced by the weaker hypothesis that f is not a right zero-divisor in $\mathbb{Z}\Gamma$ (Theorem 1.2). Finally, if Γ is torsion-free, not virtually cyclic, and satisfies Linnell's *analytic zero-divisor conjecture*, then α_f is ergodic for every $f \in \mathbb{Z}\Gamma$ (Remark 1.5).

1. Principal Algebraic Group Actions

Let Γ be a countably infinite discrete group with integral group ring $\mathbb{Z}\Gamma$. Every $g \in \mathbb{Z}\Gamma$ is written as a formal sum $g = \sum_\gamma g_\gamma \cdot \gamma$, where $g_\gamma \in \mathbb{Z}$ for every $\gamma \in \Gamma$ and $\sum_{\gamma \in \Gamma} |g_\gamma| < \infty$. The set $\mathrm{supp}(g) = \{\gamma \in \Gamma : g_\gamma \neq 0\}$ is called the

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support of g. For $g=\sum_{\gamma\in\Gamma} g_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$ we denote by $g^*=\sum_{\gamma\in\Gamma} g_\gamma\cdot\gamma^{-1}$ the *adjoint* of g. The map $g\mapsto g^*$ is an *involution* on $\mathbb{Z}\Gamma$, i.e., $(gh)^*=h^*g^*$ for all $g,h\in\mathbb{Z}\Gamma$, where the product fg of two elements $f=\sum_\gamma f_\gamma\cdot\gamma$ and $g=\sum_\gamma g_\gamma\cdot\gamma$ in $\mathbb{Z}\Gamma$ is given by $fg=\sum_{\gamma,\gamma'\in\Gamma} f_\gamma g_{\gamma'}\cdot\gamma\gamma'$.

An *algebraic* Γ*-action* is a homomorphism $\alpha\colon\Gamma\longrightarrow\mathrm{Aut}(X)$ from Γ to the group of (continuous) automorphisms of a compact metrizable abelian group X. If α is an algebraic Γ-action, then $\alpha^\gamma\in\mathrm{Aut}(X)$ denotes the image of $\gamma\in\Gamma$, and $\alpha^{\gamma\gamma'}=\alpha^\gamma\alpha^{\gamma'}$ for every $\gamma,\gamma'\in\Gamma$. The Γ-action α induces an action of $\mathbb{Z}\Gamma$ by group homomorphisms $\alpha^f\colon X\longrightarrow X$, where $\alpha^f=\sum_{\gamma\in\Gamma} f_\gamma\alpha^\gamma$ for every $f=\sum_{\gamma\in\Gamma} f_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$. Clearly, if $f,g\in\mathbb{Z}\Gamma$, then $\alpha^{fg}=\alpha^f\alpha^g$.

Let $\hat{X}$ be the dual group of X. If $\hat{\alpha}^\gamma$ is the automorphism of $\hat{X}$ dual to α^γ, then the map $\hat{\alpha}\colon\Gamma\longrightarrow\mathrm{Aut}(\hat{X})$ satisfies that $\hat{\alpha}^{\gamma\gamma'}=\hat{\alpha}^{\gamma'}\hat{\alpha}^\gamma$ for all $\gamma,\gamma'\in\Gamma$. We write $\hat{\alpha}^f\colon\hat{X}\longrightarrow\hat{X}$ for the group homomorphism dual to α^f and set $f\cdot a=\hat{\alpha}^{f^*}a$ for every $f\in\mathbb{Z}\Gamma$ and $a\in\hat{X}$. The resulting map $(f,a)\mapsto f\cdot a$ from $\mathbb{Z}\Gamma\times\hat{X}$ to $\hat{X}$ satisfies that $(fg)\cdot a=f\cdot(g\cdot a)$ for all $f,g\in\mathbb{Z}\Gamma$ and turns $\hat{X}$ into a module over the group ring $\mathbb{Z}\Gamma$. Conversely, if M is a countable module over $\mathbb{Z}\Gamma$, we set $X=\widehat{M}$ and put $\hat{\alpha}^f a=f^*\cdot a$ for $f\in\mathbb{Z}\Gamma$ and $a\in M$. The maps $\alpha^f\colon\widehat{M}\longrightarrow\widehat{M}$ dual to $\hat{\alpha}^f$, $f\in\mathbb{Z}\Gamma$, define an action of $\mathbb{Z}\Gamma$ by homomorphisms of $\widehat{M}$, which in turn induces an algebraic action α of Γ on $X=\widehat{M}$.

The simplest examples of algebraic Γ-actions arise from $\mathbb{Z}\Gamma$-modules of the form $M=\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ with $f\in\mathbb{Z}\Gamma$. Since these actions are determined by principal left ideals of $\mathbb{Z}\Gamma$ they are called *principal algebraic* Γ*-actions*. In order to describe these actions more explicitly we put $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and define the left and right shift-actions λ and ρ of Γ on $\mathbb{T}^\Gamma$ by setting

$$(\lambda^\gamma x)_{\gamma'}=x_{\gamma^{-1}\gamma'},\qquad(\rho^\gamma x)_{\gamma'}=x_{\gamma'\gamma},\tag{1.1}$$

for every $\gamma\in\Gamma$ and $x=(x_{\gamma'})_{\gamma'\in\Gamma}\in\mathbb{T}^\Gamma$. The Γ-actions λ and ρ extend to actions of $\mathbb{Z}\Gamma$ on $\mathbb{T}^\Gamma$ given by

$$\lambda^f=\textstyle\sum_{\gamma\in\Gamma} f_\gamma\lambda^\gamma,\qquad\rho^f=\sum_{\gamma\in\Gamma} f_\gamma\rho^\gamma\tag{1.2}$$

for every $f=\sum_{\gamma\in\Gamma} f_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$.

The pairing $\langle f,x\rangle=e^{2\pi i\sum_{\gamma\in\Gamma} f_\gamma x_\gamma}$, $f=\sum_{\gamma\in\Gamma} f_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$, $x=(x_\gamma)\in\mathbb{T}^\Gamma$, identifies $\mathbb{Z}\Gamma$ with the dual group $\widehat{\mathbb{T}^\Gamma}$ of $\mathbb{T}^\Gamma$. We claim that, under this identification,

$$X_f\coloneqq\ker\rho^f=\bigl\{x\in\mathbb{T}^\Gamma:\rho^f x=\textstyle\sum_{\gamma\in\Gamma} f_\gamma\rho^\gamma x=0\bigr\}=(\mathbb{Z}\Gamma f)^\perp\subset\widehat{\mathbb{Z}\Gamma}=\mathbb{T}^\Gamma.\tag{1.3}$$

Indeed,

$$\langle h,\rho^f x\rangle=\Bigl\langle h,\textstyle\sum_{\gamma'\in\Gamma} f_{\gamma'}\rho^{\gamma'}x\Bigr\rangle=e^{2\pi i\sum_{\gamma\in\Gamma} h_\gamma\sum_{\gamma'\in\Gamma} f_{\gamma'}x_{\gamma\gamma'}}$$

$$= e^{2\pi i \sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} h_{\gamma\gamma'^{-1}} f_{\gamma'} x_\gamma} = e^{2\pi i \sum_{\gamma\in\Gamma} (hf)_\gamma x_\gamma} = \langle hf, x\rangle$$

for every $h \in \mathbb{Z}\Gamma$ and $x \in \mathbb{T}^\Gamma$, so that $x \in \ker \rho^f$ if and only if $x \in (\mathbb{Z}\Gamma f)^\perp$.

Since the Γ-actions λ and ρ on $\mathbb{T}^\Gamma$ commute, the group $X_f = \ker \rho^f \subset \mathbb{T}^\Gamma$ is invariant under λ, and we denote by α_f the restriction of λ to X_f. In view of this we adopt the following terminology.

Definition 1.1. (X_f, α_f) is the principal algebraic Γ-action defined by $f \in \mathbb{Z}\Gamma$.

In [**2**] the author calls a countably infinite discrete group Γ *principally ergodic* if every principal algebraic Γ-action α_f, $f \in \mathbb{Z}\Gamma$, is ergodic w.r.t. Haar measure on X_f and proves that the following classes of groups are principally ergodic: torsion-free nilpotent groups which are not virtually cyclic,[1] free groups on more than one generator, and groups which are not finitely generated.

In order to state our extensions of these results we denote by $\ell^\infty(\Gamma,\mathbb{R}) \subset \mathbb{R}^\Gamma$ the space of bounded real-valued maps $v = (v_\gamma)$ on Γ, where v_γ is the value of v at γ, and we write $\|v\|_\infty = \sup_{\gamma\in\Gamma} |v_\gamma|$ for the supremum norm on $\ell^\infty(\Gamma,\mathbb{R})$. For $1 \le p < \infty$ we set $\ell^p(\Gamma,\mathbb{R}) = \{v = (v_\gamma) \in \ell^\infty(\Gamma,\mathbb{R}) : \|v\|_p = \bigl(\sum_{\gamma\in\Gamma} |v_\gamma|^p\bigr)^{1/p} < \infty\}$. By $\ell^p(\Gamma,\mathbb{Z}) = \ell^p(\Gamma,\mathbb{R}) \cap \mathbb{Z}^\Gamma$ we denote the additive subgroup of integer-valued elements of $\ell^p(\Gamma,\mathbb{R})$; for $1 \le p < \infty$, $\ell^p(\Gamma,\mathbb{Z}) = \ell^1(\Gamma,\mathbb{Z})$ is identified with $\mathbb{Z}\Gamma$ by viewing each $g = \sum_\gamma g_\gamma \cdot \gamma \in \mathbb{Z}\Gamma$ as the element $(g_\gamma)_{\gamma\in\Gamma} \in \ell^1(\Gamma,\mathbb{Z})$.

The group Γ acts on $\ell^p(\Gamma,\mathbb{R})$ isometrically by left and right translations: for every $v \in \ell^p(\Gamma,\mathbb{R})$ and $\gamma \in \Gamma$ we denote by $\tilde\lambda^\gamma v$ and $\tilde\rho^\gamma v$ the elements of $\ell^p(\Gamma,\mathbb{R})$ satisfying $(\tilde\lambda^\gamma v)_{\gamma'} = v_{\gamma^{-1}\gamma'}$ and $(\tilde\rho^\gamma v)_{\gamma'} = v_{\gamma'\gamma}$, respectively, for every $\gamma' \in \Gamma$. Note that $\tilde\lambda^{\gamma\gamma'} = \tilde\lambda^\gamma\tilde\lambda^{\gamma'}$ and $\tilde\rho^{\gamma\gamma'} = \tilde\rho^\gamma\tilde\rho^{\gamma'}$ for every $\gamma,\gamma' \in \Gamma$.

The Γ-actions $\tilde\lambda$ and $\tilde\rho$ extend to actions of $\ell^1(\Gamma,\mathbb{R})$ on $\ell^p(\Gamma,\mathbb{R})$ which will again be denoted by $\tilde\lambda$ and $\tilde\rho$: for $h = (h_\gamma) \in \ell^1(\Gamma,\mathbb{R})$ and $v \in \ell^p(\Gamma,\mathbb{R})$ we set

$$\tilde\lambda^h v = \sum\nolimits_{\gamma\in\Gamma} h_\gamma \tilde\lambda^\gamma v, \qquad \tilde\rho^h v = \sum\nolimits_{\gamma\in\Gamma} h_\gamma \tilde\rho^\gamma v. \tag{1.4}$$

These definitions correspond to the usual convolutions

$$\tilde\lambda^h v = h \cdot v, \qquad \tilde\rho^h v = v \cdot h^*, \tag{1.5}$$

where $h \mapsto h^*$ is the involution on $\ell^1(\Gamma,\mathbb{C})$ defined as for $\mathbb{Z}\Gamma$: $h^*_\gamma = \overline{h_{\gamma^{-1}}}$, $\gamma \in \Gamma$, for every $h = (h_\gamma) \in \ell^1(\Gamma,\mathbb{C})$. For $p = 2$, the bounded linear operators $\tilde\lambda^h, \tilde\rho^h \colon \ell^2(\Gamma,\mathbb{R}) \longrightarrow \ell^2(\Gamma,\mathbb{R})$ in (1.4) can be viewed as elements of the right (resp. left) equivariant group von Neumann algebra of Γ.

[1] A discrete group Γ is *virtually cyclic* if it has a cyclic finite-index subgroup. Virtually cyclic groups can obviously not be principally ergodic: if $\Gamma = \mathbb{Z}$, and if $\mathbb{Z}\Gamma$ is identified with the ring of Laurent polynomials $\mathbb{Z}[u^{\pm1}]$ in the obvious manner, then the principal algebraic $\mathbb{Z}$-action α_f defined by $f = 1 - u$ is trivial — and hence nonergodic — on $X_f = \mathbb{T}$.

THEOREM 1.2. *Let Γ be a countably infinite discrete group which satisfies one of the following conditions:*

(1) *Γ contains a finitely generated amenable subgroup which is not virtually cyclic, or more generally, a finitely generated subgroup with a single end,*
(2) *Γ is not finitely generated,*
(3) *Γ contains an infinite property T subgroup, or more generally, a nonamenable subgroup Γ_0 with vanishing first ℓ^2-Betti number $\beta_1^{(2)}(\Gamma_0) = 0$.*

If $f \in \mathbb{Z}\Gamma$ is not a right zero-divisor, then the principal Γ-action α_f on X_f is ergodic (with respect to the normalized Haar measure of X_f).

THEOREM 1.3. *Let Γ be a countably infinite discrete group which is not virtually cyclic. If $f \in \mathbb{Z}\Gamma$ satisfies that*

$$\ker \tilde{\rho}^{f^*} = \{v \in \ell^2(\Gamma, \mathbb{R}) : \tilde{\rho}^{f^*}(v) = v \cdot f = 0\} = \{0\}, \tag{1.6}$$

then the principal Γ-action α_f on X_f is ergodic.

In view of the hypotheses on f in the Theorems 1.2 and 1.3 it is useful to recall the following result.

PROPOSITION 1.4. *Let Γ be a countably infinite discrete amenable group. For every $f \in \mathbb{Z}\Gamma$ the following conditions are equivalent.*

(1) *f is a right zero-divisor in $\mathbb{Z}\Gamma$,*
(2) *$\{v \in \ell^2(\Gamma, \mathbb{R}) : f^* \cdot v = 0\} \neq \{0\}$,*
(3) *f is a left zero-divisor in $\mathbb{Z}\Gamma$,*
(4) *$\{v \in \ell^2(\Gamma, \mathbb{R}) : f \cdot v = 0\} \neq \{0\}$,*
(5) *$\ker \tilde{\rho}^f = \{v \in \ell^2(\Gamma, \mathbb{R}) : \tilde{\rho}^f(v) = 0\} \neq \{0\}$.*

PROOF. (4) $\Leftrightarrow$ (5): This follows from $(f \cdot v)^* = v^* \cdot f^*$ for all $v \in \ell^2(\Gamma, \mathbb{R})$.

(2) $\Leftrightarrow$ (3) $\Rightarrow$ (4): This is part of [**5**, Proposition 4.16].

(1) $\Leftrightarrow$ (2): Taking $*$ we see that (1) holds if and only if f^* is a left zero-divisor in $\mathbb{Z}\Gamma$. Applying (3) $\Leftrightarrow$ (4) to f^*, we see that the latter condition is equivalent to (2). □

REMARK 1.5. Linnell's *analytic zero-divisor conjecture* is the conjectural statement that for any torsion-free discrete group Γ and any nonzero $f \in \mathbb{C}\Gamma$, $\ker \tilde{\rho}^{f^*} = \{0\}$ [**6**, Conjecture 1]. Linnell has shown that this conjecture holds for Γ if G_1 is a normal subgroup of Γ, G_2 is a normal subgroup of G_1, Γ is torsion-free, G_2 is free, G_1/G_2 is elementary amenable, and Γ/G_1 is right orderable [**7**, Proposition 1.4].

If a countably infinite, torsion-free, and not virtually cyclic group Γ satisfies Linnell's analytic zero-divisor conjecture, then the principal Γ-action α_f on X_f is ergodic for every $f \in \mathbb{Z}\Gamma$ by Theorem 1.3.

As a corollary to the Theorems 1.2 – 1.3 and Remark 1.5 we obtain the following results by Hayes.

Corollary 1.6 ([**2**, Theorem 2.3.6 and Corollary 2.5.5]). *Suppose that* Γ *satisfies either of the following conditions.*

(1) Γ *is an infinite, torsion-free, nilpotent group not isomorphic to the integers,*
(2) Γ *is the free group with* $k \geq 2$ *generators.*

Then the principal Γ*-action* (X_f, α_f) *is ergodic for every* $f \in \mathbb{Z}\Gamma$*.*

Proof. If $f = 0$, then α_f is the left shift-action by Γ on $X_f = \mathbb{T}^\Gamma$, which is obviously ergodic. Suppose therefore that $f \neq 0$. Since Γ is either free or torsion-free nilpotent, $\ker \tilde{\rho}^{f^*} = \{0\}$ by Remark 1.5, so that α_f is ergodic by either Theorem 1.2 or 1.3. □

Whereas the proofs of these results in [**2**] use structure theory of Γ, the proofs in this paper employ cohomological methods.

2. Cohomological results

Let Γ be a countably infinite discrete group and $\mathcal{M}$ a left $\mathbb{Z}\Gamma$-module. A map $c\colon \Gamma \longrightarrow \mathcal{M}$ is a 1-*cocycle* (or, for our purposes here, simply a *cocycle*) if

$$c(\gamma\gamma') = c(\gamma) + \gamma c(\gamma') \tag{2.1}$$

for all $\gamma, \gamma' \in \Gamma$. A cocycle $c\colon \Gamma \longrightarrow \mathcal{M}$ is a *coboundary* (or *trivial*) if there exists a $b \in \mathcal{M}$ such that

$$c(\gamma) = b - \gamma b \tag{2.2}$$

for every $\gamma \in \Gamma$.

A finitely generated group G has two ends if and only if it is infinite and virtually cyclic, i.e., if and only if it contains a finite-index subgroup $G' \cong \mathbb{Z}$. Stallings' theorem ([**13**]) implies that a finitely generated group G has a single end whenever it is amenable and not virtually cyclic (see [**8**] for a short proof).

Proposition 2.1. *Let* Γ *be a countably infinite discrete group and* $\Delta \subset \Gamma$ *a finitely generated subgroup with a single end. Then every cocycle* $c\colon \Delta \longrightarrow \mathbb{Z}\Gamma$ *is a coboundary.*

Proof. By [**3**, Theorem 4.6] if Δ has a single end, then every 1-cocycle $\Delta \longrightarrow \mathbb{Z}\Delta$ is a coboundary.[2] It follows that for each $\gamma \in \Gamma$ there is some $b_\gamma \in \mathbb{Z}[\Delta\gamma]$ such that the restriction of $c(\delta)$ on $\Delta\gamma$ is equal to $b_\gamma - \delta b_\gamma$ for all $\delta \in \Delta$.

[2] The authors are grateful to Andreas Thom for alerting us to this reference.

For each $\delta \in \Delta$, there is a finite set W_δ of right cosets of Δ in Γ such that the support of $c(\delta)$ is contained in $\bigcup_{\Delta\gamma \in W_\delta} \Delta\gamma$. If F is a finite symmetric set of generators of Δ, then for any $\Delta\gamma \notin \bigcup_{\delta' \in F} W_{\delta'}$, one has $(1-\delta)\cdot b_\gamma = 0$ for every $\delta \in F$ and hence for every $\delta \in \Delta$. Therefore $c(\delta)$ is equal to 0 on $\Delta\gamma$ for all $\Delta\gamma \notin \bigcup_{\delta' \in F} W_{\delta'}$ and $\delta \in \Delta$. Set $b = \sum_{\Delta\gamma \in \bigcup_{\delta' \in F} W_{\delta'}} b_\gamma \in \mathbb{Z}\Gamma$. Then $c(\delta) = (1-\delta)\cdot b$ for all $\delta \in \Delta$. □

Next we prove an analogous result for nonamenable groups with vanishing first ℓ^2-Betti number, e.g., infinite property T groups [**1**, Corollary 6].

Proposition 2.2. *Let Γ be a countably infinite discrete group and $\Delta \subset \Gamma$ a nonamenable subgroup with $\beta_1^{(2)}(\Delta) = 0$. Then every cocycle $c\colon \Delta \longrightarrow \mathbb{Z}\Gamma$ is a coboundary.*

For the proof of Proposition 2.2 we have to discuss cocycles of Γ which take values in a Hilbert space $\mathcal{H}$ carrying a unitary action $U\colon \gamma \mapsto U^\gamma$ of Γ. A map $c\colon \Gamma \longrightarrow \mathcal{H}$ is a 1-*cocycle* for U if

$$c(\gamma\gamma') = c(\gamma) + U^\gamma c(\gamma') \tag{2.3}$$

for all $\gamma, \gamma' \in \Gamma$, and such a cocycle is a *coboundary* if and only if there exists a $b \in \mathcal{H}$ with

$$c(\gamma) = b - U^\gamma b \tag{2.4}$$

for every $\gamma \in \Gamma$. The cocycle c is an *approximate coboundary* if there exists a sequence $(c_n)_{n\ge 1}$ of coboundaries $c_n\colon \Gamma \longrightarrow \mathcal{H}$ such that

$$\lim_{n\to\infty} \|c_n(\gamma) - c(\gamma)\| = 0 \tag{2.5}$$

for every $\gamma \in \Gamma$.

The following lemma is well-known (cf. [**11**, Proposition 1.6]). For convenience of the reader, we give a proof here.

Lemma 2.3. *Let U be a unitary representation of Γ on $\mathcal{H}$ which does not contain the trivial representation weakly. Then every approximate coboundary $c\colon \Gamma \longrightarrow \mathcal{H}$ for U is a coboundary.*

Proof. Since U does not weakly contain the trivial representation of Γ, we can find a finite subset $F \subset \Gamma$ and some $\varepsilon > 0$ such that

$$\sum\nolimits_{\delta\in F} \|v - U^\delta v\| \ge \varepsilon \|v\|$$

for all $v \in \mathcal{H}$.

Let c be an approximate coboundary of Γ taking values in $\mathcal{H}$. Let $(b_n)_{n\ge1}$ be a sequence in $\mathcal{H}$ such that the coboundaries $c_n(\gamma) = b_n - U^\gamma b_n,\ \gamma \in \Gamma$, approximate c in the sense of (2.5). Then

$$\sum\nolimits_{\delta\in F} \|c(\delta)\| = \lim_{n\to\infty} \sum\nolimits_{\delta\in F} \|c_n(\delta)\| \ge \varepsilon \limsup_{n\to\infty} \|b_n\|,$$

and hence

$$\|c(\gamma)\| = \lim_{n\to\infty} \|c_n(\gamma)\| \le 2 \limsup_{n\to\infty} \|b_n\| \le 2\varepsilon^{-1} \sum\nolimits_{\delta\in F} \|c(\delta)\|$$

for all $\gamma \in \Gamma$.

For a bounded subset Y of $\mathcal{H}$ and $v \in \mathcal{H}$, set $d(v, Y) = \sup_{y\in Y} \|v - y\|$. Since $\mathcal{H}$ is a Hilbert space, the function $v \mapsto d(v, Y)$ on $\mathcal{H}$ takes a minimal value at exactly one point, namely the Chebyshev center of Y, which we denote by center(Y).

Consider the affine isometric action V of Γ on $\mathcal{H}$ defined by $V^\gamma v = U^\gamma v + c(\gamma)$ for all $\gamma \in \Gamma$ and $v \in \mathcal{H}$. Set $Y = \{c(\gamma') : \gamma' \in \Gamma\}$, and let $\gamma \in \Gamma$. Since $V^\gamma(Y) = Y$, we obtain that $V^\gamma(\mathrm{center}(Y)) = \mathrm{center}(Y)$ and hence that $U^\gamma(\mathrm{center}(Y)) + c(\gamma) = \mathrm{center}(Y)$. Thus $c(\gamma) = \mathrm{center}(Y) - U^\gamma(\mathrm{center}(Y))$ for all $\gamma \in \Gamma$, so that c is a coboundary. $\square$

Proof of Proposition 2.2. By [**1**] in the finitely generated case, and [**9**, Corollary 2.4] in general, if Δ is nonamenable and $\beta_1^{(2)}(\Delta) = 0$, then every 1-cocycle $\Delta \longrightarrow \ell^2(\Delta, \mathbb{R})$ for the left regular representation is a coboundary. It follows that for each $\gamma \in \Gamma$ there is some $b_\gamma \in \ell^2(\Delta\gamma, \mathbb{R})$ such that the restriction of $c(\delta)$ on $\Delta\gamma$ is equal to $b_\gamma - \delta b_\gamma$ for all $\delta \in \Delta$. Since $c(\delta)$ has finite support for each $\delta \in \Delta$, we conclude that the cocycle $c\colon \Delta \longrightarrow \ell^2(\Gamma, \mathbb{R})$ is an approximate coboundary.

Because Δ is nonamenable, its left regular representation on $\ell^2(\Delta, \mathbb{R})$ does not contain the trivial representation weakly. Since the restriction of the left regular representation of Γ on $\ell^2(\Gamma, \mathbb{R})$ to Δ is a direct sum of copies of the left regular representation of Δ, it does not contain the trivial representation of Δ weakly either. By Lemma 2.3 there exists $v \in \ell^2(\Gamma, \mathbb{R})$ satisfying

$$c(\delta) = v - \tilde{\lambda}^\delta v = (1 - \delta)v \tag{2.6}$$

for every $\delta \in \Delta$.

Since Δ is nonamenable, it is infinite. It follows that that $v \in \ell^2(\Gamma, \mathbb{Z}) = \mathbb{Z}\Gamma$. $\square$

If a subgroup $\Delta \subset \Gamma$ has more than one end then there exist nontrivial cocycles $c\colon \Delta \longrightarrow \mathbb{Z}\Delta$ (cf. [**12**, 5.2. Satz IV] or [**14**, Lemma 3.5]), which immediately implies the existence of nontrivial cocycles $c\colon \Delta \longrightarrow \mathbb{Z}\Gamma$. For example, if Δ is the free group on $k \ge 2$ generators, it has nontrivial cocycles. However, Proposition 2.4 below guarantees triviality of cocycles which become trivial under right multiplication by an element $f \in \mathbb{Z}\Gamma$ satisfying (1.6) (cf. Remark 1.5).

Proposition 2.4. *Let Γ be a countably infinite discrete group, $\Delta \subset \Gamma$ a nonamenable subgroup, and let $f \in \mathbb{Z}\Gamma$ satisfy that* $\ker \tilde{\rho}^{f^*} = \{0\}$*. If $c\colon \Delta \longrightarrow \mathbb{Z}\Gamma$ is a cocycle such that cf is a coboundary, then c is a coboundary.*

Lemma 2.5. *Let Γ be a countably infinite discrete group, $\Delta \subset \Gamma$ a nonamenable subgroup, and let $f \in \mathbb{Z}\Gamma$. We write $\tilde{\lambda}_\Delta$ for the unitary representation of Δ obtained by restricting the left regular representation $\tilde{\lambda}$ of Γ on $\ell^2(\Gamma, \mathbb{C})$ to Δ.*

If $c\colon \Delta \longrightarrow \ell^2(\Gamma, \mathbb{C})$ is a cocycle for $\tilde{\lambda}_\Delta$ such that $c \cdot f = \tilde{\rho}^{f^*} c$ is a coboundary and $c(\Delta)$ is contained in the orthogonal complement V of $\ker \tilde{\rho}^{f^*}$ *in $\ell^2(\Gamma, \mathbb{C})$ (cf. (1.6)), then c is a coboundary.*

Proof. By assumption there exists a $b \in \ell^2(\Gamma, \mathbb{C})$ such that $(1-\delta)\cdot b = c(\delta) \cdot f$ for every $\delta \in \Delta$. Let $\tilde{\rho}^{f^*} = UH$ be the polar decomposition [**4**, Theorem 6.1.2] of $\tilde{\rho}^{f^*}$, where U is a partial isometry on $\ell^2(\Gamma, \mathbb{C})$, $H = \big(\tilde{\rho}^{ff^*}\big)^{1/2} = (\tilde{\rho}^f \tilde{\rho}^{f^*})^{1/2}$, and both U and H lie in the left-equivariant group von Neumann algebra $\mathcal{N}\Gamma$.

Note that $\ker H = \ker \tilde{\rho}^{f^*}$. We write $H = \int_0^{\|\tilde{\rho}^{f^*}\|} \lambda \, dE_\lambda$ for the spectral decomposition of the positive self-adjoint operator H and consider, for each $0 < \varepsilon < \|\tilde{\rho}^{f^*}\|$, the projection operator $P_\varepsilon = P - E_\varepsilon$, where P is the orthogonal projection $\ell^2(\Gamma, \mathbb{C}) \longrightarrow V$. Then one has $P_\varepsilon \to P$ in the strong operator topology as $\varepsilon \searrow 0$.

Put $Q_\varepsilon = UP_\varepsilon U^*$ for every ε with $0 < \varepsilon < \|\rho^{f^*}\|$. Then

$$\begin{aligned} P_\varepsilon(c(\delta)) \cdot f = \tilde{\rho}^{f^*} P_\varepsilon(c(\delta)) &= UHP_\varepsilon(c(\delta)) = UP_\varepsilon H(c(\delta)) = Q_\varepsilon UH(c(\delta)) \\ &= Q_\varepsilon \tilde{\rho}^{f^*}(c(\delta)) = Q_\varepsilon(c(\delta) \cdot f) = Q_\varepsilon((1-\delta)\cdot b) = (1-\delta) \cdot Q_\varepsilon(b) \end{aligned}$$

for every $\delta \in \Delta$. Since $\|\tilde{\rho}^{f^*} v\| \geq \varepsilon \|v\|$ for every $v \in \mathrm{range}(P_\varepsilon)$, there exists $V_\varepsilon \in \mathcal{N}\Gamma$ vanishing on the orthogonal complement of $\mathrm{range}(\tilde{\rho}^{f^*} P_\varepsilon)$ and satisfying that $V_\varepsilon \, \tilde{\rho}^{f^*} v = v$ for every $v \in \mathrm{range}(P_\varepsilon)$. Therefore

$$P_\varepsilon(c(\delta)) = V_\varepsilon \, \tilde{\rho}^{f^*} P_\varepsilon(c(\delta)) = V_\varepsilon Q_\varepsilon((1-\delta)\cdot b)) = (1-\delta) \cdot V_\varepsilon Q_\varepsilon(b).$$

The 1-cocycle $\delta \mapsto P_\varepsilon c(\delta) = (1-\delta) \cdot V_\varepsilon Q_\varepsilon(b)$ for $\tilde{\lambda}_\Delta$ is thus a coboundary. Since $P_\varepsilon(c(\delta)) \to c(\delta)$ in $\ell^2(\Gamma, \mathbb{C})$ as $\varepsilon \searrow 0$ for every $\delta \in \Delta$, we conclude that the 1-cocycle $c\colon \Delta \longrightarrow \ell^2(\Gamma, \mathbb{C})$ for $\tilde{\lambda}_\Delta$ is an approximate coboundary.

Since Δ is nonamenable, the left regular representation of Δ on $\ell^2(\Delta, \mathbb{C})$ does not weakly contain the trivial representation of Δ. Thus, the representation $\tilde{\lambda}_\Delta$ of Δ on $\ell^2(\Gamma, \mathbb{C})$, as a direct sum of copies of the left regular representation of Δ, does not weakly contain the trivial representation of Δ.

From Lemma 2.3 we conclude that there is some $b \in \ell^2(\Gamma, \mathbb{C})$ satisfying $c(\delta) = (1-\delta)b$ for every $\delta \in \Delta$. $\square$

Proof of Proposition 2.4. Suppose that $f \in \mathbb{Z}\Gamma$ satisfies (1.6), and that $c\colon \Delta \longrightarrow \mathbb{Z}\Gamma$ is a 1-cocycle such that cf is a coboundary. Then cf is also a coboundary when c is viewed as an $\ell^2(\Gamma,\mathbb{C})$-valued cocycle for the unitary representation $\tilde{\lambda}_\Delta$ on $\ell^2(\Gamma,\mathbb{C})$. Lemma 2.5 shows that there exists a $b\in\ell^2(\Gamma,\mathbb{C})$ such that $c(\delta)=(1-\delta)\cdot b$ for every $\delta\in\Delta$. In order to prove that $b\in\mathbb{Z}\Gamma$ we set, for every $\varepsilon>0$, $F_\varepsilon(b)=\{\gamma\in\Gamma:|b_\gamma|\geq\varepsilon\}$. Then F_ε is finite, and so is the set $\{\delta\in\Delta:|(\delta\cdot b)_\gamma|=|b_{\delta^{-1}\gamma}|\geq\varepsilon\}=\{\delta\in\Delta:\delta^{-1}\gamma\in F_\varepsilon\}=\gamma F_\varepsilon^{-1}\cap\Delta$ for every $\gamma\in\Gamma$. Since Δ is nonamenable, it is infinite, and by varying ε we see that $\lim_{\delta\to\infty}(\delta\cdot b)_\gamma=0$ for every $\gamma\in\Gamma$. Since $c(\delta)_\gamma=b_\gamma-(\delta\cdot b)_\gamma\in\mathbb{Z}$ we conclude, by letting $\delta\to\infty$, that $b_\gamma\in\mathbb{Z}$ for every $\gamma\in\Gamma$. This completes the proof of the proposition. □

3. Ergodicity of principal actions

We recall the following result from [**10**, Lemma 1.2 and Theorem 1.6].

Theorem 3.1. *If α is an algebraic action of a countably infinite discrete group Γ on a compact abelian group X with dual group $\hat{X}$, then α is ergodic if and only if the orbit $\{\hat{\alpha}^\gamma a:\gamma\in\Gamma\}$ is infinite for every nontrivial $a\in\hat{X}$.*

Corollary 3.2. *Let Γ be a countably infinite discrete group, $f\in\mathbb{Z}\Gamma$, and let α_f be the principal algebraic Γ-action on the group X_f with Haar measure μ_f (cf. Definition 1.1). For $a\in\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f=\widehat{X_f}$ let $S(a)=\{\gamma\in\Gamma:\gamma\cdot a=a\}$ be its stabilizer.*

Then α_f is ergodic with respect to μ_f if and only if $S(a)$ has infinite index in Γ for every nonzero $a\in\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$.

Proof of Theorem 1.2. Suppose that $f\in\mathbb{Z}\Gamma$ is not a right zero-divisor, but that α_f is nonergodic. By Corollary 3.2 there exists an $h\in\mathbb{Z}\Gamma$ such that $h\notin\mathbb{Z}\Gamma f$ and the Γ-orbit $D=\{\gamma h+\mathbb{Z}\Gamma f:\gamma\in\Gamma\}$ of $a=h+\mathbb{Z}\Gamma f$ in $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ is finite. We denote by

$$\Delta=\{\delta\in\Gamma:\delta h-h\in\mathbb{Z}\Gamma f\}\tag{3.1}$$

the stabilizer of a, which has finite index in Γ by hypothesis, and consider the cocycle $c\colon\Delta\longrightarrow\mathbb{Z}\Gamma$ given by

$$h-\delta h=c(\delta)f\tag{3.2}$$

for every $\delta\in\Delta$ (here we are using that f is not a right zero-divisor). If $\Delta_0\subset\Delta$ is an infinite subgroup on which c is a coboundary then $c(\delta)=b-\delta b$ for some $b\in\mathbb{Z}\Gamma$ and every $\delta\in\Delta_0$. Hence $c(\delta)f=(1-\delta)bf=(1-\delta)h$ for every $\delta\in\Delta_0$. Since Δ_0 is infinite, this implies that $h=bf\in\mathbb{Z}\Gamma f$, contrary to our choice of h. In other words, if c is a coboundary when restricted to any infinite subgroup, we run into a contradiction with our assumption that α_f is nonergodic.

Proof of (1). If $\Gamma_0 \subset \Gamma$ is a finitely generated subgroup with a single end, then the same is true for its finite-index subgroup $\Delta \cap \Gamma_0$ where Δ is from (3.1). Proposition 2.1 shows that c is a coboundary on $\Delta \cap \Gamma_0$.

As was explained at the beginning of the proof of this theorem this contradicts the non-ergodicity of α_f.

Proof of (2). This is [**2**, Theorem 2.4.1]. For convenience of the reader we include the proof. Let $\Gamma_0 \subset \Gamma$ be the subgroup generated by $\mathrm{supp}(h) \cup \mathrm{supp}(f)$. Since Γ is not finitely generated there exists an increasing sequence of subgroups $\Gamma_n \subset \Gamma$, $n \geq 1$, such that Γ_{n+1} is generated over Γ_n by a single element $\gamma_{n+1} \in \Gamma_{n+1} \smallsetminus \Gamma_n$. Put $D = \{\gamma h + \mathbb{Z}\Gamma f : \gamma \in \Gamma\} \subset \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ and $D_n = \{\gamma h + \mathbb{Z}\Gamma f : \gamma \in \Gamma_n\} \subset \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$, $n \geq 0$. Then $|D_0| \leq |D_1| \leq \cdots \leq |D_n| \leq \cdots \leq |D| < \infty$. Hence there exists an $N \geq 0$ with $\gamma_{N+1} h + \mathbb{Z}\Gamma f = \gamma' h + \mathbb{Z}\Gamma f$ for some $\gamma' \in \Gamma_N$. Then $(\gamma_{N+1} - \gamma')h = gf$ for some $g \in \mathbb{Z}\Gamma$. We write $g = g_1 + g_2$ with $\mathrm{supp}(g_1) \subset \Gamma_N$ and $\mathrm{supp}(g_2) \cap \Gamma_N = \varnothing$. Then

$$\gamma_{N+1} h - g_2 f = g_1 f + \gamma' h. \tag{3.3}$$

All the terms on the right hand side of (3.3) are supported in Γ_N, whereas the supports of the terms on the left hand side of (3.3) are disjoint from Γ_N. Hence both sides of (3.3) have to vanish, which means that $\gamma_{N+1} h = g_2 f$ and $h \in \gamma_{N+1}^{-1} g_2 f \in \mathbb{Z}\Gamma f$, contrary to our choice of h. As explained above, this contradiction proves the ergodicity of α_f.

Proof of (3). If $\Gamma_0 \subset \Gamma$ is a nonamenable subgroup with $\beta_1^{(2)}(\Gamma_0) = 0$, then the same is true for its finite-index subgroup $\Delta \cap \Gamma_0$. By Proposition 2.2, the cocycle $c \colon \Delta \cap \Gamma_0 \longrightarrow \mathbb{Z}\Gamma$ is a coboundary, which leads to a contradiction as in (1). □

Proof of Theorem 1.3. If Γ is amenable, use Theorem 1.2 (1) or (2). If Γ is nonamenable, combine the argument at the beginning of the proof of Theorem 1.2 with Proposition 2.4. □

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