ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C*–ALGEBRAS

REVISED AND EXPANDED VERSION

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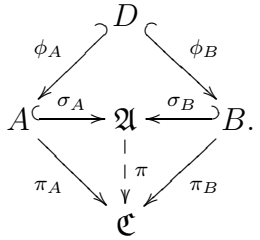
ABSTRACT. We examine the question of when the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ of full amalgamated free product C*–algebras, arising from compatible inclusions of C*–algebras $A \subseteq \widetilde{A}$, $B \subseteq \widetilde{B}$ and $D \subseteq \widetilde{D}$, is an embedding. Results giving sufficient conditions for λ to be injective, as well of classes of examples where λ fails to be injective, are obtained. As an application, we give necessary and sufficient condition for the full amalgamated free product of finite dimensional C*–algebras to be residually finite dimensional.

1. INTRODUCTION

Given C*–algebras A, B and D with injective $*$–homomorphisms $\phi_A : D \to A$ and $\phi_B : D \to B$, the corresponding full amalgamated free product C*–algebra (see [1] or [9, Chapter 5]) is the C*–algebra $\mathfrak{A}$, equipped with injective $*$–homomorphisms $\sigma_A : A \to \mathfrak{A}$ and $\sigma_B : B \to \mathfrak{A}$ such that $\sigma_A \circ \phi_A = \sigma_B \circ \phi_B$, such that $\mathfrak{A}$ is generated by $\sigma_A(A) \cup \sigma_B(B)$ and satisfying the universal property that whenever $\mathfrak{C}$ is a C*–algebra and $\pi_A : A \to \mathfrak{C}$ and $\pi_B : B \to \mathfrak{C}$ are $*$–homomorphisms satisfying $\pi_A \circ \phi_A = \pi_B \circ \phi_B$, there is a $*$–homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi \circ \sigma_A = \pi_A$ and $\pi \circ \sigma_B = \pi_B$. This situation is illustrated by the following commuting diagram:



(1)

The full amalgamated free product C*–algebra $\mathfrak{A}$ is commonly denoted by $A *_D B$, although this notation hides the dependence of $\mathfrak{A}$ on the embeddings ϕ_A and ϕ_B.

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Question 1.1. *Let D, A, B, $\widetilde{D}$, $\widetilde{A}$ and $\widetilde{B}$ be C*–algebras and suppose there are injective $*$–homomorphisms making the following diagram commute:*

$$\begin{array}{ccccc} \widetilde{A} & \xleftarrow{\phi_{\tilde{A}}} & \widetilde{D} & \xrightarrow{\phi_{\tilde{B}}} & \widetilde{B}. \\ {\scriptstyle\lambda_A}\uparrow & & {\scriptstyle\lambda_D}\uparrow & & {\scriptstyle\lambda_B}\uparrow \\ A & \xleftarrow{\phi_A} & D & \xrightarrow{\phi_B} & B \end{array}$$

Let $A*_D B$ and $\widetilde{A}*_{\widetilde{D}}\widetilde{B}$ be the corresponding full amalgamated free product C–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from λ_A and λ_B via the universal property. When is λ injective?*

We prove in §2 that λ is injective when either (i) $D = \widetilde{D}$, (or more precisely, when the $*$–homomorphism λ_D is surjective), or (ii) there are conditional expectations $E_A : \widetilde{A} \to A$ and $E_B : \widetilde{B} \to B$ that send $\widetilde{D}$ onto D and agree on $\widetilde{D}$. Injectivity in the case $D = \widetilde{D}$ was previously proved by G.K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map λ is injective when $D = \widetilde{D}$ and when there are conditional expectations

$$\widetilde{A} \overset{E_A^{\tilde{A}}}{\to} A \overset{E_D^A}{\to} D \overset{E_D^B}{\leftarrow} B \overset{E_B^{\tilde{B}}}{\leftarrow} \widetilde{B};$$

an argument for the case $D = \widetilde{D} = \mathbf{C}$, which uses Boca's results, is outlined in [3, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In §3, we consider some general conditions and give some concrete examples when λ fails to be injective. Finally, in §4, we apply this embedding result to extend a result from [5] about residual finite dimensionality of full amalgamated free products of finite dimensional C*–algebras.

2. Embeddings of full free products

The following result is of course well known. We include a proof for completeness.

Lemma 2.1. *Let A be a C*–subalgebra of a C*–algebra $\widetilde{A}$ and let $\pi : A \to B(\mathcal{H})$ be a $*$–representation. Then there is a Hilbert space $\mathcal{K}$ and a $*$–representation $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ such that*

$$\tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0, \qquad (a \in A,\, h \in \mathcal{H}). \tag{2}$$

Proof. Since in general π is a direct sum of cyclic representations, we may without loss of generality assume π is a cyclic representation with cyclic vector ξ. Let ϕ be the vector state $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi\rangle$ of A. Then $\mathcal{H}$ is identified with $L^2(A, \phi)$ and π is the associated GNS representation. Let $\tilde{\phi}$ be an extension of ϕ to a state of $\widetilde{A}$ and let $\widetilde{\mathcal{H}} = L^2(\widetilde{A}, \tilde{\phi})$. Then the inclusion $A \hookrightarrow \widetilde{A}$ gives rise to an isometry $\mathcal{H} \to \widetilde{\mathcal{H}}$, and we may thus write $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$ for a Hilbert space $\mathcal{K}$. If $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ is the GNS representation associated to $\tilde{\phi}$, then (2) holds. □

The following result was first proved by G.K. Pedersen [10]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.

Proposition 2.2. *Let*

$$\widetilde{A} \supseteq A \supseteq D \subseteq B \subseteq \widetilde{B}$$

be inclusions of C–algebras and let $A *_D B$ and $\widetilde{A} *_D \widetilde{B}$ be the corresponding full amalgamated free product C*–algebras. Let $\lambda : A *_D B \to \widetilde{A} *_D \widetilde{B}$ be the $*$–homomorphism arising via the universal property from the inclusions $A \hookrightarrow \widetilde{A}$ and $B \hookrightarrow \widetilde{B}$. Then λ is injective.*

Proof. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful $*$–homomorphism. We will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_D \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \qquad (x \in A *_D B,\, h \in \mathcal{H}). \tag{3}$$

This will imply λ is injective.

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$. Let

$$\sigma_{A,0} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K}_{A,0}),$$
$$\sigma_{B,0} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K}_{B,0})$$

be $*$–representations obtained from Lemma 2.1 such that

$$\sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0, \qquad (a \in A,\, h \in \mathcal{H}),$$

and similarly with A replaced by B. Note that $0 \oplus \mathcal{K}_{A,0}$ is reducing for $\sigma_{A,0}(D)$. Using Lemma 2.1, find Hilbert spaces $\mathcal{K}_{B,1}$ and $\mathcal{K}_{A,1}$ and $*$–representations

$$\sigma_{B,1} : \widetilde{B} \to B(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1})$$
$$\sigma_{A,1} : \widetilde{A} \to B(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1})$$

such that

$$\sigma_{B,1}(d)(k \oplus 0) = \sigma_{A,0}(d)(0 \oplus k), \qquad (d \in D,\, k \in \mathcal{K}_{A,0}),$$
$$\sigma_{A,1}(d)(k \oplus 0) = \sigma_{B,0}(d)(0 \oplus k), \qquad (d \in D,\, k \in \mathcal{K}_{B,0}).$$

Proceeding recursively, for every integer $n \geq 2$ we find $*$–representations

$$\sigma_{B,n} : \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}),$$
$$\sigma_{A,n} : \widetilde{B} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n})$$

such that

$$\sigma_{B,n}(d)(k \oplus 0) = \sigma_{A,n-1}(d)(0 \oplus k), \qquad (d \in D,\, k \in \mathcal{K}_{A,n-1}),$$
$$\sigma_{A,n}(d)(k \oplus 0) = \sigma_{B,n-1}(d)(0 \oplus k), \qquad (d \in D,\, k \in \mathcal{K}_{B,n-1}).$$

We now define the Hilbert spaces

$$\begin{aligned}\widetilde{\mathcal{H}}_A &= \overbrace{\mathcal{H}\oplus\mathcal{K}_{A,0}}^{\sigma_{A,0}}\oplus\overbrace{\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}}^{\sigma_{A,1}}\oplus\overbrace{\mathcal{K}_{B,1}\oplus\mathcal{K}_{A,2}}^{\sigma_{A,2}}\oplus\cdots,\\ \widetilde{\mathcal{H}}_B &= \underbrace{\mathcal{H}\oplus\mathcal{K}_{B,0}}_{\sigma_{B,0}}\oplus\underbrace{\mathcal{K}_{A,0}\oplus\mathcal{K}_{B,1}}_{\sigma_{B,1}}\oplus\underbrace{\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,2}}_{\sigma_{B,2}}\oplus\cdots,\end{aligned} \tag{4}$$

where the brackets indicate where the constructed representations act, and we let $\sigma_{\widetilde{A}}:\widetilde{A}\to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}}:\widetilde{B}\to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\begin{aligned}\sigma_{\widetilde{A}} &= \sigma_{A,0}\oplus\sigma_{A,1}\oplus\sigma_{A,2}\oplus\cdots,\\ \sigma_{\widetilde{B}} &= \sigma_{B,0}\oplus\sigma_{B,1}\oplus\sigma_{B,2}\oplus\cdots,\end{aligned}$$

where the summands act as indicated by brackets in (4). Consider the unitary $U:\widetilde{\mathcal{H}}_A\to\widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccccccccccc}\widetilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,2} & \oplus & \cdots\\ U\downarrow & & \downarrow & & & \searrow\!\swarrow & & & & \searrow\!\swarrow & & & & \searrow\!\swarrow & \\ \widetilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,2} & \oplus & \cdots.\end{array}$$

Let $\mathcal{K}=\mathcal{K}_{A,0}\oplus\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,1}\oplus\cdots$ and identify $\mathcal{H}\oplus\mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}}=\sigma_{\widetilde{A}}:\widetilde{A}\to B(\mathcal{H}\oplus\mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}}:\widetilde{B}\to B(\mathcal{H}\oplus\mathcal{K})$, the latter defined by $\tilde{\pi}_{\widetilde{B}}(\cdot)=U^*\sigma_{\widetilde{B}}(\cdot)U$. By construction, the restrictions of $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ to D agree, and we have

$$\begin{aligned}\tilde{\pi}_{\widetilde{A}}(a)(h\oplus 0) &= (\pi_A(a)h)\oplus 0, \qquad (a\in A,\ h\in\mathcal{H}),\\ \tilde{\pi}_{\widetilde{B}}(b)(h\oplus 0) &= (\pi_B(b)h)\oplus 0, \qquad (b\in B,\ h\in\mathcal{H}).\end{aligned}$$

Letting $\tilde{\pi}:\widetilde{A}*_D\widetilde{B}\to B(\mathcal{H}\oplus\mathcal{K})$ be the $*$–homomorphism obtained from $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ via the universal property, we have that (3) holds. □

For a C*–algebra A, unital or not, let A^u denote the unitization of A. Thus, as a vector space, $A^u=A\oplus\mathbf{C}$ with multiplication defined by $(a,\mu)\cdot(a',\mu')=(aa'+\mu a+\mu' a,\mu\mu')$. We identify A with the ideal $A\oplus 0$ of A^u, which has codimension 1.

Lemma 2.3. *Let $A\supseteq D\subseteq B$ be inclusions of C*–algebras. Consider the unitizations and corresponding inclusions*

$$\begin{array}{ccccc}A^u & \hookleftarrow & D^u & \hookrightarrow & B^u\\ \uparrow & & \uparrow & & \uparrow\\ A & \hookleftarrow & D & \hookrightarrow & B.\end{array}$$

Let $\lambda:A*_D B\to A^u*_{D^u}B^u$ be the resulting $*$–homomorphism between full amalgamated free products. Then there is an isomorphism $\pi:A^u*_{D^u}B^u\to(A*_D B)^u$ such

that $\pi \circ \lambda : A *_D B \to (A *_D B)^u$ is the canonical embedding arising in the definition of the unitization.

Proof. Since any $*$–representations of A and B that agree on D extend to $*$–representations of A^u and B^u that agree on D^u, the $*$–homomorphism λ is injective. Let $e \in A^u *_{D^u} B^u$ be the unit of A^u, which is of course identified with the units of B^u and D^u. Clearly, $A^u *_{D^u} B^u$ is generated by the image of λ together with e. One easily sees

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu\lambda(x') + \mu'\lambda(x) + \mu\mu' e.$$

Moreover, if $\rho : A^u *_{D^u} B^u \to \mathbf{C}$ is the $*$–homomorphism arising from the unital $*$–homomorphisms $A^u \to \mathbf{C}$ and $B^u \to \mathbf{C}$, then $\rho(e) = 1$ and $\lambda(A *_D B) \subseteq \ker \rho$. Hence $\lambda(A *_D B)$ has codimension 1 in $A^u *_{D^u} B^u$. Now π can be defined by $\pi(\lambda(x) + \mu e) = (x, \mu)$. □

Proposition 2.4. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{5}$$

is a commuting diagram of inclusions of C–algebras. Let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$–homomorphism of full free product C*–algebras. Suppose there are conditional expectations $E_A : \widetilde{A} \to A$, $E_D : \widetilde{D} \to D$ and $E_B : \widetilde{B} \to B$ onto A, D and B, respectively, such that the diagram*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \downarrow{\scriptstyle E_A} & & \downarrow{\scriptstyle E_D} & & \downarrow{\scriptstyle E_B} \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{6}$$

commutes. Then λ is injective.

Proof. By appealing to Lemma 2.3, we may without loss of generality assume all the algebras and $*$–homomorphisms in (5) are unital. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful, unital $*$–representation. As in the proof of Proposition 2.2, in order to show λ is injective, we will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_{\widetilde{D}} \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \qquad (x \in A *_D B,\ h \in \mathcal{H}). \tag{7}$$

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$, and let $\pi_D : D \to B(\mathcal{H})$ be their common restriction to D. Consider the canonical left action of $\widetilde{D}$ on the right Hilbert D–module $L^2(\widetilde{D}, E_D)$, which is obtained from $\widetilde{D}$ by separation and completion with respect to the D–valued inner product $\langle \tilde{d}_1, \tilde{d}_2 \rangle = E_D(\tilde{d}_1^* \tilde{d}_2)$. Consider the Hilbert space $L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H}$, where the left action of D on $\mathcal{H}$ is via π_D. Since π_D is unital, $\mathcal{H}$ embeds as a subspace, and we can write

$$L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D. \tag{8}$$

Consider the left action of $\widetilde{D}$ on the Hilbert space $\mathcal{H} \oplus \mathcal{K}_D$. The subspace $\mathcal{H}$ is reducing for the restriction of σ_D to D, and we have $\sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0$ for every $d \in D$ and $h \in \mathcal{H}$.

In a similar way, consider the Hilbert spaces

$$L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H}, \qquad L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} \tag{9}$$

and the associated left actions $\sigma_{A,0}$ of $\widetilde{A}$, respectively $\sigma_{B,0}$ of $\widetilde{B}$. As the diagram (6) commutes, the Hilbert space (8) embeds canonically as a subspace of both spaces (9). We may thus write

$$L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0}$$
$$L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0},$$

the subspace $\mathcal{H} \oplus \mathcal{K}_D \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ and $\sigma_{B,0}$ to $\widetilde{D}$, and we have $\sigma_{A,0}(\tilde{d})(\eta \oplus 0) = (\sigma_D(\tilde{d})\eta) \oplus 0 = \sigma_{B,0}(\tilde{d})(\eta \oplus 0)$ for every $\tilde{d} \in \widetilde{D}$ and $\eta \in \mathcal{H} \oplus \mathcal{K}_D$. Moreover, $\mathcal{H} \oplus 0 \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ to A and $\sigma_{B,0}$ to B, and we have

$$\sigma_{A,0}(a)(h \oplus 0 \oplus 0) = (\pi_A(a)h) \oplus 0 \oplus 0 \qquad (a \in A,\ h \in \mathcal{H})$$
$$\sigma_{B,0}(b)(h \oplus 0 \oplus 0) = (\pi_B(b)h) \oplus 0 \oplus 0 \qquad (b \in B,\ h \in \mathcal{H}).$$

Let $\sigma_{A,0,\widetilde{D}}$ denote the action of $\widetilde{D}$ on $\mathcal{K}_{A,0}$ obtained by restricting $\sigma_{A,0}$ to $\widetilde{D}$ and compressing, and similarly for $\sigma_{B,0,\widetilde{D}}$.

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces $\mathcal{K}_{A,n-1}$ and $\mathcal{K}_{B,n-1}$ have been constructed with actions $\sigma_{A,n-1,\widetilde{D}}$ and $\sigma_{B,n-1,\widetilde{D}}$, respectively, of $\widetilde{D}$, use Lemma 2.1 to construct Hilbert spaces $\mathcal{K}_{B,n}$ and $\mathcal{K}_{A,n}$ and $*$–homomorphisms

$$\sigma_{B,n} : \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n})$$
$$\sigma_{A,n} : \widetilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}),$$

such that

$$\sigma_{B,n}(\tilde{d})(k \oplus 0) = (\sigma_{A,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\ k \in \mathcal{K}_{A,n-1})$$
$$\sigma_{A,n}(\tilde{d})(k \oplus 0) = (\sigma_{B,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\ k \in \mathcal{K}_{B,n-1}).$$

Then let $\sigma_{B,n,\widetilde{D}}$ be the action of $\widetilde{D}$ on $\mathcal{K}_{B,n}$ obtained from the restriction of $\sigma_{B,n}$ to $\widetilde{D}$ by compressing, and similarly define the action $\sigma_{A,n,\widetilde{D}}$ of $\widetilde{D}$ on $\mathcal{K}_{A,n}$.

We may now define the Hilbert spaces

$$\begin{aligned}\widetilde{\mathcal{H}}_A &= \overbrace{\overbrace{\mathcal{H}\oplus\mathcal{K}_D}^{\sigma_D}\oplus\mathcal{K}_{A,0}}^{\sigma_{A,0}}\oplus\overbrace{\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}}^{\sigma_{A,1}}\oplus\overbrace{\mathcal{K}_{B,1}\oplus\mathcal{K}_{A,2}}^{\sigma_{A,2}}\oplus\cdots,\\ \widetilde{\mathcal{H}}_B &= \underbrace{\underbrace{\mathcal{H}\oplus\mathcal{K}_D}_{\sigma_D}\oplus\mathcal{K}_{B,0}}_{\sigma_{B,0}}\oplus\underbrace{\mathcal{K}_{A,0}\oplus\mathcal{K}_{B,1}}_{\sigma_{B,1}}\oplus\underbrace{\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,2}}_{\sigma_{B,2}}\oplus\cdots,\end{aligned}\tag{10}$$

where the brackets indicate where the constructed representations act. We let $\sigma_{\widetilde{A}}:\widetilde{A}\to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}}:\widetilde{B}\to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\begin{aligned}\sigma_{\widetilde{A}} &= \sigma_{A,0}\oplus\sigma_{A,1}\oplus\sigma_{A,2}\oplus\cdots,\\ \sigma_{\widetilde{B}} &= \sigma_{B,0}\oplus\sigma_{B,1}\oplus\sigma_{B,2}\oplus\cdots,\end{aligned}$$

where the summands act as indicated by brackets in (10). Consider the unitary $U:\widetilde{\mathcal{H}}_A\to\widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccccccccccccc}\widetilde{\mathcal{H}}_A &=& \mathcal{H} &\oplus& \mathcal{K}_D &\oplus& \mathcal{K}_{A,0} &\oplus& \mathcal{K}_{B,0} &\oplus& \mathcal{K}_{A,1} &\oplus& \mathcal{K}_{B,1} &\oplus& \mathcal{K}_{A,2} &\oplus& \cdots\\ {\scriptstyle U}\downarrow & & \downarrow & & \downarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow & & \\ \widetilde{\mathcal{H}}_B &=& \mathcal{H} &\oplus& \mathcal{K}_D &\oplus& \mathcal{K}_{B,0} &\oplus& \mathcal{K}_{A,0} &\oplus& \mathcal{K}_{B,1} &\oplus& \mathcal{K}_{A,1} &\oplus& \mathcal{K}_{B,2} &\oplus& \cdots.\end{array}$$

Let $\mathcal{K}=\mathcal{K}_D\oplus\mathcal{K}_{A,0}\oplus\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,1}\oplus\cdots$ and identify $\mathcal{H}\oplus\mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}}=\sigma_{\widetilde{A}}:\widetilde{A}\to B(\mathcal{H}\oplus\mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}}:\widetilde{B}\to B(\mathcal{H}\oplus\mathcal{K})$, the latter defined by $\tilde{\pi}_{\widetilde{B}}(\cdot)=U^*\sigma_{\widetilde{B}}(\cdot)U$. By construction, the restrictions of $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ to $\widetilde{D}$ agree, and we have

$$\begin{aligned}\tilde{\pi}_{\widetilde{A}}(a)(h\oplus 0) &= (\pi_A(a)h)\oplus 0 \qquad (a\in A,\, h\in\mathcal{H})\\ \tilde{\pi}_{\widetilde{B}}(b)(h\oplus 0) &= (\pi_B(b)h)\oplus 0 \qquad (b\in B,\, h\in\mathcal{H}).\end{aligned}$$

Letting $\tilde{\pi}:\widetilde{A}*_{\widetilde{D}}\widetilde{B}\to B(\mathcal{H}\oplus\mathcal{K})$ be the $*$–homomorphism obtained from $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ via the universal property, we have that (7) holds. □

3. Examples of non–embedding

In this section, we give some examples when the map λ of Question 1.1 fails to be injective. (In contrast, it is known [2] that in the more stringent situation of *reduced* amalgamated free products, the map analogous to λ is always injective.)

We begin with a trivial class of examples.

Examples 3.1. Let A and B be C*-subalgebras of a C*-algebra E with $A\not\subseteq B$ and $B\not\subseteq A$. Let $D=A\cap B$, $\widetilde{A}=E$ and $\widetilde{D}=\widetilde{B}=B$, equipped with the natural inclusions. Then the map $\lambda:A*_D B\to\widetilde{A}*_{\widetilde{D}}\widetilde{B}=E$ is injective if and only if $A*_D B$ is exactly the C*-subalgebra of E generated by A and B. This doesn't hold in general. Notice that in these examples, $B\cap\widetilde{D}=B\supsetneq D$.

Proposition 3.2. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array}$$

is a commuting diagram of inclusions of C–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$–homomorphism of full free product C*–algebras. Suppose there are conditional expectations $E_D^A : A \to D$ and $E_D^B : B \to D$ with E_D^B faithful. Suppose there are $\tilde{d} \in \widetilde{D}$, $a \in A$ and $b \in B$ satisfying $a\tilde{d} \in A$, $\tilde{d}b \in B$,*

$$D(\tilde{d}b) \cap Db = \{0\} \tag{11}$$

$$E_D^A(\tilde{d}^* a^* a d)b \neq 0. \tag{12}$$

Then λ is not injective.

Proof. Letting

$$\begin{aligned} &\sigma_A : A \hookrightarrow A *_D B, \qquad &&\sigma_B : B \hookrightarrow A *_D B, \\ &\sigma_{\widetilde{A}} : \widetilde{A} \hookrightarrow \widetilde{A} *_{\widetilde{D}} \widetilde{B}, \qquad &&\sigma_{\widetilde{B}} : \widetilde{B} \hookrightarrow \widetilde{A} *_{\widetilde{D}} \widetilde{B} \end{aligned} \tag{13}$$

be the embeddings as in (1), we have

$$\lambda(\sigma_A(a\tilde{d})\sigma_B(b)) = \sigma_{\widetilde{A}}(a\tilde{d})\sigma_{\widetilde{B}}(b) = \sigma_{\widetilde{A}}(a)\sigma_{\widetilde{B}}(\tilde{d}b) = \lambda(\sigma_A(a)\sigma_B(\tilde{d}b)).$$

Thus we need only show

$$\sigma_A(a\tilde{d})\sigma_B(b) \neq \sigma_A(a)\sigma_B(\tilde{d}b). \tag{14}$$

We consider the reduced amalgamated free product of C*–algebras (see [11] or [12]),

$$(A *_D^{\mathrm{red}} B, E_D) = (A, E_D^A) *_D (B, E_D^B)$$

and the natural quotient $*$–homomorphism $A *_D B \to A *_D^{\mathrm{red}} B$. Let $L^2(A *_D^{\mathrm{red}} B, E_D)$ be the right Hilbert D–module obtained by separation and completion from $A *_D^{\mathrm{red}} B$ with respect to the D–valued inner product $\langle x, y\rangle = E_D(x^* y)$, and given $x \in A *_D^{\mathrm{red}} B$, let $\hat{x}$ denote the corresponding element in $L^2(A *_D^{\mathrm{red}} B, E_D)$. Let $\mathcal{H}_A = L^2(A, E_D^A)$ and $\mathcal{H}_B = L^2(B, E_D^B)$ be similarly defined. Then in $L^2(A *_D^{\mathrm{red}} B, E_D)$, the closure of the subspace spanned by elements of the form $(ab)\hat{}$ for $a \in A$ and $b \in B$ is isomorphic to the tensor product $\mathcal{H}_A \otimes_D \mathcal{H}_B$ of Hilbert D–modules. In order to show (14), it will suffice to show

$$(a\tilde{d})\hat{} \otimes \hat{b} \neq \hat{a} \otimes (\tilde{d}b)\hat{}$$

in $\mathcal{H}_A \otimes_D \mathcal{H}_B$. Let $\zeta_B \in \mathcal{H}_B$. Then

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, (a\tilde{d})\hat{} \otimes \hat{b}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a\tilde{d})b)\hat{}\rangle \tag{15}$$

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, \hat{a} \otimes (\tilde{d}b)\hat{}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a)\tilde{d}b)\hat{}\rangle. \tag{16}$$

From assumptions (11) and (12), we obtain $E_D^A(\tilde{d}^* a^* a\tilde{d})b \neq E_D^A(\tilde{d}^* a^* a)\tilde{d}b$. Since E_D^B is faithful, there is $\zeta_B \in \mathcal{H}_B$ such that the right–hand–sides of (15) and (16) are not equal. □

Remark 3.3. From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions (11) and (12) can be dropped, and E_D^B need not be assumed faithful, but instead one must assume

$$E_D^B\big(b^*\big(E_D^A(\tilde{d}^*a^*a\tilde{d}) - E_D^A(\tilde{d}^*a^*a)\tilde{d} - \tilde{d}^*E_D^A(a^*a\tilde{d}) + \tilde{d}^*E_D^A(a^*a)\tilde{d}\big)b^*\big) \neq 0. \tag{17}$$

Note that the LHS of (17) is nothing other than

$$\langle (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{}, (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{} \rangle.$$

Corollary 3.4. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{18}$$

is a commuting diagram of inclusions of C^*–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$–homomorphism of full free product C^*–algebras. Suppose one of the following holds:

(i) $D = 0$
(ii) $D = \mathbf{C}$, *A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$, $a \in A$ and $b \in B$ such that $a\tilde{d} \in A\backslash\{0\}$, $\tilde{d}b \in B$ and $\tilde{d}b \notin \mathbf{C}b$. Then λ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume A and B are separable. Letting $E_D^A : A \to \mathbf{C}$ and $E_D^B : B \to \mathbf{C}$ be faithful states, we find the hypotheses of Proposition 3.2 are satisfied. □

From this corollary, we have the following class of concrete examples, which shows that λ may be non–injective even if

$$B \cap \widetilde{D} = D = A \cap \widetilde{D}. \tag{19}$$

Example 3.5. Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space. Inside $B(\mathcal{H})$, let $D = \mathbf{C}1$ and let $A = B = D + K(\mathcal{H})$, where $K(\mathcal{H})$ is the compact operators. Let $u \in B(\mathcal{H})$ be a unitary operator that does not belong to D and let $\widetilde{D} = C^*(u)$, $\widetilde{A} = \widetilde{B} = \widetilde{D} + K(\mathcal{H})$. Let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from the inclusions (18). Then λ is not injective.

Proof. Take $\tilde{d} = u$ and $a \in K(\mathcal{H})\backslash\{0\}$. Since $u \notin \mathbf{C}1$, there is $b \in K(\mathcal{H})$ such that $ub \notin \mathbf{C}b$. Now apply Corollary 3.4. One can choose u so that $C^*(u) \cap (\mathbf{C}1 + K(\mathcal{H})) = \mathbf{C}1$, in order to get (19). □

Proposition 3.6. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{20}$$

is a commuting diagram of inclusions of C–algebras and let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting* $*$*–homomorphism of full free product C*–algebras. Suppose one of the following holds:*

(i) $D = 0$
(ii) $D = \mathbf{C}$*, A and B are unital and the inclusions* $D \hookrightarrow A$ *and* $D \hookrightarrow B$ *are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$*,* $a_1, a_2 \in A$ *and* $b \in B \backslash D$ *such that* $a_1\tilde{d}, \tilde{d}a_2 \in A$*,* $a_1\tilde{d} \notin \mathbf{C}$ *and* $\tilde{d}b = b\tilde{d}$*. Then* λ *is not injective.*

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We use the same notation as in (13). We have

$$\begin{aligned}\lambda(\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2)) &= \sigma_{\widetilde{A}}(a_1\tilde{d})\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(a_2) \\ &= \sigma_{\widetilde{A}}(a_1)\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(\tilde{d}a_2) = \lambda(\sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2)),\end{aligned}$$

and we must only show

$$\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2) \neq \sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2). \tag{21}$$

Without loss of generality, assume A and B are separable. Let $\phi_A : A \to \mathbf{C}$ and $\phi_B : B \to \mathbf{C}$ be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume $\phi_B(b) = 0$. Let

$$(A *_{\mathbf{C}}^{\mathrm{red}} B, \phi) = (A, \phi_A) *_{\mathbf{C}} (B, \phi_B)$$

be the reduced free product of C*–algebras. Using arguments and notation as in the proof of Proposition 3.2, the closure of the subspace of $L^2(A *_{\mathbf{C}}^{\mathrm{red}} B, \phi)$ spanned by elements of the form $(aba')\hat{}$ for $a, a' \in A$ is isomorphic to $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. To show (21), it will suffice to show

$$(a_1\tilde{d})\hat{} \otimes \hat{b} \otimes \hat{a}_2 \neq \hat{a}_1 \otimes \hat{b} \otimes (\tilde{d}a_2)\hat{}$$

in $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. However, this follows from the assumptions. □

From the above proposition, we get the following example, which requires only "bad" relations between A and $\widetilde{D}$, not between B and $\widetilde{D}$.

Example 3.7. Let D, $\widetilde{D}$, A and $\widetilde{A}$ be as in Example 3.5. Let B be any unital C*–algebra of dimension greater than 1 and let $\widetilde{B} = B \otimes \widetilde{D}$, (for the unique C*–tensor norm). Then the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ arising from the inclusions (20) is not injective.

Remark 3.8. The problem with injectivity of λ in Examples 3.5 and 3.7 arises already at the algebraic level

$$A *_D^{\mathrm{alg}} B \to \widetilde{A} *_{\widetilde{D}}^{\mathrm{alg}} \widetilde{B}. \tag{22}$$

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while λ fails to be injective, e.g. by taking E to be a reduced free product. However, we do not know of an example where λ fails to be injective and where the algebraic map (22) is injective, but where $A \cap \widetilde{D} = D = B \cap \widetilde{D}$.

4. An application to residual finite dimensionality

A C*–algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite dimensional $*$–representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi's proof [6] that the full group C*–algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d. C*–algebras A and B with amalgamation over either the zero C*–algebra or over the scalar multiples of the identity (if A and B are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ over a unital subalgebra D is r.f.d. provided that the normalized traces on $M_k(\mathbf{C})$ and $M_\ell(\mathbf{C})$ restrict to the same trace on D. In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite dimensional algebras.

Lemma 4.1. *Let $S = \{x \in \mathbf{R}^n \mid Ax = 0\}$, where A is an $m \times n$ matrix having only rational entries. Then vectors having only rational entries are dense in S.*

Proof. By considering the reduced row–echelon form of A, we see that there is a basis for S consisting of rational vectors. □

Theorem 4.2. *Consider unital inclusions of C*–algebras $A \supseteq D \subseteq B$ with A and B finite dimensional. Let $A *_D B$ be the corresponding full amalgamated free product. Then $A *_D B$ is residually finite dimensional if and only if there are faithful tracial states τ_A on A and τ_B on B whose restrictions to D agree.*

Proof. Since every separable r.f.d. C*–algebra has a faithful tracial state, the necessity of the existence of τ_A and τ_B is clear.

Let us recall some well known facts about a unital inclusion $D \subseteq A$ of finite dimensional C*–algebras (see e.g. Chapter 2 of [8]). Let $p_1, \ldots, p_m$ be the minimal central projections of A and $q_1, \ldots, q_n$ the minimal central projections of D. Then the inclusion matrix Λ_D^A is a $m \times n$ integer matrix whose (i,j)th entry is $\operatorname{rank}(q_j p_i A q_j)/\operatorname{rank}(q_j D)$, where the rank of a matrix algebra $M_k(\mathbf{C})$ is k. To a trace τ on A, we associate the column vector s of length m whose ith entry is the trace of a minimal projection in $p_i A$. Then the restriction of τ to D has associated column vector $(\Lambda_D^A)^t s$, where the superscript t indicates transpose.

Thus, given $A \supseteq D \subseteq B$ as in the statement of the theorem, the existence of faithful tracial states τ_A and τ_B agreeing on D is equivalent to the existence of column vectors s_A and s_B, none of whose components are zero, such that $(\Lambda_D^A)^t s_A = (\Lambda_D^B)^t s_B$, i.e.

$$\begin{bmatrix} (\Lambda_D^A)^t, & -(\Lambda_D^B)^t \end{bmatrix} \begin{bmatrix} s_A \\ s_B \end{bmatrix} = 0. \tag{23}$$

Supposing now that such traces τ_A and τ_B exist, by Lemma 4.1 there is a solution $\left[\begin{smallmatrix} s_A \\ s_B \end{smallmatrix}\right]$ to (23) whose entries are all strictly positive and rational. Therefore, the traces τ_A and τ_B agreeing on D can be chosen to take only rational values on minimal projections of A and, respectively, B. Hence there are unital inclusions into matrix algebras,

$$M_k(\mathbf{C}) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(\mathbf{C}),$$

so that τ_A is the restriction of the tracial state on $M_k(\mathbf{C})$ to A and τ_B is the restriction of the tracial state on $M_\ell(\mathbf{C})$ to B. By Proposition 2.2, $A *_D B$ is a subalgebra of $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$. By Theorem 2.3 of [5], $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ is r.f.d. Therefore, $A *_D B$ is r.f.d. □

References

[1] B. Blackadar, *Weak expectations and nuclear C*–algebras,* Indiana Univ. Math. J. **27** (1978), 1021-1026.

[2] E. Blanchard and K. Dykema, *Embeddings of reduced free products of operator algebras,* Pacific J. Math. **199** (2001), 1-19.

[3] D. Blecher and V. Paulsen, *Explicit construction of universal operator algebras and applications to polynomial factorization,* Proc. Amer. Math. Soc. **112** (1991), 839-850.

[4] F. Boca, *Free products of completely positive maps and spectral sets,* J. Funct. Anal. **97** (1991), 251-263.

[5] N.P. Brown and K. Dykema, *Popa algebras in free group factors,* preprint.

[6] M.-D. Choi, *The full C*-algebra of the free group on two generators,* Pacific J. Math. **87** (1980), 41-48.

[7] R. Exel and T. Loring, *Finite-dimensional representations of free product C*-algebras*, Internat. J. Math. **3** (1992), 469-476.

[8] F.M. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter graphs and towers of algebras,* MSRI Publications, vol. 14, Springer–Verlag, New York, 1987.

[9] T. Loring, *Lifting solutions to perturbing problems in C*–algebras,* Fields Institute Monographs vol. 8, American Mathematical Society, Providence, 1997.

[10] G.K. Pedersen, *Pullback and pushout constructions in C*-algebra theory,* J. Funct. Anal. **167** (1999), 243-344.

[11] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Mathematics, Volume 1132, Springer-Verlag, 1985, 556–588.

[12] D.V. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series **1**, American Mathematical Society, 1992.

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