θ-DEFORMATIONS AS COMPACT QUANTUM METRIC SPACES

HANFENG LI

ABSTRACT. Let M be a compact spin manifold with a smooth action of the n-torus. Connes and Landi constructed θ-deformations M_θ of M, parameterized by $n \times n$ real skew-symmetric matrices θ. The M_θ's together with the canonical Dirac operator $(D, \mathcal{H})$ on M are an isospectral deformation of M. The Dirac operator D defines a Lipschitz seminorm on $C(M_\theta)$, which defines a metric on the state space of $C(M_\theta)$. We show that when M is connected, this metric induces the weak-$*$ topology. This means that M_θ is a compact quantum metric space in the sense of Rieffel.

1. INTRODUCTION

In noncommutative geometry there are many examples of noncommutative spaces deformed from commutative spaces. However, for many of them the Hochschild dimension, which corresponds to the commutative notion of dimension, is different from that of the original commutative space. For instance, the C^*-algebras of the standard Podleś quantum 2-spheres and of the quantum 4-spheres of [1] are isomorphic to each other, and their Hochschild dimension is zero [17].

In [8] Connes and Landi introduced a one-parameter deformation S^4_θ of the 4-sphere with the property that the Hochschild dimension of S^4_θ equals that of S^4. They also considered general θ-deformations, which was studied further by Connes and Dubois-Violette in [7] (see also [28]). In general, the θ-deformation M_θ of a manifold M equipped with a smooth action of the n-torus T^n is determined by defining the algebra of smooth functions $C^\infty(M_\theta)$ as the invariant subalgebra (under the diagonal action of T^n) of the algebra $C^\infty(M \times T_\theta) := C^\infty(M)\hat{\otimes}C^\infty(T_\theta)$ of smooth functions on $M \times T_\theta$; here θ is a real skew-symmetric $n \times n$ matrix and T_θ is the corresponding noncommutative n-torus. This construction is a special case of the strict deformation quantization constructed in [21]. When M is a compact spin manifold, Connes and Landi showed that the canonical Dirac operator $(D, \mathcal{H})$ on M and a deformed anti-unitary operator J_θ together gives a spectral triple for $C^\infty(M_\theta)$, fitting it into Connes' noncommutative Riemannian geometry framework [5, 6]. In [7] Connes and Dubois-Violette also showed how θ-deformations lead to compact quantum groups which are deformations of various classical groups (see also [30, Section 4]).

In this paper we investigate the metric aspect of θ-deformation. The study of metric spaces in noncommutative setting was initiated by Connes in [4] in the framework of his spectral triple. The main ingredient of a spectral triple is a Dirac operator D. On the one hand, it captures the differential structure by setting $df = [D, f]$. On the other hand, it enables us to recover the Lipschitz seminorm L,

Date: February 21, 2005.
2000 *Mathematics Subject Classification.* Primary 46L87; Secondary 53C23, 53C27, 58B34.

which is usually defined as

$$L(f) := \sup\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y\}, \tag{1}$$

where ρ is the geodesic metric on the Riemannian manifold, instead by means of $L(f) = \| [D, f] \|$, and then one recovers the metric ρ by

$$\rho(x, y) = \sup_{L(f) \leq 1} |f(x) - f(y)|. \tag{2}$$

In [4, Section 1] Connes went further by considering the (possibly $+\infty$-valued) metric on the state space of the algebra defined by (2). Motivated by what happens to ordinary compact metric spaces, in [22, 23, 24] Rieffel introduced "compact quantum metric spaces" (see Definition 2.9 below) which requires the metric on the state space to induce the w^*-topology. Many examples of compact quantum metric spaces have been constructed, mostly from ergodic actions of compact groups [22] or group algebras [26, 18]. Usually it is quite difficult to find out whether a specific seminorm L on a unital C^*-algebra gives a quantum metric, i.e., whether the metric defined by (2) on the state space induces the w^*-topology.

Denote by L_θ the seminorm on $C(M_\theta)$ determined by the Dirac operator D (see Definition 3.11 below for detail). Notice that when M is connected the geodesic distance makes M into a metric space. Then our main theorem in this paper is:

Theorem 1.1. *Let M be a connected compact spin manifold with a smooth action of $\mathbb{T}^n$. For every real skew-symmetric $n \times n$ matrix θ the pair $(C(M_\theta), L_\theta)$ is a C^*-algebraic compact quantum metric space.*

Motivated by questions in string theory, Rieffel also introduced a notion of quantum Gromov-Hausdorff distance for compact quantum metric spaces [24, 25]. It has many nice properties. Using the quantum Gromov-Hausdorff distance one can discuss the continuity of θ-deformations (with respect to the parameter θ) in a concrete way. This will be done in [16].

This paper is organized as follows. We shall use heavily the theory of locally convex topological vector spaces (LCTVS). In Section 2 we review some facts about LCTVS, Clifford algebras, and Rieffel's theory of compact quantum metric spaces. Connes and Dubois-Violette's formulation of θ-deformations is reviewed in Section 3. In Section 4 we prove a general theorem showing that in the presence of a compact group action, sometimes we can reduce the study of a given seminorm to its behavior on the isotypic components of this group action. Section 5 contains the main part of our proof of Theorem 1.1, where we study various differential operators to derive certain formulas. Finally, Theorem 1.1 is proved in Section 6.

Throughout this paper G will be a nontrivial compact group with identity e_G, endowed with the normalized Haar measure. Denote by $\hat{G}$ the dual of G, and by γ_0 the trivial representation. For any $\gamma \in \hat{G}$ let χ_γ be the corresponding character on G, and let $\bar{\gamma}$ be the contragradient representation . For any $\gamma \in \hat{G}$ and any representation of G on some complex vector space V, we denote by V_γ the γ-isotypic component of V. If $\mathcal{J}$ is a finite subset of $\hat{G}$, we also let $V_{\mathcal{J}} = \sum_{\gamma \in \mathcal{J}} V_\gamma$, and let $\bar{\mathcal{J}} = \{\bar{\gamma} : \gamma \in \mathcal{J}\}$.

Acknowledgments. This is part of my Ph.D. dissertation submitted to UC Berkeley in 2002. I am indebted to my advisor, Professor Marc Rieffel, for many helpful discussions, suggestions, and for his support throughout my time at Berkeley. I also thank Thomas Hadfield and Frédéric Latrémolière for valuable conversations.

2. Preliminaries

In this section we review some facts about locally convex topological vector spaces (LCTVS), Clifford algebras, and Rieffel's theory of compact quantum metric spaces.

2.1. Locally convex topological vector spaces. We recall first some facts about LCTVS. The reader is referred to [29, Chapters 5 and 43] for detailed information about completion and tensor products of LCTVS. Throughout this paper, our LCTVS will all be Hausdorff.

For any LCTVS V and W, one can define the *projective tensor product* of V and W, denoted by $V \otimes_\pi W$, as the vector space $V \otimes W$ equipped with the so called projective topology. $V \otimes_\pi W$ is also a LCTVS, and one can form the completion $V \hat{\otimes}_\pi W$.

For continuous linear maps $\psi_j : V_j \to W_j$ $(j = 1, 2)$ between LCTVS, the tensor product linear map $\psi_1 \otimes_\pi \psi_2 : V_1 \otimes_\pi V_2 \to W_1 \otimes_\pi W_2$ is also continuous and extends to a continuous linear map $\psi_1 \hat{\otimes}_\pi \psi_2 : V_1 \hat{\otimes}_\pi V_2 \to W_1 \hat{\otimes}_\pi W_2$.

Let V be a LCTVS, and let α be an action of a topological G on V by automorphisms. We say that the action α is *continuous* if the map $G \times V \to V$ given by $(x, v) \mapsto \alpha_x(v)$ is (jointly) continuous. Let V (resp. W) be a LCTVS and α (resp. β) be a continuous action of G on V (resp. W). Then the tensor product action $\alpha \hat{\otimes}_\pi \beta$ of G on $V \hat{\otimes}_\pi W$ is easily seen to be continuous.

A *locally convex algebra* (LCA) [3] is a LCTVS V with an algebra structure such that the multiplication $V \times V \to V$ is (jointly) continuous. If furthermore V is a $*$-algebra and the $*$-operation $* : V \to V$ is continuous, let us say that V is a *locally convex $*$-algebra* (LC$*$A). A *locally convex left V-module* of V is a left V-module W such that the action $V \times W \to W$ is (jointly) continuous. For a smooth manifold M, the space of (possibly unbounded) smooth functions $C^\infty(M)$ equipped with usual Fréchet space topology is a LC$*$A. For a smooth vector bundle E over M, the space of smooth sections $C^\infty(M, E)$ is a locally convex $C^\infty(M)$-bimodule. If furthermore E is an algebra bundle with fibre algebras being finite-dimensional, then $C^\infty(M, E)$ is also a LCA. We shall need Proposition 2.3 below.

Lemma 2.1. *Let V and W be two* LCTVS. *Denote by $\hat{V}$ and $\hat{W}$ the completion of V and W respectively. Then*

$$\hat{V} \hat{\otimes}_\pi \hat{W} = V \hat{\otimes}_\pi W.$$

Proof. The natural linear maps $\iota_V : V \hookrightarrow \hat{V}$ and $\iota_W : W \hookrightarrow \hat{W}$ are continuous, so we have the continuous linear map $\iota_V \hat{\otimes}_\pi \iota_W : V \hat{\otimes}_\pi W \to \hat{V} \hat{\otimes}_\pi \hat{W}$, which is the unique continuous extension of $\iota_V \otimes \iota_W : V \otimes W \to \hat{V} \otimes \hat{W}$.

Let $v_0 \in \hat{V}$ (resp. $w_0 \in \hat{W}$) and a net $\{v_j\}_{j \in I}$ (resp. $\{w_j\}_{j \in I}$) in V (resp. W) converging to v_0 (resp. w_0). Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be a continuous seminorm on V (resp. W). Consider the continuous tensor product seminorm $\mathfrak{p} \hat{\otimes}_\pi \mathfrak{q}$ on $V \hat{\otimes}_\pi W$ defined by

$$(\mathfrak{p} \hat{\otimes}_\pi \mathfrak{q})(\eta) = \inf \sum_j \mathfrak{p}(v'_j)\mathfrak{q}(w'_j)$$

for all $\eta \in V \otimes_\pi W$, where the infimum is taken over all finite sets of pairs (v'_k, w'_k) such that

$$\eta = \sum_k v'_k \otimes w'_k.$$

It satisfies

$$(\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})(v \otimes w) = \mathfrak{p}(v)\mathfrak{q}(w)$$

for all $v \in V$ and $w \in W$ [29, Proposition 43.1]. In particular, we have

$$\begin{aligned}(\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})(v_j \otimes w_j - v_{j'} \otimes w_{j'}) &= (\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})((v_j - v_{j'}) \otimes w_j + v_{j'} \otimes (w_j - w_{j'})) \\ &\leq \mathfrak{p}(v_j - v_{j'})\mathfrak{q}(w_j) + \mathfrak{p}(v_{j'})\mathfrak{q}(w_j - w_{j'}) \to 0\end{aligned}$$

as $j, j' \to \infty$. Since such $\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q}$ form a basis of continuous seminorms on $V\hat{\otimes}_\pi W$ [29, page 438], the net $\{v_j \otimes w_j\}_{j \in I}$ is a Cauchy net in $V\hat{\otimes}_\pi W$. Then it converges to some element in $V\hat{\otimes}_\pi W$. Let $\varphi(v_0, w_0) = \lim_{j\to\infty}(v_j \otimes w_j)$. Clearly $\varphi(v_0, w_0)$ doesn't depend on the choice of the nets $\{v_j\}_{j\in I}$ and $\{w_j\}_{j\in I}$. So the map $\varphi : \hat{V} \times \hat{W} \to V\hat{\otimes}_\pi W$ is well-defined. It is easy to see that φ is bilinear and is an extension of the natural map $V \times W \to V\hat{\otimes}_\pi W$. Denote the extension of $\mathfrak{p}$ (resp. $\mathfrak{q}$) on $\hat{V}$ (resp. $\hat{W}$) still by $\mathfrak{p}$ (resp. $\mathfrak{q}$). Notice that

$$\begin{aligned}(\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})(\varphi(v_0, w_0)) &= (\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})(\lim_{j\to\infty}(v_j \otimes w_j)) = \lim_{j\to\infty}(\mathfrak{p}\hat{\otimes}_\pi \mathfrak{q})(v_j \otimes w_j) \\ &= \lim_{j\to\infty} \mathfrak{p}(v_j)\mathfrak{q}(w_j) = \mathfrak{p}(v)\mathfrak{q}(w).\end{aligned}$$

So φ is continuous, and hence the associated linear map $\hat{V} \otimes_\pi \hat{W} \to V\hat{\otimes}_\pi W$ is continuous [29, Proposition 43.4]. Consequently, we have the continuous extension $\psi : \hat{V}\hat{\otimes}_\pi \hat{W} \to V\hat{\otimes}_\pi W$ [29, Theorem 5.2].

Notice that $V \otimes W$ is dense in both $\hat{V}\hat{\otimes}_\pi \hat{W}$ and $V\hat{\otimes}_\pi W$. Clearly ψ and $\iota_V \hat{\otimes}_\pi \iota_W$ are inverse to each other when restricted to $V \otimes W$. It follows immediately that ψ and $\iota_V \hat{\otimes}_\pi \iota_W$ are isomorphisms inverse to each other between $\hat{V}\hat{\otimes}_\pi \hat{W}$ and $V\hat{\otimes}_\pi W$. □

Lemma 2.2. *Let V_j, W_j, H_j $(j = 1, 2)$ be* LCTVS*, and let $\psi_j : V_j \times W_j \to H_j$ be continuous bilinear maps; then the bilinear map*

$$\psi_1 \otimes \psi_2 : (V_1 \otimes V_2) \times (W_1 \otimes W_2) \to H_1 \otimes H_2$$

extends to a continuous bilinear map

$$\psi_1 \hat{\otimes}_\pi \psi_2 : (V_1 \hat{\otimes}_\pi V_2) \times (W_1 \hat{\otimes}_\pi W_2) \to H_1 \hat{\otimes}_\pi H_2.$$

Proof. We have the associated continuous linear map $\varphi_j : V_j \otimes_\pi W_j \to H_j, j = 1, 2$ [29, Proposition 43.4] and hence the continuous linear map

$$\varphi_1 \hat{\otimes}_\pi \varphi_2 : (V_1 \otimes_\pi W_1)\hat{\otimes}_\pi (V_2 \otimes_\pi W_2) \to H_1 \hat{\otimes}_\pi H_2.$$

By the associativity of the projective tensor product and Lemma 2.1 we have

$$\begin{aligned}&(V_1 \otimes_\pi W_1)\hat{\otimes}_\pi (V_2 \otimes_\pi W_2) \\ =\ &((V_1 \otimes_\pi W_1) \otimes_\pi V_2)\hat{\otimes}_\pi W_2 = ((V_1 \otimes_\pi V_2) \otimes_\pi W_1)\hat{\otimes}_\pi W_2 \\ =\ &(V_1 \otimes_\pi V_2)\hat{\otimes}_\pi (W_1 \otimes_\pi W_2) = (V_1 \hat{\otimes}_\pi V_2)\hat{\otimes}_\pi (W_1 \hat{\otimes}_\pi W_2).\end{aligned}$$

So we get a continuous linear map $(V_1 \hat{\otimes}_\pi V_2)\hat{\otimes}_\pi (W_1 \hat{\otimes}_\pi W_2) \to H_1 \hat{\otimes}_\pi H_2$, which is equivalent to a continuous bilinear map $(V_1 \hat{\otimes}_\pi V_2) \times (W_1 \hat{\otimes}_\pi W_2) \to H_1 \hat{\otimes}_\pi H_2$. Clearly this extends the bilinear map $\psi_1 \otimes \psi_2 : (V_1 \otimes V_2) \times (W_1 \otimes W_2) \to H_1 \otimes H_2$. □

Proposition 2.3. *Let V and W be* LCA*. Then $V\hat{\otimes}_\pi W$ is also a* LCA *extending the natural algebra structure on $V \otimes W$. If both V and W are* LC∗A*, so is $V\hat{\otimes}_\pi W$. If H is a locally convex left V-module, then $H\hat{\otimes}_\pi W$ is a locally convex left $V\hat{\otimes}_\pi W$-module.*

Proof. By Lemma 2.2 we have the continuous bilinear map

$$(V\hat{\otimes}_\pi W) \times (V\hat{\otimes}_\pi W) \to V\hat{\otimes}_\pi W$$

extending the multiplication of $V \otimes W$. Since $V \otimes W$ is dense in $V\hat{\otimes}_\pi W$, clearly the above bilinear map is associative. In other words, $V\hat{\otimes}_\pi W$ is a LCA. The assertion about modules can be proved in the same way.

If both V and W are LC∗A, then we have the tensor product of the ∗-operations $V\hat{\otimes}_\pi W \to V\hat{\otimes}_\pi W$. Since it extends the natural ∗-operation on $V \otimes W$, it is easy to check that it is compatible with the algebra structure. So $V\hat{\otimes}_\pi W$ is a LC∗A. □

For any LCTVS V and W, one can also define the *injective tensor product* $V \otimes_\epsilon W$ of V and W, and form the completion $V\hat{\otimes}_\epsilon W$. Let us say that a continuous linear map $\psi : V \to W$ is an isomorphism of V into W if ψ is injective and $\psi : V \to \psi(V)$ is a homeomorphism of topological spaces. The only property about injective tensor product we shall need is that if ψ_j is an isomorphism of V_j into W_j for $j = 1, 2$, then the corresponding tensor product linear map $\psi_1\hat{\otimes}_\epsilon\psi_2$ is an isomorphism of $V_1\hat{\otimes}_\epsilon V_2$ into $W_1\hat{\otimes}_\epsilon W_2$ [29, Proposition 43.7].

Let $n \geq 2$, and let θ be a real skew-symmetric $n \times n$ matrix. Denote by $\mathcal{A}_\theta$ the corresponding quantum torus [19, 20]. It could be described as follows. Let ω_θ denote the skew-symmetric bicharacter on $\mathbb{Z}^n$ defined by

$$\omega_\theta(p, q) = e^{i\pi p\cdot\theta q}.$$

For each $p \in \mathbb{Z}^n$ there is a unitary u_p in $\mathcal{A}_\theta$. And $\mathcal{A}_\theta$ is generated by these unitaries with the relation

$$u_p u_q = \omega_\theta(p, q) u_{p+q}.$$

So one may think of vectors in $\mathcal{A}_\theta$ as some kind of functions on $\mathbb{Z}^n$. The n-torus $\mathbb{T}^n$ has a canonical ergodic action τ on $\mathcal{A}_\theta$. Notice that $\mathbb{Z}^n$ is the dual group of $\mathbb{T}^n$. We denote the duality by $\langle p, x\rangle$ for $x \in \mathbb{T}^n$ and $p \in \mathbb{Z}^n$. Then τ is determined by

$$\tau_x(u_p) = \langle p, x\rangle\, u_p.$$

The set $\mathcal{A}^\infty_\theta$ of smooth vectors for the action τ is exactly the Schwarz space $\mathcal{S}(\mathbb{Z}^n)$ [2]. Let $X_1, \cdots, X_n$ be a basis for the Lie algebra of $\mathbb{T}^n$. Then we have the differential $\partial_{X_j}(f)$ for each $f \in \mathcal{A}^\infty_\theta$ and $1 \leq j \leq n$. For each $k \in \mathbb{N}$ define a seminorm, $\mathfrak{q}_k$, on $\mathcal{A}^\infty_\theta$ by

$$\mathfrak{q}_k := \max_{|\overrightarrow{m}|\leq k} \parallel \partial^{m_1}_{X_1} \cdots \partial^{m_n}_{X_n}(f) \parallel .$$

Clearly $\mathcal{A}^\infty_\theta$ is a complete LC∗A equipped with the topology defined by these $\mathfrak{q}_k$'s. On the other hand, it is easy to see that this topology is the same as the usual topology on $\mathcal{S}(\mathbb{Z}^n)$. Thus $\mathcal{A}^\infty_\theta$ is a *nuclear* space [29, Theorem 51.5], which means that for every LCTVS V the injective and projective topologies on $V \otimes \mathcal{A}^\infty_\theta$ coincide [29, Theorem 50.1]. So we shall simply use $V \otimes \mathcal{A}^\infty_\theta$ to denote the (projective or injective) topological tensor product. The algebraic tensor product will be denoted by $V \otimes_{alg} \mathcal{A}^\infty_\theta$.

We shall need to integrate continuous functions with values in a LCTVS. For our purpose, it suffices to use the Riemann integral. Though this should be well-known, we have not been able to find any reference in the literature. So we include a definition here.

Lemma 2.4. *Let X be a compact space with a probability measure μ. Let*

$$\begin{aligned} I := \{ & \{X_1, \cdots, X_k\} : X_1, \cdots, X_k \text{ are disjoint measurable subsets of } X, \\ & k \in \mathbb{N}, \cup_{j=1}^{k} X_j = X\} \end{aligned}$$

be the set of all finite partitions of X into measurable subsets with the fine order, i.e.

$\{X_1, \cdots, X_k\} \geq \{X'_1, \cdots, X'_{k'}\}$ if and only if every X_j is contained in some $X'_{j'}$.

Let V be a complete LCTVS, *and let $f : X \to V$ be a continuous map. For each $\{X_1, \cdots, X_k\}$ in I pick an $x_j \in X_j$ for each j, and let*

$$v_{\{X_1,\cdots,X_k\}} = \sum_{j=1}^{k} \mu(X_j) f(x_j).$$

Then $\{v_{\{X_1,\cdots,X_k\}}\}_{\{X_1,\cdots,X_k\}\in I}$ is a Cauchy net in V, and its limit doesn't depend on the choice of the representatives $x_1, \cdots, x_k$.

Proof. Let a continuous seminorm $\mathfrak{p}$ on V and an $\epsilon > 0$ be given. For each $x \in X$ there is an open neighborhood $\mathcal{U}_x$ of x such that $\mathfrak{p}(f(x) - f(y)) \leq \epsilon$ for all $y \in \mathcal{U}_x$. Since X is compact, we can cover X with finitely many such $\mathcal{U}_x$, say $\mathcal{U}_{x_1}, \cdots, \mathcal{U}_{x_k}$. Let $X_1 = \mathcal{U}_{x_1}$ and $X_j = \mathcal{U}_{x_j} \setminus \cup_{s=1}^{j-1} X_s$ inductively for all $2 \leq j \leq k$. Then $\{X_1, \cdots, X_k\}$ is a finite partition of X. For any $\{X'_1, \cdots, X'_{k'}\} \geq \{X_1, \cdots, X_k\}$, clearly $\mathfrak{p}(v_{\{X'_1,\cdots,X'_{k'}\}} - v_{\{X_1,\cdots,X_k\}}) \leq 2\epsilon$ no matter how we choose the representatives for $\{X'_1, \cdots, X'_{k'}\}$ and $\{X_1, \cdots, X_k\}$. This gives the desired result. □

Definition 2.5. Let X be a compact space with a probability measure μ, and let f be a continuous function from X into a complete LCTVS V. The integration of f over X, denoted by $\int_X f\, d\mu$, is defined as the limit in Lemma 2.4.

The next proposition is obvious:

Proposition 2.6. *Let X be a compact space with a probability measure μ, and let f_1, f_2 be continuous functions from X into a complete* LCTVS *V. Then*

$$\begin{aligned} \int_X (f_1 + f_2)\, d\mu &= \int_X f_1\, d\mu + \int_X f_2\, d\mu, \\ \int_X \lambda f_1\, d\mu &= \lambda \int_X f_1\, d\mu \end{aligned}$$

for any scalar λ. If $\psi : V \to W$ is a continuous linear map from V into another complete LCTVS *W, then*

$$\int_X \psi \circ f_1\, d\mu = \psi(\int_X f_1\, d\mu).$$

It is also easy to verify the analogue of the fundamental theorem of calculus:

Proposition 2.7. *Let f be a continuous map from $[0,1]$ to a complete* LCTVS V. *Then*

$$f(0) = \lim_{t\to 0} \frac{\int_0^t f(s)\,ds}{t}.$$

2.2. **Clifford algebras.** Next we recall some facts about Clifford algebras [11, Chapter 1] [12, Section 1.8].

Let V be a real vector space of dimension m equipped with a positive-definite inner product. The corresponding *Clifford algebra*, denoted by $Cl(V)$, is the quotient of the tensor algebra $\oplus_{k\geq 0} V\otimes\cdots\otimes V$ generated by V by the two sided ideal generated by all elements of the form $v\otimes v + \parallel v \parallel^2$ for $v\in V$. The *complexified Clifford algebra*, denoted by $Cl^{\mathbb{C}}(V)$, is defined as $Cl^{\mathbb{C}}(V) := Cl(V)\otimes_{\mathbb{R}}\mathbb{C}$.

$Cl^{\mathbb{C}}(V)$ has a natural finite-dimensional C^*-algebra structure [11, Theorem 1.7.35]. Denote by $SO(V)$ the group of isometries of V preserving the orientation. For each $g\in SO(V)$ the isometry $g : V\to V$ induces an algebra isomorphism $Cl(V)\to Cl(V)$ and a C^*-algebra isomorphism $Cl^{\mathbb{C}}(V)\to Cl^{\mathbb{C}}(V)$. In this way $SO(V)$ acts on $Cl(V)$ and $Cl^{\mathbb{C}}(V)$.

Recall that a state φ on a C^*-algebra $\mathcal{A}$ is said to be *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b\in\mathcal{A}$.

Lemma 2.8. *When m is even, there is a unique tracial state tr on $Cl^{\mathbb{C}}(V)$. When m is odd, let $\gamma := i^{\frac{m+1}{2}} e_1\cdots e_m$ be the* chirality operator*, where $e_1,\cdots,e_m$ is an orthonormal basis of V. Then γ is fixed under the action of $SO(V)$ (equivalently, γ doesn't depend on the choice of the ordered orthonormal basis $e_1,\cdots,e_m$), and there is a unique tracial state tr on $Cl^{\mathbb{C}}(V)$ such that $tr(\gamma) = 0$. In both cases, tr is $SO(V)$-invariant.*

Proof. In both cases, the $SO(V)$-invariance of tr follows from the uniqueness. So we just need to show the uniqueness of tr.

When m is even, $Cl^{\mathbb{C}}(V)$ is isomorphic to the C^*-algebra of $2^{\frac{m}{2}}$ by $2^{\frac{m}{2}}$ matrices [11, Theorem 1.3.2]. The uniqueness of tr follows from the fact that for any $n\in\mathbb{N}$ the C^*-algebra of n by n matrices has a unique tracial state [13, Example 8.1.2].

Assume that m is odd now. Then $Cl^{\mathbb{C}}(V)$ is isomorphic to the direct sum of two copies of the C^*-algebra of $2^{\frac{m-1}{2}}$ by $2^{\frac{m-1}{2}}$ matrices [11, Theorem 1.3.2]. Say $Cl^{\mathbb{C}}(V) = \mathcal{A}_1\oplus\mathcal{A}_2$, where both $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic to the C^*-algebra of $2^{\frac{m-1}{2}}$ by $2^{\frac{m-1}{2}}$ matrices. Let p_j be the projection of $Cl^{\mathbb{C}}(V)$ to $\mathcal{A}_j$, and let φ_j be the unique tracial state of $\mathcal{A}_j$. Then the tracial states of $Cl^{\mathbb{C}}(V)$ are exactly $\lambda\varphi_1\circ p_1 + (1-\lambda)\varphi_2\circ p_2$ for $0\leq\lambda\leq 1$. It is easily verified that γ belongs to the center of $Cl^{\mathbb{C}}(V)$. So γ must be in $\mathbb{C}\cdot 1_{\mathcal{A}_1} + \mathbb{C}\cdot 1_{\mathcal{A}_2}$. It's also clear that $\gamma^2 = 1$ and $\gamma\notin\mathbb{C}$. So γ must be $\pm(1_{\mathcal{A}_1} - 1_{\mathcal{A}_2})$. It follows immediately that $Cl^{\mathbb{C}}(V)$ has a unique tracial state tr satisfying $tr(\gamma) = 0$, namely, $tr = \frac{1}{2}(\varphi_1\circ p_1 + \varphi_2\circ p_2)$. It is easy to check that γ is fixed under the action of $SO(V)$. $\square$

There is a natural injective map $V\hookrightarrow Cl(V)$. So one may think of V as a subspace of $Cl(V)$. The C^*-algebra norm on $Cl^{\mathbb{C}}(V)$ extends the norm on V induced from the inner product (see [11, Theorem 1.7.22(iv)] for the corresponding statement for the real C^*-algebra norm; the proofs are similar). Let M be an oriented Riemannian manifold of dimension m. Then we have the smooth algebra bundles ClM and $Cl^{\mathbb{C}}M$ over M with fibre algebras $Cl(TM_x)$ and $Cl^{\mathbb{C}}(TM_x)$ respectively, where TM_x is the tangent space at $X\in M$. These are called the *Clifford algebra* *bundle* and the *complexified Clifford algebra bundle*. Since $TM_x \subseteq Cl(TM_x)$, the complexified tangent bundle $TM^{\mathbb{C}}$ is a subbundle of $Cl^{\mathbb{C}}M$. Since $Cl^{\mathbb{C}}(TM_x)$ is unital, $C^\infty(M)$ is a subalgebra of $C^\infty(M, Cl^{\mathbb{C}}M)$.

2.3. **Compact quantum metric spaces.** Finally, we review Rieffel's theory of compact quantum metric spaces [22, 23, 24, 27]. Though Rieffel has set up his theory in the general framework of order-unit spaces, we shall need it only for C^*-algebras. See the discussion preceding Definition 2.1 in [24] for the reason of requiring the reality condition (3) below.

Definition 2.9. [24, Definition 2.1] By a *C^*-algebraic compact quantum metric space* we mean a pair $(\mathcal{A}, L)$ consisting of a unital C^*-algebra $\mathcal{A}$ and a (possibly $+\infty$-valued) seminorm L on $\mathcal{A}$ satisfying the *reality condition*

$$L(a) \quad = \quad L(a^*) \tag{3}$$

for all $a \in \mathcal{A}$, such that L vanishes exactly on $\mathbb{C}$ and the metric ρ_L on the state space $S(\mathcal{A})$ defined by (2) induces the w^*-topology. The *radius* of $(\mathcal{A}, L)$ is defined to be the radius of $(S(\mathcal{A}), \rho_L)$. We say that L is a *Lip-norm*.

Let $\mathcal{A}$ be a unital C^*-algebra and let L be a (possibly $+\infty$-valued) seminorm on $\mathcal{A}$ vanishing on $\mathbb{C}$. Then L and $\| \cdot \|$ induce (semi)norms $\tilde{L}$ and $\| \cdot \|^{\sim}$ respectively on the quotient space $\tilde{\mathcal{A}} = \mathcal{A}/\mathbb{C}$.

Notation 2.10. For any $r \geq 0$, let

$$\mathcal{D}_r(\mathcal{A}) := \{a \in \mathcal{A} : L(a) \leq 1, \| a \| \leq r\}.$$

The main criterion for when a seminorm L is a Lip-norm is the following:

Proposition 2.11. [22, Proposition 1.6, Theorem 1.9] *Let $\mathcal{A}$ be a unital C^*-algebra and let L be a (possibly $+\infty$-valued) seminorm on $\mathcal{A}$ satisfying the reality condition (3). Assume that L takes finite values on a dense subspace of $\mathcal{A}$, and that L vanishes exactly on $\mathbb{C}$. Then L is a Lip-norm if and only if*

(1) there is a constant $K \geq 0$ such that $\| \cdot \|^{\sim} \leq K\tilde{L}$ on $\tilde{\mathcal{A}}$;

and (2) for any $r \geq 0$, the ball $\mathcal{D}_r(\mathcal{A})$ is totally bounded in $\mathcal{A}$ for $\| \cdot \|$;

or (2') for some $r > 0$, the ball $\mathcal{D}_r(\mathcal{A})$ is totally bounded in $\mathcal{A}$ for $\| \cdot \|$.

In this event, $r_{\mathcal{A}}$ is exactly the minimal K such that $\| \cdot \|^{\sim} \leq K\tilde{L}$ on $(\tilde{\mathcal{A}})_{sa}$.

3. Connes and Dubois-Violette's formulation of θ-deformations

Though the Dirac operator does not depend on θ in Connes and Landi's formulation of θ-deformations in [8, Section 5], it does in Connes and Dubois-Violette's formulation in [7]. In this section we review the formulation of θ-deformations by Connes and Dubois-Violette [7, Sections 11 and 13], including the deformation of both the algebra and the Dirac operator.

Let M be a smooth manifold with a smooth action σ_M of $\mathbb{T}^n$. We denote by σ the induced action of $\mathbb{T}^n$ on the LC$*$A $C^\infty(M)$. Then σ is continuous. By Proposition 2.3 the tensor product completion $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ is a LC$*$A. The tensor product action $\sigma\hat{\otimes}\tau^{-1}$ of $\mathbb{T}^n$ on $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ is also continuous. The deformed smooth algebra [7, Section 11], denoted by $C^\infty(M_\theta)$, is then defined as the fixed-point space of this action, *i.e.* $C^\infty(M_\theta) = (C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty)^{\sigma\hat{\otimes}\tau^{-1}}$. Clearly, this is a LC$*$A.

Suppose M is equipped with a σ_M-invariant Riemannian metric. (For any Riemannian metric on M, we can always integrate it over $\mathbb{T}^n$ to make it σ_M-invariant.) Also assume that M is a spin manifold and that σ_M lifts to a smooth action σ_S of $\mathbb{T}^n$ on the spin bundle S, *i.e.* the following diagram

$$\begin{array}{ccc} S & \xrightarrow[\sigma_{S,x}]{} & S \\ \downarrow & & \downarrow \\ M & \xrightarrow[\sigma_{M,x}]{} & M \end{array}$$

is commutative for every $x \in \mathbb{T}^n$. (Usually σ_M doesn't lift directly to S, but lifts only modulo $\pm I$, *i.e.* there is a twofold covering $\mathbb{T}^n \to \mathbb{T}^n$ such that σ_M lifts to an action of the two-folding covering on S. Correspondingly, Connes and Dubois-Violette defined the various deformed structures using tensor product with $\mathcal{A}_{\frac{1}{2}\theta}$ instead of $\mathcal{A}_\theta$. But for the deformed algebras and Dirac operators, the difference is just a matter of parameterization.) We denote the induced continuous action of $\mathbb{T}^n$ on $C^\infty(M,S)$ also by σ. Then $C^\infty(M,S)$ is a locally convex left $C^\infty(M)$-module and

$$\sigma_x(f\psi) = \sigma_x(f)\sigma_x(\psi)$$

for all $f \in C^\infty(M), \psi \in C^\infty(M,S)$ and $x \in \mathbb{T}^n$. We also have the tensor product completion $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$, which is a locally convex left module over $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ by Proposition 2.3. The tensor product action $\sigma\hat{\otimes}\tau^{-1}$ of $\mathbb{T}^n$ on $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ is still continuous. The deformed spin bundle, denoted by $C^\infty(M_\theta,S)$, is then defined as the fixed-point space of this action, *i.e.* $C^\infty(M_\theta,S) = (C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty)^{\sigma\hat{\otimes}\tau^{-1}}$. This is a locally convex left $C^\infty(M_\theta)$-module. Let D be the Dirac operator on $C^\infty(M,S)$. This is a first-order linear differential operator. So it is easy to see that D is continuous with respect to the locally convex topology on $C^\infty(M,S)$. Then we have the tensor product linear map $D\hat{\otimes}I$ from $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ to itself. Notice that D commutes with the action σ, so $D\hat{\otimes}I$ commutes with the action $\sigma\hat{\otimes}\tau^{-1}$. Therefore $C^\infty(M_\theta,S)$ is stable under $D\hat{\otimes}I$. Denote by D_θ the restriction of $D\hat{\otimes}I$ to $C^\infty(M_\theta,S)$.

Assume further that M is compact. As usual, one defines a positive-definite scalar product on $C^\infty(M,S)$ by

$$<\psi,\psi'>=\int_M (\psi,\psi')\, vol,$$

where *vol* is the Riemannian volume form. Denote by $\mathcal{H} = L^2(M,S)$ the Hilbert space obtained by completion. Then $C(M)$ has a natural faithful representation on $\mathcal{H}$ by multiplication, and we shall think of $C(M)$ as a subalgebra of $B(\mathcal{H})$, the C^*-algebra of all bounded operators on $\mathcal{H}$. The action σ uniquely extends to a continuous unitary representation of $\mathbb{T}^n$ in $\mathcal{H}$, which will be still denoted by σ. On the other hand, $\mathcal{A}_\theta$ has an inner product induced by the unique τ-invariant tracial state. Denote by $L^2(\mathcal{A}_\theta)$ the Hilbert space obtained by completion. Then $\mathcal{A}_\theta$ acts on $L^2(\mathcal{A}_\theta)$ faithfully by the GNS construction, and we shall also think of $\mathcal{A}_\theta$ as a subalgebra of $B(L^2(\mathcal{A}_\theta))$. The action τ also extends to a continuous unitary representation of $\mathbb{T}^n$ in $L^2(\mathcal{A}_\theta)$. Let $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ be the Hilbert space tensor product. Then we have the continuous tensor product action $\sigma\bar{\otimes}\tau^{-1}$ on $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$. The deformed Hilbert space, denoted by $\mathcal{H}_\theta$, is defined as the fixed-point space of $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ under the action $\sigma\bar{\otimes}\tau^{-1}$. Clearly the maps $C^\infty(M,S) \to \mathcal{H}$ and $\mathcal{A}_\theta^\infty \to L^2(\mathcal{A}_\theta)$ are continuous with respect to the locally convex topologies on $C^\infty(M,S)$, $\mathcal{A}_\theta^\infty$ and the norm topologies on $\mathcal{H}$, $L^2(\mathcal{A}_\theta)$. Then we have the sequence of continuous linear maps

$$C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty \xrightarrow{\phi} \mathcal{H}\hat{\otimes}_\pi L^2(\mathcal{A}_\theta) \xrightarrow{\psi} \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta),$$

where $\mathcal{H}\hat{\otimes}_\pi L^2(\mathcal{A}_\theta)$ is the completion of the projective tensor product of $\mathcal{H}$ and $L^2(\mathcal{A}_\theta)$. Let $\Phi : C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty \to \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ be the composition. Then Φ is $\mathbb{T}^n$-equivariant. So Φ maps $C^\infty(M_\theta,S)$ into $\mathcal{H}_\theta$. Let Φ_θ be the restriction of Φ to $C^\infty(M_\theta,S)$.

Lemma 3.1. *Both maps $\phi : C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty \to \mathcal{H}\hat{\otimes}_\pi L^2(\mathcal{A}_\theta)$ and ψ : $\mathcal{H}\hat{\otimes}_\pi L^2(\mathcal{A}_\theta) \to \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ are injective. Consequently, Φ and Φ_θ are injective.*

Proof. We'll prove the injectivity of ϕ. The proof for ψ is similar. Recall the notation at the end of Section 1. We shall need the following well-known fact several times. We omit the proof.

Lemma 3.2. *Let G be a compact group. Let α be a continuous action of G on a complex complete* LCTVS *V. For a continuous $\mathbb{C}$-valued function φ on G let*

$$\alpha_\varphi(v) = \int_G \varphi(x)\alpha_x(v)\,dx$$

for $v \in V$. Then $\alpha_\varphi : V \to V$ is a continuous linear map. If $\mathcal{J}$ is a finite subset of $\hat{G}$ and if φ is a linear combination of the characters of $\gamma \in \bar{\mathcal{J}}$, then $\alpha_\varphi(V) \subseteq V_{\mathcal{J}}$. Let

$$\alpha_{\mathcal{J}} = \alpha_{\sum_{\gamma\in\mathcal{J}} \dim(\gamma)\overline{\chi_\gamma}}.$$

(When $\mathcal{J}$ is a one-element set $\{\gamma\}$, we'll simply write α_γ for $\alpha_{\{\gamma\}}$.) Then $\alpha_{\mathcal{J}}(v) = v$ for all $v \in V_{\mathcal{J}}$, and $\alpha_{\mathcal{J}}(v) = 0$ for all $v \in V_\gamma$ with $\gamma \in \hat{G} \setminus \mathcal{J}$.

From Proposition 2.6 we also have:

Lemma 3.3. *Let G be a compact group with continuous actions α and β on complex complete* LCTVS *V and W. Let $\phi : V \to W$ be a continuous G-equivariant linear map, and let $\varphi : G \to \mathbb{C}$ be a continuous function. Then*

$$\phi \circ \alpha_\varphi = \beta_\varphi \circ \phi. \tag{4}$$

In particular, let $\mathcal{J}$ be a finite subset of $\hat{G}$. Then

$$\phi \circ \alpha_{\mathcal{J}} = \beta_{\mathcal{J}} \circ \phi.$$

We shall need the following lemma a few times:

Lemma 3.4. *Let G be a compact group, and let h be a continuous $\mathbb{C}$-valued function on G with $h(e_G) = 0$. Then for any $\epsilon > 0$ there is a nonnegative function φ on G such that φ is a linear combination of finitely many characters, $\| \varphi \|_1 = 1$, and $\| \varphi \cdot h \|_1 < \epsilon$.*

Proof. Notice that the left regular representation of G on $L^2(G)$ is faithful. Since the left regular representation is a Hilbert space direct sum of irreducible representations, we see that any $x \neq e_G$ acts nontrivially in some $\gamma \in \hat{G}$. Let $\mathcal{U}$ be an open neighborhood of e_G such that $|h(x)| < \epsilon/2$ for all $x \in \mathcal{U}$. For any $x \in G\backslash\mathcal{U}$, suppose that x acts nontrivially in $\gamma_x \in \hat{G}$. Then there is some open neighborhood $\mathcal{U}_x$ of x

such that x' acts nontrivially in γ_x for all $x' \in \mathcal{U}_X$. Since $G \setminus \mathcal{U}$ is compact, we can find $x_1, \cdots, x_m \in G \setminus \mathcal{U}$ so that $U_{x_1}, \cdots, U_{x_m}$ cover $G \setminus \mathcal{U}$. Let $\mathcal{J}_\mathcal{U} = \{\gamma_{x_1}, \cdots, \gamma_{x_m}\}$. Then no element in $G \setminus \mathcal{U}$ acts trivially in all $\gamma \in \mathcal{J}_\mathcal{U}$. Let π_1 be the direct sum of one copy for each γ in $\mathcal{J}_\mathcal{U} \cup \{\gamma_0\}$, and let χ_{π_1} be the character of π_1.

Let $\pi = \pi_1 \otimes \overline{\pi_1}$. Also let χ be the character of π. Note that $\chi(x) = |\chi_{\pi_1}(x)|^2 \geq 0$ for all $x \in G$. Let $\varphi_n = \chi^n / \parallel \chi^n \parallel_1$. Then each φ_n is a linear combination of finitely many characters. Since every element in $G \setminus \mathcal{U}$ acts nontrivially in π, $\chi(x) < \chi(e_G)$ on $G \setminus \mathcal{U}$. Therefore it's easy to see (cf. the proof of Theorem 8.2 in [24]) that $\int_{G\setminus\mathcal{U}} \varphi_n(x)\, dx \to 0$ as $n \to \infty$, and hence

$$\limsup_{n\to+\infty} \int_G |\varphi_n(x)h(x)|\, dx \leq \sup_{x\in\mathcal{U}} |h(x)| < \epsilon.$$

So when n is big enough, we have that $\parallel \varphi_n \cdot h \parallel_1 < \epsilon$. □

As a corollary of Lemma 3.4 we have:

Lemma 3.5. *Let G be a compact group. Let α be a continuous action of G on a complex complete* LCTVS *V. Let $v \in V$. If $\alpha_\gamma(v) = 0$ for all $\gamma \in \hat{G}$, then $v = 0$.*

Proof. Let $\mathfrak{p}$ be a continuous seminorm on V, and let $\epsilon > 0$. Define a function h on G by $h(x) = \mathfrak{p}(v - \alpha_x(v))$. Then h is continuous on G, and $h(e_G) = 0$. Pick φ for h and ϵ in Lemma 3.4. According to the assumption we have $\alpha_\varphi(v) = 0$. Then

$$\mathfrak{p}(v) = \mathfrak{p}(v - \alpha_\varphi(v)) = \mathfrak{p}(\int_G \varphi(x)(v - \alpha_x(v))\, dx) \leq \int_G \varphi(x)h(x)\, dx < \epsilon.$$

Since the topology on V is defined by all the continuous seminorms, we see that $v = 0$. □

We are ready to prove Lemma 3.1. Let $\alpha = I\hat{\otimes}\tau$ acting on $V = C^\infty(M, S)\hat{\otimes}\mathcal{A}_\theta^\infty$, and let $\beta = I\hat{\otimes}\tau$ acting on $\mathcal{H}\hat{\otimes}_\pi L^2(\mathcal{A}_\theta)$. Let ϕ be as in Lemma 3.1. Then $\phi \circ \alpha = \beta \circ \phi$. Recall the notation about $\mathcal{A}_\theta$ in subsection 2.1. For any $q \in \mathbb{Z}^n = \widehat{\mathbb{T}^n}$ clearly α_q maps $C^\infty(M) \otimes_{alg} \mathcal{A}_\theta^\infty$ onto $C^\infty(M) \otimes u_q$. Since α_q is continuous, by Lemma 3.2 it follows immediately that $V_q = \alpha_q(V) = C^\infty(M, S) \otimes u_q$. Let $f \in ker(\phi)$. For any $q \in \mathbb{Z}^n$ by Lemma 3.3 $\phi(\alpha_q(f)) = \beta_q(\phi(f)) = 0$. Now $\alpha_q(f) \in C^\infty(M, S) \otimes u_q$, and clearly ϕ restricted to $C^\infty(M, S) \otimes u_q$ is injective. So $\alpha_q(f) = 0$. From Lemma 3.5 we see that $f = 0$. □

Lemma 3.6. *The image $\Phi_\theta(C^\infty(M_\theta, S))$ is dense in $\mathcal{H}_\theta$.*

Clearly $\Phi(C^\infty(M, S)\hat{\otimes}\mathcal{A}_\theta^\infty)$ is dense in $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$, so this is an immediate consequence of the following:

Lemma 3.7. *Let G be a compact group. Let α and β be continuous actions of G on complex complete* LCTVS *V and W respectively. Let $\phi : V \to W$ be a continuous G-equivariant linear map such that $\phi(V)$ is dense in W. Then $\phi(V^\alpha)$ is dense in W^β.*

Proof. Recall that γ_0 is the trivial representation of G. By Lemma 3.2 β_{γ_0} is continuous. So $\beta_{\gamma_0}(\phi(V))$ is dense in $\beta_{\gamma_0}(W) = W^\beta$. But $\beta_{\gamma_0}(\phi(V)) = \phi(\alpha_{\gamma_0}(V)) = \phi(V^\alpha)$ according to Lemma 3.3. The conclusion follows. □

The Dirac operator D is essentially self-adjoint on $\mathcal{H}$ [15, Theorem 5.7]. Then $D \otimes I$ is also essentially self-adjoint on $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ [13, Proposition 11.2.37]. Denote its closure by D^{L^2}.

Lemma 3.8. *$\Phi(C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty)$ is contained in the domain of D^{L^2}, and*

$$D^{L^2}\circ\Phi=\Phi\circ(D\hat{\otimes}I). \tag{5}$$

Proof. For any $y\in C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$, take a net y_j in $C^\infty(M,S)\otimes_{alg}\mathcal{A}_\theta^\infty$ converging to y. Then $\Phi(y_j)\to\Phi(y)$, $(D\hat{\otimes}I)(y_j)\to(D\hat{\otimes}I)(y)$ and $D^{L^2}(\Phi(y_j))=\Phi((D\hat{\otimes}I)(y_j))\to\Phi((D\hat{\otimes}I)(y))$. So $\Phi(y)$ is contained in the domain of D^{L^2}, and $D^{L^2}(\Phi(y))=\Phi((D\hat{\otimes}I)(y))$. □

So the intersection of $\mathcal{H}_\theta$ and the domain of D^{L^2} contains $\Phi_\theta(C^\infty(M_\theta,S))$, which is dense in $\mathcal{H}_\theta$ by Lemma 3.6. Clearly $D\otimes I$ commutes with the action $\sigma\bar{\otimes}\tau^{-1}$, and thus so does D^{L^2}. Hence D^{L^2} maps the intersection of $\mathcal{H}_\theta$ and the domain of D^{L^2} into $\mathcal{H}_\theta$. Therefore the restriction of D^{L^2} to $\mathcal{H}_\theta$ is also self-adjoint. The deformed Dirac operator, denoted by $D_\theta^{L^2}$, is then defined to be this restriction.

Similarly, the maps $C^\infty(M)\to C(M)$ and $\mathcal{A}_\theta^\infty\to\mathcal{A}_\theta$ are continuous with respect to the locally convex topologies on $C^\infty(M)$, $\mathcal{A}_\theta^\infty$ and the norm topologies on $C(M)$, $\mathcal{A}_\theta$. So we have the $\mathbb{T}^n$-equivariant continuous linear map $\Psi: C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty\to C(M)\otimes\mathcal{A}_\theta$, where $C(M)\otimes\mathcal{A}_\theta$ is the spatial C^*-algebraic tensor product of $C(M)$ and $\mathcal{A}_\theta$ [31, Appendix T.5].

Definition 3.9. We define the deformed continuous algebra, $C(M_\theta)$, to be the fixed-point algebra $(C(M)\otimes\mathcal{A}_\theta)^{\sigma\otimes\tau^{-1}}$.

Then Ψ maps $C^\infty(M_\theta)$ into $C(M_\theta)$. By similar arguments as in Lemma 3.1 and 3.7 we have

Lemma 3.10. *The map Ψ is injective, and $\Psi(C^\infty(M_\theta))$ is dense in $C(M_\theta)$.*

Clearly $\mathcal{H}_\theta$ is stable under the action of elements in $C(M_\theta)$. So we can define $\Psi_\theta: C^\infty(M_\theta)\to B(\mathcal{H}_\theta)$ as the composition of $C^\infty(M_\theta)\to C(M_\theta)$ and the restriction map of $C(M_\theta)$ to $B(\mathcal{H}_\theta)$. We shall see later in Proposition 5.6 that the restriction map of $C(M_\theta)$ to $B(\mathcal{H}_\theta)$ is isometric. So we may also think of $C(M_\theta)$ as a subalgebra of $B(\mathcal{H}_\theta)$. Then the closure of $\Psi_\theta(C^\infty(M_\theta))$ is just $C(M_\theta)$.

We shall see later in Proposition 5.2 that the domain of $D_\theta^{L^2}$ is stable under $\Psi_\theta(f)$, and that the commutator $[D_\theta^{L^2},\Psi_\theta(f)]$ is bounded for every $f\in C^\infty(M_\theta)$.

Definition 3.11. We define the *deformed Lip-norm*, denoted by L_θ, on $C(M_\theta)$ by

$$L_\theta(f):=\begin{cases}\|\,[D_\theta^{L^2},f]\,\|, & \text{if } f\in\Psi_\theta(C^\infty(M_\theta));\\ +\infty, & \text{otherwise .}\end{cases}$$

4. Lip-norms and Compact Group Actions

In this section we consider a general situation in which there are a seminorm and a compact group action. We show that under certain compatibility hypotheses we can use this group action to prove that the seminorm is a Lip-norm. The strategy is a generalization of the one Rieffel used to deal with Lip-norms associated to ergodic compact (Lie) group actions [22, 24]. We'll see that θ-deformations fit into this general picture.

Throughout this section we assume that G is an arbitrary compact group which has a fixed length function l, *i.e.* a continuous real-valued function, l, on G such that

$$\begin{aligned} l(xy) &\leq l(x)+l(y) \text{ for all } x,y \in G \\ l(x^{-1}) &= l(x) \text{ for all } x \in G \\ l(x) &= 0 \text{ if and only if } x = e_G, \end{aligned}$$

where e_G is the identity of G.

Theorem 4.1. *Let $\mathcal{A}$ be a unital C^*-algebra, let L be a (possibly $+\infty$-valued) seminorm on $\mathcal{A}$ satisfying the reality condition (3), and let α be a strongly continuous action of G on $\mathcal{A}$. Assume that L takes finite values on a dense subspace of $\mathcal{A}$, and that L vanishes on $\mathbb{C}$. Let L^l be the (possibly $+\infty$-valued) seminorm on $\mathcal{A}$ defined by*

$$L^l(a) = \sup\{\frac{\| \alpha_x(a) - a \|}{l(x)} : x \in G, x \neq e_G\}. \tag{6}$$

Suppose that the following conditions are satisfied:

(1) there is some constant $C > 0$ such that $L^l \leq C \cdot L$ on $\mathcal{A}$;

(2) for any linear combination φ of finitely many characters on G we have $L \circ \alpha_\varphi \leq \| \varphi \|_1 \cdot L$ on $\mathcal{A}$, where α_φ is the linear map on $\mathcal{A}$ defined in Lemma 3.2;

(3) for each $\gamma \in \hat{G}$ with $\gamma \neq \gamma_0$ the ball $\mathcal{D}_r(\mathcal{A}_\gamma) := \{a \in \mathcal{A}_\gamma : L(a) \leq 1, \| a \| \leq r\}$ is totally bounded for some $r > 0$, and the only element in A_γ vanishing under L is 0;

(4) there is a unital C^*-algebra $\mathcal{B}$ containing $\mathcal{A}_{\gamma_0} = \mathcal{A}^\alpha$, with a Lip-norm $L_\mathcal{B}$, such that $L_\mathcal{B}$ extends the restriction of L to $\mathcal{A}_{\gamma_0}$.

Then $(\mathcal{A}, L)$ is a C^*-algebraic compact quantum metric space with $r_\mathcal{A} \leq r_\mathcal{B} + C\int_G l(x)\,dx$.

Remark 4.2. (1) We assume the existence of $(\mathcal{B}, L_\mathcal{B})$ in the condition (4) only for the convenience of application. In fact, conditions (2) and (4) imply that L restricted to $\mathcal{A}_{\gamma_0}$ is a Lip-norm on $\mathcal{A}_{\gamma_0}$: for any $a \in \mathcal{A}_{\gamma_0}$ and $\epsilon > 0$ pick $a' \in \mathcal{A}$ with $L(a') < \infty$ and $\| a - a' \| < \epsilon$. Then by Lemma 3.2 $\alpha_{\gamma_0}(a') \in \mathcal{A}_{\gamma_0}$ and $\| a - \alpha_{\gamma_0}(a') \| = \| \alpha_{\gamma_0}(a - a') \| < \epsilon$. By the condition (2) $L(\alpha_{\gamma_0}(a')) < \infty$. Therefore L takes finite values on a dense subspace of $\mathcal{A}_{\gamma_0}$. Then from Proposition 2.11 it is easy to see that L restricted to $\mathcal{A}_{\gamma_0}$ is a Lip-norm on $\mathcal{A}_{\gamma_0}$. Consequently, we may take $\mathcal{B}$ to be $\mathcal{A}_{\gamma_0}$ itself.

(2) Conditions (1) and (2) in Theorem 4.1 enable us to reduce the study of L to that of the restriction of L to each $\mathcal{A}_\gamma$. Conditions (3) and (4) say roughly that L restricted to each $\mathcal{A}_\gamma$ is a Lip-norm.

(3) Usually it is not hard to verify the condition (2). In particular, by Lemma 4.3 it holds when L is α-invariant and lower semicontinuous on $\{a \in \mathcal{A} : L(a) < +\infty\}$, and $\{a \in \mathcal{A} : L(a) < +\infty\}$ is stable under α_γ for every $\gamma \in \hat{G}$.

Lemma 4.3. *Let α be a strongly continuous action of G on a C^*-algebra $\mathcal{A}$, and let L be a (possibly $+\infty$-valued) seminorm on $\mathcal{A}$. Suppose that L is α-invariant and lower semicontinuous on $\{a \in \mathcal{A} : L(a) < +\infty\}$. For any continuous function $\varphi : G \to \mathbb{C}$, if $\{a \in \mathcal{A} : L(a) < +\infty\}$ is stable under the map $\alpha_\varphi : \mathcal{A} \to \mathcal{A}$ defined in Lemma 3.2, then*

$$L \circ \alpha_\varphi \leq \| \varphi \|_1 \cdot L$$

on $\mathcal{A}$.

Proof. We only need to show $L(\alpha_\varphi(a)) \le\| \varphi \|_1 \cdot L(a)$ for each $a \in \mathcal{A}$ with $L(a) < +\infty$. But

$$\alpha_\varphi(a) = \lim_{\Delta\to 0} \sum_{j=1}^{k} \alpha_{g_j}(a)\mu(E_j)\varphi(g_j),$$

where μ is the normalized Haar measure on G, $(E_1, \cdots, E_k)$ is a partition of G, $g_j \in E_j$, $\Delta(E_j) := \sup\{\max(|\varphi(x) - \varphi(y)|, |\alpha_x(a) - \alpha_y(a)|) : x, y \in E_j\}$ and $\Delta = \max_{1\le j\le k} \Delta(E_j)$. By the assumptions we have

$$\begin{aligned} L(\alpha_\varphi(a)) &\le \liminf_{\Delta\to 0} L(\sum_{j=1}^{k} \alpha_{g_j}(a)\mu(E_j)\varphi(g_j)) \\ &\le L(a) \liminf_{\Delta\to 0} \sum_{j=1}^{k} \mu(E_j)|\varphi(g_j)| = L(a) \| \varphi \|_1 . \end{aligned}$$

□

For θ-deformations of course $\mathcal{A}$ is $C(M_\theta)$. Notice that $\mathbb{T}^n$ has a natural action $I \otimes \tau$ on $C(M_\theta)$. They will be our G and α.

The following lemma is a generalization of Lemmas 8.3 and 8.4 in [24].

Lemma 4.4. *For any $\epsilon > 0$ there is a finite subset $\mathcal{J} = \bar{\mathcal{J}}$ in $\hat{G}$, containing γ_0, depending only on l and ϵ/C, such that for any strongly continuous isometric action α on a complex Banach space V with a (possibly $+\infty$-valued) seminorm L on V satisfying conditions (1) and (2) (with $\mathcal{A}$ replaced by V) in Theorem 4.1, and for any $v \in V$, there is some $v' \in V_{\mathcal{J}}$ with*

$$\| v' \|\le\| v \|, \quad L(v') \le L(v), \quad \text{and } \| v - v' \|\le \epsilon L(v).$$

If V has an isometric involution $*$ invariant under α, then when v is self-adjoint we can choose v' also to be self-adjoint.

Proof. Pick φ for l and ϵ/C as in Lemma 3.4. Then there is a finite subset $\mathcal{J} \subseteq \hat{G}$ such that φ is a linear combination of characters χ_γ for $\gamma \in \mathcal{J}$. Replacing $\mathcal{J}$ by $\mathcal{J} \cup \bar{\mathcal{J}}$, we may assume that $\mathcal{J} = \bar{\mathcal{J}}$. For any $v \in V$ clearly

$$\| \alpha_\varphi(v) \|\le\| \varphi \|_1 \cdot \| v \|=\| v \| .$$

A simple calculation as in the proof of [24, Lemma 8.3] tells us that

$$\| v - \alpha_\varphi(v) \| \quad \le \quad L^l(v) \int_G \varphi(x)l(x)\, dx \le \frac{\epsilon}{C} L^l(v).$$

Then it follows from the condition (1) in Theorem 4.1 that $\| v - \alpha_\varphi(v) \|\le \epsilon L(v)$. Also from the condition (2) we see that $L(\alpha_\varphi(v)) \le L(v)$. So for any $v \in \mathcal{A}$, the element $v' = \alpha_\varphi(v)$ satisfies the requirement.

Notice that φ is real-valued, so when v is self-adjoint, so is $\alpha_\varphi(v)$. □

Proof of Theorem 4.1. We verify the conditions in Proposition 2.11 for $(\mathcal{A}, L)$ to be a compact quantum metric space one by one.

Lemma 4.5. *For any $a \in \mathcal{A}$ if $L(a) = 0$ then a is a scalar.*

Proof. For any $\gamma \in \mathcal{J}$ by the condition (2) we have

$$L(\alpha_\gamma(a)) \leq \parallel \dim(\gamma)\overline{\chi_\gamma} \parallel_1 \cdot L(a) = 0.$$

By conditions (3) and (4) we see that $\alpha_\gamma(a) = 0$ for $\gamma \neq \gamma_0$ and that $\alpha_{\gamma_0}(a) \in \mathbb{C}$. Hence $\alpha_\gamma(a - \alpha_{\gamma_0}(a)) = 0$ for all $\gamma \in \hat{G}$. Then Lemma 3.5 tells us that $a = \alpha_{\gamma_0}(a) \in \mathbb{C}$. □

Lemma 4.6. *For any $R \geq 0$ the ball*

$$\mathcal{D}_R(\mathcal{A}) = \{a \in \mathcal{A} : L(a) \leq 1, \parallel a \parallel \leq R\}$$

is totally bounded.

Proof. For any $\epsilon > 0$ by Lemma 4.4 there is some finite subset $\mathcal{J} \subseteq \hat{G}$ such that for every $v \in \mathcal{D}_R(\mathcal{A})$ there exists $v' \in \mathcal{D}_R(\mathcal{A}_\mathcal{J})$ with $\parallel v - v' \parallel < \epsilon$. Let $M = \max\{\parallel \dim(\gamma)\overline{\chi_\gamma} \parallel_1 : \gamma \in \mathcal{J}\}$. For any $a = \sum_{\gamma \in \mathcal{J}} a_\gamma \in \mathcal{D}_R(\mathcal{A}_\mathcal{J})$ and $\gamma \in \mathcal{J}$ we have

$$\parallel a_\gamma \parallel = \parallel \alpha_{\dim(\gamma)\overline{\chi_\gamma}}(a) \parallel \leq \parallel \dim(\gamma)\overline{\chi_\gamma} \parallel_1 \cdot \parallel a \parallel \leq M \cdot R,$$

and by the condition (2)

$$L(a_\gamma) = L(\alpha_{\dim(\gamma)\overline{\chi_\gamma}}(a)) \leq \parallel \dim(\gamma)\overline{\chi_\gamma} \parallel_1 \cdot L(a) \leq M.$$

Therefore

$$\mathcal{D}_R(\mathcal{A}_\mathcal{J}) \subseteq \{\sum_{\gamma \in \mathcal{J}} a_\gamma \in \mathcal{A}_\mathcal{J} : a_\gamma \in \mathcal{A}_\gamma,\ L(a_\gamma) \leq M,\ \parallel a_\gamma \parallel \leq M \cdot R\}.$$

By the conditions (3), (4) and Proposition 2.11 the latter set is totally bounded. Then $\mathcal{D}_R(\mathcal{A}_\mathcal{J})$ is totally bounded. Since ϵ is arbitrary, $\mathcal{D}_R(\mathcal{A})$ is also totally bounded. □

Lemma 4.7. *We have*

$$\parallel \cdot \parallel^\sim \leq \Big(r_\mathcal{B} + C \int_G l(x)\,dx\Big) L^\sim$$

on $\mathcal{A}_{sa}/\mathbb{R}e$.

Proof. Let $a \in \mathcal{A}_{sa}$ with $L(a) = 1$. Let φ be the constant function $\chi_{\gamma_0} = 1$ on G. Then $\alpha_\varphi = \alpha_{\gamma_0}$ and $\parallel \varphi \parallel_1 = 1$. As in the proof of Lemma 4.4 we have $\alpha_\varphi(a) \in (\mathcal{A}^\alpha)_{sa}$ and

$$\parallel a - \alpha_\varphi(a) \parallel \leq L^l(a) \int_G \varphi(x) l(x)\,dx \leq C \cdot L(a) \int_G l(x)\,dx = C \int_G l(x)\,dx,$$

where the second inequality comes from the condition (1). Let $b = \alpha_\varphi(a)$. By the condition (2) we have

$$L(b) \leq \parallel \varphi \parallel_1 \cdot L(a) = 1.$$

Then by Proposition 2.11

$$r_\mathcal{B} \geq \parallel \tilde{b} \parallel^\sim \geq \parallel \tilde{a} \parallel^\sim - \parallel \tilde{a} - \tilde{b} \parallel^\sim \geq \parallel \tilde{a} \parallel^\sim - \parallel a - \alpha_\varphi(a) \parallel \geq \parallel \tilde{a} \parallel^\sim - C \int_G l(x)\,dx.$$

Therefore we have $\parallel \cdot \parallel^\sim \leq (r_\mathcal{B} + C \int_G l(x)\,dx) L^\sim$. □

Now Theorem 4.1 follows from Lemmas 4.5-4.7 and Proposition 2.11 immediately. □

5. Differential Operators and Seminorms

In this section we make preparation for our proof of Theorem 1.1. In Section 6 we shall verify the conditions in Theorem 4.1 for $(C(M_\theta), L_\theta, \mathbb{T}^n, I\otimes\tau)$. The seminorm L^l_θ on $C(M_\theta)$ associated to $I\otimes\tau$ is defined in Definition 5.4. The main difficulty is to verify the condition (1). We shall see that it is much more convenient to work on the whole Hilbert space $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$ instead of $\mathcal{H}_\theta$. So we have to study the corresponding seminorms L^D and L^l on $C(M)\otimes\mathcal{A}_\theta$ (see Definitions 5.3 and 5.4). We prove the comparison formula for L^D and L^l first, in (20). Then we relate them to L_θ and L^l_θ by proving (22). The information about these various seminorms is all hidden in differential operators, which involve mainly the theory of LCTVS. Subsections 5.1 and 5.2 are devoted to analyzing these operators.

5.1. **Differential Operators.** In this subsection we assume that M is an oriented Riemannian manifold with an isometric smooth action σ_M of $\mathbb{T}^n$. Our aim is to derive the formulas (8), (11) and (12) below.

Let $Cl^{\mathbb{C}}M$ be the complexified Clifford algebra bundle on M. Then its space of smooth sections, $C^\infty(M, Cl^{\mathbb{C}}M)$, is a LCA containing $C^\infty(M)$ as a central subalgebra, and containing $C^\infty(M, TM^{\mathbb{C}})$ as a subspace, where $TM^{\mathbb{C}}$ is the complexified tangent bundle. Using the Riemannian metric, we can identify TM and T^*M canonically. Then $C^\infty(M, T^*M^{\mathbb{C}}) = C^\infty(M, TM^{\mathbb{C}})$ is also a subspace of $C^\infty(M, Cl^{\mathbb{C}}M)$. Notice that $C^\infty(M, S)$ is a locally convex left module over $C^\infty(M, Cl^{\mathbb{C}}M)$. Since $\mathcal{A}^\infty_\theta$ is nuclear, the complete tensor products $C^\infty(M)\hat{\otimes}\mathcal{A}^\infty_\theta$, $C^\infty(M, TM^{\mathbb{C}})\hat{\otimes}\mathcal{A}^\infty_\theta$ and $C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}^\infty_\theta$ can be thought of as complete injective tensor products, and hence are are all subspaces of $C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}^\infty_\theta$ (see the discussion after Proposition 2.3).

In the same way we think of $C(M, T^*M^{\mathbb{C}}) = C(M, TM^{\mathbb{C}})$ as a subspace of $C(M, Cl^{\mathbb{C}}M)$. Since the C^*-algebraic norm on $Cl^{\mathbb{C}}(TM_p)$ extends the inner-product norm on the tangent space TM_p for each $p\in M$ (see the discussion after Lemma 2.8), clearly the supremum (possibly $+\infty$-valued) norm on $C(M, Cl^{\mathbb{C}}M)$ extends that on $C(M, TM)$, which is pointwise the inner-product norm.

Clearly the action of $\mathbb{T}^n$ on the bundle TM extends to an action on the bundle $Cl^{\mathbb{C}}M$. We denote the induced continuous action on $C^\infty(M, Cl^{\mathbb{C}}M)$ also by σ. Much as in Section 3, we can define

$$\begin{aligned} C^\infty(M_\theta, Cl^{\mathbb{C}}M) &:= (C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}^\infty_\theta)^{\sigma\hat{\otimes}\tau^{-1}}, \\ C^\infty(M_\theta, TM^{\mathbb{C}}) &:= (C^\infty(M, TM^{\mathbb{C}})\hat{\otimes}\mathcal{A}^\infty_\theta)^{\sigma\hat{\otimes}\tau^{-1}}, \\ C^\infty(M_\theta, T^*M^{\mathbb{C}}) &:= (C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}^\infty_\theta)^{\sigma\hat{\otimes}\tau^{-1}}. \end{aligned}$$

The differential operator $d : C^\infty(M)\to C^\infty(M, T^*M^{\mathbb{C}})$ is a first-order linear operator, and hence easily seen to be continuous. Then we have the tensor product linear map $d\hat{\otimes}I : C^\infty(M)\hat{\otimes}\mathcal{A}^\infty_\theta \to C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}^\infty_\theta$. Notice that d commutes with the action σ. So $d\hat{\otimes}I$ commutes with $\sigma\hat{\otimes}\tau^{-1}$, and hence maps $C^\infty(M_\theta)$ into $C^\infty(M_\theta, T^*M^{\mathbb{C}})$. The deformed differential d_θ is then defined to be the restriction of $d\hat{\otimes}I$ to $C^\infty(M_\theta)$.

For any $f\in C^\infty(M)$ we have

$$[D, f] = df \text{ as linear maps on } C^\infty(M, S), \tag{7}$$

where $df \in C^\infty(M, T^*M^{\mathbb{C}}) \subseteq C^\infty(M, Cl^{\mathbb{C}}M)$ acts on $C^\infty(M, S)$ via the left $C^\infty(M, Cl^{\mathbb{C}}M)$-module structure of $C^\infty(M, S)$. Then it is easy to see that for any $f \in C^\infty(M) \otimes_{alg} \mathcal{A}_\theta^\infty$ we have

$$[D \otimes I, f] = (d \otimes I)f \text{ as linear maps on } C^\infty(M, S) \otimes_{alg} \mathcal{A}_\theta^\infty.$$

This means that the bilinear maps $(f, \psi) \mapsto [D\hat{\otimes}I, f](\psi)$ and $(f, \psi) \mapsto ((d\hat{\otimes}I)f)(\psi)$ from $W := (C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty) \times (C^\infty(M, S)\hat{\otimes}\mathcal{A}_\theta^\infty)$ to $C^\infty(M, S)\hat{\otimes}\mathcal{A}_\theta^\infty$ coincide on the dense subspace $(C^\infty(M) \otimes_{alg} \mathcal{A}_\theta^\infty) \times (C^\infty(M, S) \otimes_{alg} \mathcal{A}_\theta^\infty)$. Since both of them are (jointly) continuous, they coincide on the whole of W. In other words, for any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$[D\hat{\otimes}I, f] = (d\hat{\otimes}I)f \text{ as linear maps on } C^\infty(M, S)\hat{\otimes}\mathcal{A}_\theta^\infty. \tag{8}$$

The canonical $\mathbb{R}$-bilinear pairing $C^\infty(M, TM) \times C^\infty(M, T^*M) \to C^\infty(M)$ extends to a $\mathbb{C}$-bilinear pairing $C^\infty(M, TM^{\mathbb{C}}) \times C^\infty(M, T^*M^{\mathbb{C}}) \to C^\infty(M)$, which is clearly continuous. For any $Y \in C^\infty(M, TM^{\mathbb{C}})$ let $\mathfrak{i}_Y$ be the corresponding contraction $C^\infty(M, T^*M^{\mathbb{C}}) \to C^\infty(M)$. Then we have the tensor-product map $\mathfrak{i}_Y\hat{\otimes}I$: $C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}_\theta^\infty \to C^\infty(M)\hat{\otimes}\,\mathcal{A}_\theta^\infty$. Let $\partial_Y : C^\infty(M) \to C^\infty(M)$ be the derivation with respect to Y. Since ∂_Y is a first-order linear operator, it is continuous. Then we also have the tensor-product map $\partial_Y\hat{\otimes}I : C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty \to C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$. For any $f \in C^\infty(M)$ it is trivial to see that

$$\partial_Y(f) = \mathfrak{i}_Y(df).$$

Then for any $f \in C^\infty(M) \otimes_{alg} \mathcal{A}_\theta^\infty$ clearly

$$(\partial_Y \otimes I)(f) = ((\mathfrak{i}_Y \otimes I) \circ (d \otimes I))(f).$$

By the same argument as for (8), for any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we then have

$$(\partial_Y\hat{\otimes}I)(f) = ((\mathfrak{i}_Y\hat{\otimes}I) \circ (d\hat{\otimes}I))(f). \tag{9}$$

Since the tracial state $tr : Cl^{\mathbb{C}}(TM_p) \to \mathbb{C}$ in Lemma 2.8 is invariant under the action of $SO(TM_p)$ for each $p \in M$, we can use them pointwisely to define a linear map $C^\infty(M, Cl^{\mathbb{C}}M) \to C^\infty(M)$, which is clearly continuous. We denote this map also by tr. Then tr is still tracial in the sense that $tr(f \cdot g) = tr(g \cdot f)$ for any $f, g \in C^\infty(M, Cl^{\mathbb{C}}M)$. We have the tensor-product linear map $tr\hat{\otimes}I : C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty \to C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$. For any $Y \in C^\infty(M, TM^{\mathbb{C}}) \subseteq C^\infty(M, Cl^{\mathbb{C}}M)$ and $Z \in C^\infty(M, T^*M^{\mathbb{C}}) \subseteq C^\infty(M, Cl^{\mathbb{C}}M)$, recalling that we have a canonical identification of $C^\infty(M, TM^{\mathbb{C}})$ and $C^\infty(M, T^*M^{\mathbb{C}})$, we get

$$tr(Y \cdot Z) = \frac{1}{2}tr(Y \cdot Z + Z \cdot Y) = \frac{1}{2}tr(-2 < Y, Z >) = - < Y, Z >= -\mathfrak{i}_Y(Z),$$

where $Y \cdot Z$ is the multiplication in $C^\infty(M, Cl^{\mathbb{C}}M)$, and $< \cdot, \cdot >$ is the $C^\infty(M)$-valued $C^\infty(M)$-bilinear pairing on $C^\infty(M, TM^{\mathbb{C}})$. So $\mathfrak{i}_Y = tr \circ (-Y)$ on $C^\infty(M, T^*M^{\mathbb{C}})$. Then $\mathfrak{i}_Y \otimes I = (tr \otimes I) \circ ((-Y) \otimes 1)$ on $C^\infty(M, T^*M^{\mathbb{C}}) \otimes_{alg} \mathcal{A}_\theta^\infty$. Since both $\mathfrak{i}_Y\hat{\otimes}I$ and $(tr\hat{\otimes}I) \circ ((-Y) \otimes 1)$ are continuous maps from $C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}_\theta^\infty$ to $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$, we get

$$\mathfrak{i}_Y\hat{\otimes}I = (tr\hat{\otimes}I) \circ ((-Y) \otimes 1). \tag{10}$$

as maps $C^\infty(M, T^*M^{\mathbb{C}})\hat{\otimes}\mathcal{A}_\theta^\infty \to C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$. Combining (9) and (10) together, for any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we get

$$(\partial_Y\hat{\otimes}I)(f) = ((tr\hat{\otimes}I) \circ ((-Y) \otimes 1) \circ (d\hat{\otimes}I))(f). \tag{11}$$

Let $\mathrm{Lie}(\mathbb{T}^n)$ be the Lie algebra of $\mathbb{T}^n$. For any $X \in \mathrm{Lie}(\mathbb{T}^n)$ we denote by $X^\#$ the vector field on M generated by X.

Lemma 5.1. *For any $X \in \mathrm{Lie}(\mathbb{T}^n)$ and any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have*

$$\lim_{t\to 0} \frac{(\sigma_{e^{tX}}\hat{\otimes} I)(f) - f}{t} = (\partial_{-X^\#}\hat{\otimes} I)(f). \tag{12}$$

Proof. For any $f \in C^\infty(M)$ and $x \in \mathbb{T}^n$ clearly

$$\begin{aligned}
(\partial_{-X^\#})(f) &= \lim_{t\to 0} \frac{\sigma_{e^{tX}}(f) - f}{t}, \\
(\partial_{-X^\#})(\sigma_x(f)) &= \lim_{t\to 0} \frac{\sigma_{e^{tX}}(\sigma_x(f)) - \sigma_x(f)}{t} \\
&= \lim_{t\to 0} \sigma_x(\frac{\sigma_{e^{tX}}(f) - f}{t}) = \sigma_x(\partial_{-X^\#}(f)),
\end{aligned}$$

where the limits are taken with respect to the locally convex topology in $C^\infty(M)$. (Here we have $-X^\#$ instead of $X^\#$ in the first equation because $(\sigma_{e^{tX}}(f))(p) = f(\sigma_{e^{-tX}}(p))$ for any $p \in M$.) So we see that the map $t \mapsto \partial_{-X^\#}(\sigma_{e^{tX}}(f))$ is continuous. When M is compact, we know that

$$\sigma_{e^{tX}}(f) - f = \int_0^t \partial_{-X^\#}(\sigma_{e^{sX}}(f))\, ds = \int_0^t \sigma_{e^{sX}}(\partial_{-X^\#}(f))\, ds, \tag{13}$$

where the integral is taken with respect to the supremum norm topology in $C(M)$. Notice that the inclusion $C^\infty(M) \hookrightarrow C(M)$ is continuous when $C^\infty(M)$ is endowed with the locally convex topology and $C(M)$ is endowed with the norm topology. By Proposition 2.6 the integral $\int_0^t \sigma_{e^{sX}}(\partial_{-X^\#}(f))\, ds$ is also defined in $C^\infty(M)$, and is mapped to the corresponding integral in $C(M)$ under the inclusion $C^\infty(M) \hookrightarrow C(M)$. Therefore we see that (13) also holds with respect to the locally convex topology in $C^\infty(M)$. For noncompact M, since the locally convex topology on $C^\infty(M)$ is defined using seminorms from compact subsets of local trivializations, it is easy to see that (13) still holds.

Now for any $f \in C^\infty(M) \otimes_{alg} \mathcal{A}_\theta^\infty$ clearly we have

$$(\sigma_{e^{tX}} \otimes I)(f) - f = \int_0^t (\sigma_{e^{sX}} \otimes I)((\partial_{-X^\#} \otimes I)(f))\, ds$$

in $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$. For fixed X notice that $f \mapsto (\sigma_{e^{tX}}\hat{\otimes} I)(f) - f$ is a continuous map from $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ to itself. It is also easy to see that both $f \mapsto (\partial_{-X^\#}\hat{\otimes} I)(f)$ and $f \mapsto \int_0^t (\sigma_{e^{sX}}\hat{\otimes} I)(f)\, ds$ are continuous maps from $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ to itself. So the map $f \mapsto \int_0^t (\sigma_{e^{sX}}\hat{\otimes} I)((\partial_{-X^\#}\hat{\otimes} I)(f))\, ds$ from $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ to itself is continuous. Therefore, for any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$(\sigma_{e^{tX}}\hat{\otimes} I)(f) - f = \int_0^t (\sigma_{e^{sX}}\hat{\otimes} I)((\partial_{-X^\#}\hat{\otimes} I)(f))\, ds.$$

Now (12) follows from Proposition 2.7. $\square$

5.2. **Seminorms.** In this subsection we assume that M is an m-dimensional compact Spin manifold, and that the action σ_M lifts to an action on S. Notice that the fibres of $Cl^{\mathbb{C}}M$ are all isomorphic to the C^*-algebra $Cl^{\mathbb{C}}(\mathbb{R}^m)$, where $\mathbb{R}^m$ is the standard m-dimensional Euclidean space. Clearly $C^\infty(M, Cl^{\mathbb{C}}M)$ generates a continuous field of C^*-algebras [9, Secton 10.3] over M with continuous sections $\Gamma' = C(M, Cl^{\mathbb{C}}M)$. Recall that $\mathcal{H}$ is the Hilbert space completion of $C^\infty(M, S)$. So the algebra $C(M, Cl^{\mathbb{C}}M)$ has a natural faithful representation on $\mathcal{H}$. It is easy to see that the inclusion $C^\infty(M, Cl^{\mathbb{C}}M) \hookrightarrow C(M, Cl^{\mathbb{C}}M)$ is continuous with respect to the locally convex topology on $C^\infty(M, Cl^{\mathbb{C}}M)$ and the norm topology on $C(M, Cl^{\mathbb{C}}M)$. Just as in the case of $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty \to C(M) \otimes \mathcal{A}_\theta$ in Section 3, we have a $\mathbb{T}^n$-equivariant continuous linear map $C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty \to C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{A}_\theta$ extending this former one. We still denote it by Ψ. As in Lemmas 3.1 and 3.10, Ψ is in fact injective. Clearly Ψ is a $*$-algebra homomorphism. Let $C(M_\theta, Cl^{\mathbb{C}}M_\theta)$ be $(C(M, Cl^{\mathbb{C}}M)\otimes\mathcal{A}_\theta)^{\sigma\otimes\tau^{-1}}$. We also have the homomorphism $C^\infty(M_\theta, Cl^{\mathbb{C}}M_\theta) \to B(\mathcal{H}_\theta)$, which we still denote by Ψ_θ.

Proposition 5.2. *For any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ the domain of D^{L^2} is stable under $\Psi(f)$, and*

$$[D^{L^2}, \Psi(f)] = \Psi((d\hat{\otimes}I)f). \tag{14}$$

When f is in $C^\infty(M_\theta)$, the domain of $D_\theta^{L^2}$ is stable under $\Psi_\theta(f)$, and

$$[D_\theta^{L^2}, \Psi_\theta(f)] = \Psi_\theta(d_\theta f). \tag{15}$$

Proof. By Lemma 2.3 $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ is a locally convex left module over the algebra $C^\infty(M, Cl^{\mathbb{C}}M)\ \hat{\otimes}\mathcal{A}_\theta^\infty$. So we have the continuous maps:

$$(C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty) \times (C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty) \to C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty \xrightarrow{\Phi} \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta).$$

On the other hand, we have continuous maps:

$$\begin{aligned}&(C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty) \times (C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty)\\ &\xrightarrow{\Psi\times\Phi} B(\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)) \times \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta) \to \mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta).\end{aligned}$$

The two compositions coincide on $(C^\infty(M, Cl^{\mathbb{C}}M)\otimes_{alg}\mathcal{A}_\theta^\infty)\times(C^\infty(M,S)\otimes_{alg}\mathcal{A}_\theta^\infty)$. So they coincide on the whole of $(C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty) \times (C^\infty(M,S)\hat{\otimes}\ \mathcal{A}_\theta^\infty)$. In other words, for any $f \in C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty$ and any $\psi \in C^\infty(M,S)\hat{\otimes}\ \mathcal{A}_\theta^\infty$ we have

$$\Psi(f)\cdot\Phi(\psi) = \Phi(f\psi). \tag{16}$$

Then for any $f \in C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty$ and $\psi \in C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$\begin{aligned}&\Phi([D\hat{\otimes}I, f](\psi))\\ =\ &\Phi((D\hat{\otimes}I)(f\psi) - f((D\hat{\otimes}I)\psi)) \overset{(5)}{=} D^{L^2}(\Phi(f\psi)) - \Phi(f((D\hat{\otimes}I)\psi))\\ \overset{(16)}{=}\ &D^{L^2}((\Psi(f))(\Phi(\psi))) - \Psi(f)\cdot\Phi((D\hat{\otimes}I)\psi)\\ \overset{(5)}{=}\ &D^{L^2}((\Psi(f))(\Phi(\psi))) - \Psi(f)(D^{L^2}(\Phi(\psi))) = [D^{L^2}, \Psi(f)](\Phi(\psi)).\end{aligned}$$

So for any $f \in C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$\Phi\circ[D\hat{\otimes}I, f] = [D^{L^2}, \Psi(f)]\circ\Phi \tag{17}$$

as linear maps from $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ to $\mathcal{H}\bar{\otimes}L^2(\mathcal{A}_\theta)$. When f is in $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we also have

$$\Phi\circ[D\hat{\otimes}I, f] \overset{(8)}{=} \Phi\circ((d\hat{\otimes}I)f) \overset{(16)}{=} \Psi((d\hat{\otimes}I)f)\circ\Phi.$$

Therefore, for any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$[D^{L^2}, \Psi(f)]\circ\Phi = \Psi((d\hat{\otimes}I)f)\circ\Phi. \tag{18}$$

For any z in the domain of D^{L^2} take a net ψ_j in $C^\infty(M,S)\hat{\otimes}\mathcal{A}_\theta^\infty$ with $\Phi(\psi_j)\to z$ and $D^{L^2}(\Phi(\psi_j))\to D^{L^2}(z)$. Then

$$\begin{aligned} D^{L^2}((\Psi(f))(\Phi(\psi_j))) &\overset{(18)}{=} (\Psi(f))(D^{L^2}(\Phi(\psi_j))) + \Psi((d\hat{\otimes}I)(f))(\Phi(\psi_j)) \\ &\to (\Psi(f))(D^{L^2}(z)) + \Psi((d\hat{\otimes}I)(f))(z), \end{aligned}$$

and

$$(\Psi(f))(\Phi(\psi_j))\to(\Psi(f))(z).$$

So $(\Psi(f))(z)$ is in the domain of D^{L^2}, and

$$D^{L^2}((\Psi(f))(z)) = (\Psi(f))(D^{L^2}(z)) + \Psi((d\hat{\otimes}I)(f))(z).$$

Therefore the domain of D^{L^2} is stable under $\Psi(f)$, and $[D^{L^2},\Psi(f)]=\Psi((d\hat{\otimes}I)f)$.

The assertions about $C^\infty(M_\theta)$ follow from those about $C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$. $\square$

By Proposition 5.2 we see that the commutator $[D^{L^2},f]$ is bounded for any $f\in\Psi(C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty)$. Corresponding to L_θ defined in Definition 3.11 we have:

Definition 5.3. We define a seminorm, denoted by L^D, on $C(M)\otimes\mathcal{A}_\theta$ by

$$L^D(f) := \begin{cases} \|\,[D^{L^2},f]\,\|, & \text{if } f\in\Psi(C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty);\\ +\infty, & \text{otherwise}\ . \end{cases}$$

Fix an inner product on $\mathrm{Lie}(\mathbb{T}^n)$, and use it to get a translation-invariant Riemannian metric on $\mathbb{T}^n$ in the usual way. We get a length function l on $\mathbb{T}^n$ by setting $l(x)$ to be the geodesic distance from x to $e_{\mathbb{T}^n}$ for $x\in\mathbb{T}^n$. Notice that $I\otimes\tau=\sigma\otimes I$ is a nontrivial action of $\mathbb{T}^n$ on $C(M_\theta)$. To make use of Theorem 4.1 we define two seminorms:

Definition 5.4. We define a (possibly $+\infty$-valued) seminorm L^l on $C(M)\otimes\mathcal{A}_\theta$ for the action $\sigma\otimes I$ via (6):

$$L^l(f) := \sup\{\frac{\|\,(\sigma\otimes I)_x(f)-f\,\|}{l(x)} : x\in\mathbb{T}^n, x\neq e_{\mathbb{T}^n}\}.$$

We also define a (possibly $+\infty$-valued) seminorm L^l_θ on $C(M_\theta)$ for the action $I\otimes\tau$:

$$L^l_\theta(f) := \sup\{\frac{\|\,(I\otimes\tau)_x(f)-f\,\|}{l(x)} : x\in\mathbb{T}^n, x\neq e_{\mathbb{T}^n}\}.$$

Then

$$L^l_\theta = L^l \tag{19}$$

on $C(M_\theta)$, because there $I\otimes\tau=\sigma\otimes I$.

Our first key technical fact is the following comparison between L^l and L^D:

Proposition 5.5. *Let C be the norm of the linear map $X\mapsto X^\#$ from* $\mathrm{Lie}(\mathbb{T}^n)$ *to $C^\infty(M,TM)\subseteq C(M,Cl^{\mathbb{C}}M)$. Then on $C(M)\otimes\mathcal{A}_\theta$ we have*

$$L^l\le C\cdot L^D. \tag{20}$$

Proof. Let $X \in \mathrm{Lie}(\mathbb{T}^n)$. For any $f \in C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$ we have

$$\begin{aligned}(\Psi \circ (\partial_{-X^\#}\hat{\otimes} I))(f) &\overset{(12)}{=} \Psi(\lim_{t\to 0} \frac{(\sigma_{e^{tX}}\hat{\otimes} I)(f) - f}{t}) \\ &= \lim_{t\to 0} \Psi(\frac{(\sigma_{e^{tX}}\hat{\otimes} I)(f) - f}{t}) \\ &= \lim_{t\to 0} \frac{(\sigma_{e^{tX}} \otimes I)(\Psi(f)) - \Psi(f)}{t}.\end{aligned}$$

It follows immediately that $\Psi(f)$ is once-differentiable with respect to the action $\sigma \otimes I$. In fact, $\Psi(f)$ is easily seen to be smooth for the action $\sigma \otimes I$, though we don't need this fact here. By [24, Proposition 8.6]

$$L^l(\Psi(f)) = \sup_{\|X\|=1} \| \lim_{t\to 0} \frac{(\sigma_{e^{tX}} \otimes I)(\Psi(f)) - \Psi(f)}{t} \| .$$

Then we get

$$\begin{aligned}L^l(\Psi(f)) &= \sup_{\|X\|=1} \| (\Psi \circ (\partial_{-X^\#}\hat{\otimes} I))(f) \| \\ &\overset{(11)}{=} \sup_{\|X\|=1} \| (\Psi \circ (tr\hat{\otimes} I) \circ ((-X^\#) \otimes 1) \circ (d\hat{\otimes} I))(f) \| .\end{aligned}$$

Notice that the linear map $tr : C^\infty(M, Cl^{\mathbb{C}}M) \to C^\infty(M)$ extends to $C(M, Cl^{\mathbb{C}}M) \to C(M)$, which we still denote by tr. By Lemma 2.8 the map $tr : Cl^{\mathbb{C}}(\mathbb{R}^m) \to \mathbb{C}$ is positive. Then so is $tr : C(M, Cl^{\mathbb{C}}M) \to C(M)$. Since $C(M)$ is commutative, $tr : C(M, Cl^{\mathbb{C}}M) \to C(M)$ is completely positive [10, Lemma 5.1.4]. Then we have the tensor-product completely positive map [14, Proposition 8.2] $tr \otimes I : C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{A}_\theta \to C(M) \otimes \mathcal{A}_\theta$. Consequently, we have $\| tr \otimes I \| = \| (tr \otimes I)(1 \otimes 1) \| = 1$ [10, Lemma 5.1.1]. In fact, $tr \otimes I$ is easily seen to be a conditional expectation in the sense of [13, Exercise 8.7.23], though we don't need this fact here. Clearly

$$(tr \otimes I) \circ \Psi = \Psi \circ (tr\hat{\otimes} I) \tag{21}$$

holds on $C^\infty(M, Cl^{\mathbb{C}}M) \otimes_{alg} \mathcal{A}_\theta^\infty$. Since both maps here are continuous, (21) holds on the whole of $C^\infty(M, Cl^{\mathbb{C}}M)\hat{\otimes}\mathcal{A}_\theta^\infty$. For any $Y \in C^\infty(M, Cl^{\mathbb{C}}M) \subseteq C(M, Cl^{\mathbb{C}}M)$, we have

$$\begin{aligned}&\| (\Psi \circ (tr\hat{\otimes} I) \circ ((-Y) \otimes 1) \circ (d\hat{\otimes} I))(f) \| \\ \overset{(21)}{=} &\| ((tr \otimes I) \circ \Psi \circ ((-Y) \otimes 1) \circ (d\hat{\otimes} I))(f) \| \\ \leq &\| (\Psi \circ ((-Y) \otimes 1) \circ (d\hat{\otimes} I))(f) \| = \| \Psi((-Y) \otimes 1) \cdot \Psi((d\hat{\otimes} I)(f)) \| \\ \leq &\| \Psi((-Y) \otimes 1) \| \cdot \| \Psi((d\hat{\otimes} I)(f)) \| = \| Y \| \cdot \| \Psi((d\hat{\otimes} I)(f)) \| .\end{aligned}$$

Recall that $X^\# \in C^\infty(M, TM) \subseteq C^\infty(M, Cl^{\mathbb{C}}M)$. Therefore

$$\begin{aligned}L^l(\Psi(f)) &= \sup_{\|X\|=1} \| (\Psi \circ (tr\hat{\otimes} I) \circ ((-X^\#) \otimes 1) \circ (d\hat{\otimes} I))(f) \| \\ &\leq \sup_{\|X\|=1} \| X^\# \| \cdot \| \Psi((d\hat{\otimes} I)(f)) \| = C \| \Psi((d\hat{\otimes} I)(f)) \| \\ &\overset{(14)}{=} C \| [D^{L^2}, \Psi(f)] \| = C \cdot L^D(\Psi(f))\end{aligned}$$

as desired. ∎

5.3. **Restriction Map.** Our goal in this subsection is to prove the second key technical fact:

Proposition 5.6. *The restriction map from $C(M_\theta, Cl^{\mathbb{C}}M)$ to $B(\mathcal{H}_\theta)$ is isometric. In particular, for any $f \in C^\infty(M_\theta, Cl^{\mathbb{C}}M)$ we have*

$$\| \Psi(f) \| = \| \Psi_\theta(f) \| . \tag{22}$$

First of all, Proposition 5.6 justifies our way of taking $C(M_\theta)$ as a subalgebra of $B(\mathcal{H}_\theta)$ via restriction to $\mathcal{H}_\theta$. Secondly, it enables us to compute L_θ using our seminorm L^D in Subsection 5.2, and hence to compare it with L^l_θ:

Corollary 5.7. *On $C(M_\theta)$ we have*

$$L_\theta = L^D, \tag{23}$$

and

$$L^l_\theta \le C \cdot L_\theta. \tag{24}$$

Proof. We prove (23) first. Since Ψ is injective it suffices to show (23) on $\Psi(C^\infty(M_\theta))$. For any $f \in C^\infty(M_\theta)$ we have

$$\begin{aligned} L^D(\Psi(f)) &= \| [D^{L^2}, \Psi(f)] \| \overset{(14)}{=} \| \Psi((d\hat{\otimes} I)f) \| = \| \Psi(d_\theta f) \| \overset{(22)}{=} \| \Psi_\theta(d_\theta f) \| \\ &\overset{(15)}{=} \| [D^{L^2}_\theta, \Psi_\theta(f)] \| = L_\theta(\Psi_\theta(f)), \end{aligned}$$

which yields (23). Then on $C(M_\theta)$ we have

$$L^l_\theta \overset{(19)}{=} L^l \overset{(20)}{\le} C \cdot L^D \overset{(23)}{=} C \cdot L_\theta.$$

$\square$

Instead of proving Proposition 5.6 directly, we shall prove a slightly more general form. Let $\mathcal{A}$ be a unital C^*-algebra with a strongly continuous action σ of $\mathbb{T}^n$, which we shall set to be $C(M, Cl^{\mathbb{C}}M)$ later. Assume that $\mathcal{A} \subseteq B(\mathcal{H})$ and that $\mathbb{T}^n$ has a strongly continuous unitary representation on $\mathcal{H}$, which we still denote by σ, such that the action σ on $\mathcal{A}$ is induced by conjugation. Then the C^*-algebraic spatial tensor product $\mathcal{A} \otimes \mathcal{A}_\theta$ [31, Appendix T.5] acts on $\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta)$ faithfully. For any $q \in \mathbb{Z}^n = \widehat{\mathbb{T}^n}$ let $(\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta))_q$ be the q-isotypic subspace of $\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta)$ for the action $\sigma\bar{\otimes}\tau^{-1}$. Notice that $(\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta))_q$ is stable under the action of $(\mathcal{A} \otimes \mathcal{A}_\theta)^{\sigma\otimes\tau^{-1}}$ for each $q \in \mathbb{Z}^n$.

Proposition 5.8. *For any $f \in (\mathcal{A} \otimes \mathcal{A}_\theta)^{\sigma\otimes\tau^{-1}}$ and $q \in \mathbb{Z}^n$ we have*

$$\| f \| = \| f|_{(\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta))_q} \|, \tag{25}$$

where $(\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta))_q$ is the q-isotypic component of $\mathcal{H}\bar{\otimes} L^2(\mathcal{A}_\theta)$ under $\sigma\bar{\otimes}\tau^{-1}$.

Proof. Think of $-\theta q$ as an element of $\mathbb{T}^n$ via the natural projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. For any $p \in \mathbb{Z}^n$, recalling the skew-symmetric bicharacter ω_θ in Section 2, we have

$$u_q u_p u_{-q} = \omega_\theta(q, p)\omega_\theta(q + p, -q)u_p = \omega_\theta(q, 2p)u_p = \langle p, -\theta q\rangle\, u_p = \tau_{-\theta q}(u_p).$$

It follows immediately that for any $b \in \mathcal{A}_\theta$ we have

$$u_q b u_{-q} = \tau_{-\theta q}(b).$$

Consequently, for any $f \in \mathcal{A} \otimes \mathcal{A}_\theta$ we have

$$(1 \otimes u_q) f (1 \otimes u_{-q}) = (I \otimes \tau)_{-\theta q}(f).$$

Therefore

$$(1 \otimes u_q) \circ f \circ (1 \otimes u_{-q}) = (I \bar{\otimes} \tau)_{-\theta q} \circ f \circ (I \bar{\otimes} \tau)_{\theta q} \tag{26}$$

on $\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta)$. Clearly $1 \otimes u_{-q}$ is in the q-isotypic component of $\mathcal{A} \otimes \mathcal{A}_\theta$ under $\sigma \otimes \tau^{-1}$. So $1 \otimes u_{-q}$ restricted to $(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_0$ is a unitary map onto $(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_q$. Since $I \bar{\otimes} \tau$ and $\sigma \bar{\otimes} \tau^{-1}$ commute with each other, $I \bar{\otimes} \tau$ preserves $(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_q$. Thus (26) tells us that for any $f \in (\mathcal{A} \otimes \mathcal{A}_\theta)^{\sigma \otimes \tau^{-1}}$ the two restrictions $f|_{(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_0}$ and $f|_{(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_q}$ are unitarily conjugate to each other. Hence

$$\| f|_{(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_0} \| = \| f|_{(\mathcal{H} \bar{\otimes} L^2(\mathcal{A}_\theta))_q} \|$$

for all $q \in \mathbb{Z}^n$. Then (25) follows immediately. □

Now Proposition 5.6 is just a consequence of Proposition 5.8 applied to $\mathcal{A} = C(M, Cl^{\mathbb{C}} M)$.

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by verifying the conditions in Theorem 4.1 for the quadruple $(C(M_\theta), L_\theta, \mathbb{T}^n, I \otimes \tau)$. Clearly L_θ satisfies the reality condition (3). The condition (1) is already verified in (24).

Let $\alpha = I \otimes \tau$, and let $\hat{\alpha} = I \hat{\otimes} \tau$. Notice that α is in fact an action of $\mathbb{T}^n$ on $C(M) \otimes \mathcal{A}_\theta$, under which $C(M_\theta)$ is stable. For any $f \in C(M_\theta)$ and any continuous function $\varphi : \mathbb{T}^n \to \mathbb{C}$ clearly $\alpha_\varphi(f)$ doesn't depend on whether we think of f as being in $C(M_\theta)$ or $C(M) \otimes \mathcal{A}_\theta$, where α_φ is the linear map on $C(M) \otimes \mathcal{A}_\theta$ or $C(M_\theta)$ defined in Lemma 3.2.

Now we verify the condition (2):

Proposition 6.1. *Let $\varphi \in C(\mathbb{T}^n)$. Then $\Psi(C^\infty(M) \hat{\otimes} \mathcal{A}_\theta^\infty)$ and $\Psi_\theta(C^\infty(M_\theta))$ are both stable under α_φ. We have*

$$L^D \circ \alpha_\varphi \leq \| \varphi \|_1 \cdot L^D \tag{27}$$

on $C(M) \otimes \mathcal{A}_\theta$, and

$$L_\theta \circ \alpha_\varphi \leq \| \varphi \|_1 \cdot L_\theta \tag{28}$$

on $C(M_\theta)$.

Proof. For any $f \in C^\infty(M) \hat{\otimes} \mathcal{A}_\theta^\infty$ by Lemma 3.3 we have $\alpha_\varphi(\Psi(f)) = \Psi(\hat{\alpha}_\varphi(f)) \in \Psi(C^\infty(M, S) \hat{\otimes} \mathcal{A}_\theta^\infty)$. So $\Psi(C^\infty(M) \hat{\otimes} \mathcal{A}_\theta^\infty)$ is stable under α_φ. For any $g \in C^\infty(M_\theta)$ by Lemma 3.3 we have $\hat{\alpha}_\varphi(g) \in C^\infty(M_\theta)$. Then $\alpha_\varphi(\Psi_\theta(g)) = \alpha_\varphi(\Psi(g)) = \Psi(\hat{\alpha}_\varphi(g)) \in \Psi(C^\infty(M_\theta)) = \Psi_\theta(C^\infty(M_\theta))$. So $\Psi_\theta(C^\infty(M_\theta))$ is also stable under α_φ.

Notice that D^{L^2} is invariant under the conjugation of $\sigma \bar{\otimes} I$, and hence $D_\theta^{L^2}$ is invariant under the conjugation of the restriction of $\sigma \bar{\otimes} I$ to $\mathcal{H}_\theta$. Then clearly L^D and L_θ are invariant under α. Also notice that seminorms defined by commutators are lower semicontinuous [23, Proposition 3.7]. Then (27) and (28) follow from Remark 4.2(3). □

We proceed to verify the conditions (3) and (4). For each $q \in \mathbb{Z}^n = \widehat{\mathbb{T}^n}$ let $(C(M_\theta))_q$ be the q-isotypic component of $C(M_\theta)$ under α throughout the rest of this section. Also let $(C(M))_q$ and $(C^\infty(M))_q$ be the q-isotypic components of $C(M)$ and $C^\infty(M)$ under σ. We need:

Lemma 6.2. *For each $q \in \mathbb{Z}^n$ we have*

$$(C(M_\theta))_q = (C(M))_q \otimes u_q, \tag{29}$$

and

$$(C(M_\theta))_q \cap \Psi_\theta(C^\infty(M_\theta)) = (C^\infty(M))_q \otimes u_q. \tag{30}$$

Proof. Let $V = C^\infty(M)\hat{\otimes}\mathcal{A}_\theta^\infty$, and let $W = C(M) \otimes \mathcal{A}_\theta$. Let V_q and W_q be the q-isotypic component of V and W under $\hat{\alpha}$ and α respectively. By similar arguments as in Lemma 3.1, we have $V_q = C^\infty(M) \otimes u_q$ and $W_q = C(M) \otimes u_q$. Then

$$(C(M_\theta))_q = W_q \cap W^{\sigma\otimes\tau^{-1}} = (C(M) \otimes u_q)^{\sigma\otimes\tau^{-1}} = (C(M))_q \otimes u_q.$$

Since Ψ is injective, we also have

$$\begin{aligned}(C(M_\theta))_q \cap \Psi_\theta(C^\infty(M_\theta)) &= \Psi(V_q \cap V^{\sigma\hat{\otimes}\tau^{-1}}) = \Psi((C^\infty(M) \otimes u_q)^{\sigma\hat{\otimes}\tau^{-1}}) \\ &= \Psi((C^\infty(M))_q \otimes u_q)) = (C^\infty(M))_q \otimes u_q.\end{aligned}$$

□

The geodesic distance on M defines a seminorm L_ρ on $C(M)$ via (1). This makes $C(M)$ into a compact quantum metric space (see the discussion after Lemma 4.6 in [24]). Let r_M be the radius. Define a new seminorm L on $C(M)$ by $L = L_\rho$ on $C^\infty(M)$, and $L = +\infty$ on $C(M) \setminus C^\infty(M)$. Since $L \geq L_\rho$, by Proposition 2.11 clearly L is also a Lip-norm and has radius no bigger than r_M. It is well known [4, 5] that

$$L(f) = \parallel df \parallel = \parallel [D, f] \parallel \tag{31}$$

for all $f \in C^\infty(M)$, where we denote the closure of D on $\mathcal{H}$ also by D. Notice that for any $f = f_q \otimes u_q \in (C^\infty(M))_q \otimes u_q$ we have

$$L^D(f) = \parallel [D^{L^2}, f] \parallel = \parallel [D, f_q] \otimes u_q \parallel = \parallel [D, f_q] \parallel \overset{(31)}{=} L(f_q).$$

Combining this with (23), we get

$$L_\theta(f_q \otimes u_q) = L(f_q) \tag{32}$$

for $f_q \otimes u_q \in (C^\infty(M))_q \otimes u_q$. From (32), (29) and (30) we see that L_θ restricted to $(C(M_\theta))_q$ can be identified with L restricted to $(C(M))_q$. Then conditions (3) and (4) of Theorem 4.1 follow immediately. Then Theorem 1.1 is just a consequence of Theorem 4.1 applied to $(C(M_\theta), L_\theta, \mathbb{T}^n, \alpha)$. We also see that $(C(M_\theta), L_\theta)$ has radius no bigger than $r_M + C\int_{\mathbb{T}^n} l(x)\,dx$.

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Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3, CANADA
E-mail address: hli@fields.toronto.edu