INTRINSIC ERGODICITY, GENERATORS AND SYMBOLIC REPRESENTATIONS OF ALGEBRAIC GROUP ACTIONS

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Dedicated to Anatole M. Vershik on the occasion of his 90th birthday

ABSTRACT. We construct natural symbolic representations of intrinsically ergodic, but not necessarily expansive, principal algebraic actions of countably infinite amenable groups and use these representations to find explicit generating partitions (up to null-sets) for such actions.

1. INTRODUCTION

Let Γ be a countably infinite discrete group with integral and real group rings $\mathbb{Z}\Gamma \subset \mathbb{R}\Gamma$. Every $g \in \mathbb{R}\Gamma$ is written as a finite formal sum $g = \sum_{\gamma\in\Gamma} g_\gamma \cdot \gamma$ with g_γ in $\mathbb{Z}$ or $\mathbb{R}$, respectively, for every γ. We write $\operatorname{supp}(g) = \{\gamma \in \Gamma \mid g_\gamma \neq 0\}$ for the *support* of g and set $g^+ = \sum_{\gamma\in\Gamma} \max\{g_\gamma, 0\} \cdot \gamma$ and $g^- = \sum_{\gamma\in\Gamma} \min\{g_\gamma, 0\} \cdot \gamma$.

For $g = \sum_{\gamma\in\Gamma} g_\gamma \cdot \gamma$, $h = \sum_{\gamma\in\Gamma} h_\gamma \cdot \gamma$ in $\mathbb{R}\Gamma$ we denote by $g + h = \sum_{\gamma\in\Gamma}(g_\gamma + h_\gamma) \cdot \gamma$ their *sum*, by $gh = \sum_{\gamma,\delta\in\Gamma} g_\gamma h_\delta \cdot \gamma\delta$ their *product*, and by $g^* = \sum_{\gamma\in\Gamma} g_\gamma \cdot \gamma^{-1}$ and $h^* = \sum_{\gamma\in\Gamma} h_\gamma \cdot \gamma^{-1}$ their *adjoints*. The adjoint map $g \mapsto g^*$ is an *involution* on $\mathbb{R}\Gamma$, i.e., $(gh)^* = h^* g^*$.

An *algebraic* Γ-*action* is a homomorphism $\tau\colon \Gamma \to \operatorname{Aut}(X)$ from Γ to the group of continuous automorphisms of a compact metrizable abelian group X. If τ is such an algebraic Γ-action, then $\tau^\gamma \in \operatorname{Aut}(X)$ denotes the image of $\gamma \in \Gamma$, and $\tau^{\gamma\delta} = \tau^\gamma\tau^\delta$ for every $\gamma, \delta \in \Gamma$. The action τ induces an action of $\mathbb{Z}\Gamma$ by group homomorphisms $\tau^f\colon X \to X$, where $\tau^f = \sum_{\gamma\in\Gamma} f_\gamma \tau^\gamma$ for every $f = \sum_{\gamma\in\Gamma} f_\gamma \cdot \gamma \in \mathbb{Z}\Gamma$. Clearly, if $f, g \in \mathbb{Z}\Gamma$, then $\tau^{fg} = \tau^f\tau^g$.

Let $\hat{X}$ be the dual group of X. If $\hat{\tau}^\gamma$ is the automorphism of $\hat{X}$ dual to τ^γ, then the map $\hat{\tau}\colon \Gamma \to \operatorname{Aut}(\hat{X})$ satisfies that $\hat{\tau}^{\gamma\delta} = \hat{\tau}^\delta\hat{\tau}^\gamma$ for all $\gamma, \delta \in \Gamma$. We denote by $\hat{\tau}^f\colon \hat{X} \to \hat{X}$ the group homomorphism dual to τ^f and set $f \cdot a = \hat{\tau}^{f^*} a$ for every $f \in \mathbb{Z}\Gamma$ and $a \in \hat{X}$. The resulting map $(f, a) \mapsto f \cdot a$ from $\mathbb{Z}\Gamma \times \hat{X}$ to $\hat{X}$ satisfies that $(fg) \cdot a = f \cdot (g \cdot a)$ for all $f, g \in \mathbb{Z}\Gamma$ and turns $\hat{X}$ into a left module over the group ring $\mathbb{Z}\Gamma$. Conversely, if M is a countable left module over $\mathbb{Z}\Gamma$, we set $X = \widehat{M}$ and put $\hat{\tau}^f a = f^* \cdot a$ for $f \in \mathbb{Z}\Gamma$ and $a \in M$. The maps $\tau^f\colon \widehat{M} \to \widehat{M}$ dual to $\hat{\tau}^f$ define an action of $\mathbb{Z}\Gamma$ by homomorphisms of $\widehat{M}$, which in turn induces an algebraic action τ of Γ on $X = \widehat{M}$.

The simplest examples of algebraic Γ-actions arise from $\mathbb{Z}\Gamma$-modules of the form $M = \mathbb{Z}\Gamma/(f)$, where $(f) = \mathbb{Z}\Gamma f$ is the principal left ideal generated by f: such actions are called *principal*. For an explicit description of such actions we put $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and define the left and right shift-actions λ and ρ of Γ on $\mathbb{T}^\Gamma$ by

$$(\lambda^\gamma x)_\delta = x_{\gamma^{-1}\delta} \quad \text{and} \quad (\rho^\gamma x)_\delta = x_{\delta\gamma} \tag{1.1}$$

for every $\gamma \in \Gamma$ and $x = (x_\delta)_{\delta\in\Gamma} \in \mathbb{T}^\Gamma$. The actions λ and ρ extend to $\mathbb{Z}\Gamma$-actions on $\mathbb{T}^\Gamma$ given by

$$\lambda^f = \textstyle\sum_{\gamma\in\Gamma} f_\gamma \lambda^\gamma, \qquad \rho^f = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma,$$

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for every $f=\sum_{\gamma\in\Gamma}f_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$. These $\mathbb{Z}\Gamma$-actions obviously commute: for every $f,g\in\mathbb{Z}\Gamma$ and $x\in\mathbb{T}^\Gamma$,

$$(\lambda^f\circ\rho^g)\,x=(\rho^g\circ\lambda^f)\,x.$$

The pairing $\langle f,x\rangle=\sum_{\gamma\in\Gamma}f_\gamma x_\gamma=(\rho^f x)_{1_\Gamma}$ with $f=\sum_{\gamma\in\Gamma}f_\gamma\cdot\gamma\in\mathbb{Z}\Gamma$ and $x=(x_\gamma)\in\mathbb{T}^\Gamma$, identifies $\mathbb{Z}\Gamma$ with the dual group $\widehat{\mathbb{T}^\Gamma}$ of $\mathbb{T}^\Gamma$ and has the property that

$$\begin{aligned}\langle h,\rho^f x\rangle&=\Big\langle h,\sum\nolimits_{\delta\in\Gamma}f_\delta\rho^\delta x\Big\rangle=\sum\nolimits_{\gamma\in\Gamma}h_\gamma\sum\nolimits_{\delta\in\Gamma}f_\delta x_{\gamma\delta}\\&=\sum\nolimits_{\gamma\in\Gamma}\sum\nolimits_{\delta\in\Gamma}h_{\gamma\delta^{-1}}f_\delta x_\gamma=\sum\nolimits_{\gamma\in\Gamma}(hf)_\gamma x_\gamma=\langle hf,x\rangle\end{aligned}$$

for every $f,h\in\mathbb{Z}\Gamma$ and $x\in\mathbb{T}^\Gamma$. Every $f\in\mathbb{Z}\Gamma$ defines a λ-invariant closed subgroup

$$\begin{aligned}X_f&=\ker\rho^f=\{x\in\mathbb{T}^\Gamma\mid\rho^f x=0\}\\&=\{x\in\mathbb{T}^\Gamma\mid\langle h,\rho^f x\rangle=\langle hf,x\rangle=0\ \text{ for every }\ h\in\mathbb{Z}\Gamma\}=(f)^\perp\subset\widehat{\mathbb{Z}\Gamma}=\mathbb{T}^\Gamma.\end{aligned}\tag{1.2}$$

We denote by $\lambda_f=\lambda_{X_f}$ the restriction of λ to X_f and note that the normalized Haar measure μ_f on X_f is invariant under λ_f.

Definition 1.1. For every $f\in\mathbb{Z}\Gamma$ we call the left shift-action λ_f on the probability space (X_f,μ_f) the *principal algebraic* Γ*-action* defined by f.

Dynamical properties of algebraic actions of countably infinite groups — and, in particular, of principal actions — have been investigated at various levels of generality (cf. e.g., [29], [4], [19], or [14]). In this paper we focus on symbolic representations of principal algebraic actions of countably infinite amenable groups and on generators of such actions arising from these representations.

Symbolic representations of algebraic actions have a long history. The first such representations arose from geometrically constructed Markov partitions around 1967–1970 (cf. [32], [2]) and helped to provide a crucial link between smooth and symbolic dynamics. A different approach to symbolic representations of toral automorphisms had its origins in the paper [34] by Vershik from 1992, where he represented the hyperbolic toral automorphism $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ by the *Golden Mean* shift by using homoclinic points rather than Markov partitions. Vershik's original construction was subsequently extended to arbitrary hyperbolic toral and solenoidal automorphisms in [15] and [17], and to the 'homoclinic' construction of symbolic covers of expansive[1] principal algebraic $\mathbb{Z}^d$-actions (cf. [10]). What these constructions have in common is that they use a *summable homoclinic point* $w\in X_f$ of a principal algebraic action λ_f of a countably infinite discrete group Γ to define a shift-equivariant surjective map $\xi_w\colon\ell^\infty(\Gamma,\mathbb{Z})\to X_f$, and to restrict this map to a suitable compact shift-invariant subset $\mathcal{V}\subset\ell^\infty(\Gamma,\mathbb{Z})$ (cf. e.g. [10], [13], or [23]).

While expansive principal algebraic actions always have summable homoclinic points permitting such a construction, this is generally not the case for nonexpansive actions (cf. [4], [8], [21]). In this paper we obviate the need for summable homoclinic points and show directly that, for every countably infinite discrete amenable group Γ and every $f\in\mathbb{Z}\Gamma$ for which the principal algebraic action λ_f on (X_f,μ_f) is *intrinsically ergodic* there exists a natural isomorphism

[1] A continuous Γ-action τ on a compact metrizable space Y with compatible metric d is *expansive* if there exists a $\delta>0$ such that $\sup_{\gamma\in\Gamma}\mathsf{d}(\tau^\gamma y,\tau^\gamma y')\geq\delta$ whenever y,y' are distinct points in Y.

(mod μ_f) of λ_f with the left shift action $\bar{\lambda}$ of Γ on a closed, shift-invariant subset $\bar{Z}_f$ of the symbolic space $\{-\|f^-\|_1, \dots, \|f^+\|_1\}^\Gamma$, furnished with a shift-invariant Borel probability measure $\nu_f^\#$ (Theorem 3.18). As an obvious consequence of this isomorphism one obtains that the 'alphabet' $\mathcal{B}_f \subsetneq \{-\|f^-\|_1, \dots, \|f^+\|_1\}$ of $\bar{Z}_f$ determines a natural generator[2] for λ_f on (X_f, μ_f) (Corollary 4.1). As a further corollary of this construction we see that the partition $\mathcal{C}_f = \{C_j \mid j = 0, \dots, \|f\|_1 - 1\}$ of X_f, defined by

$$C_j = \left\{x = (x_\gamma)_{\gamma \in \Gamma} \in X_f \mid \tfrac{j}{\|f\|_1} \le x_{1_\Gamma} < \tfrac{j+1}{\|f\|_1} \pmod 1\right\}$$

for $j = 0, \dots, \|f\|_1 - 1$, is a generator (mod μ_f) for λ_f (Corollary 4.2). If λ_f is *expansive*, $\mathcal{C}_f$ is obviously a generator without any conditions on Γ (cf. Subsection 5.2.1), but for nonexpansive actions this result is nontrivial.

In Section 5 we present examples of intrinsically ergodic principal algebraic actions λ_f, $f \in \mathbb{Z}\Gamma$, of countably infinite discrete amenable groups Γ. If $\Gamma = \mathbb{Z}^d$, $d \ge 1$, or if Γ is arbitrary and λ_f is expansive, the situation is well-understood (Subsections 5.1 or 5.2.1). For nonexpansive principal algebraic actions intrinsic ergodicity is a more elusive property. A sufficient condition for intrinsic ergodicity is that the group $\Delta^1(X_f)$ of summable homoclinic points of λ_f is dense in the group X_f carrying the action (Proposition 5.3). In Theorem 6.1 we verify the latter condition for *well-balanced* polynomials $f \in \mathbb{Z}\Gamma$, provided that Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ and the center of Γ contains an element of infinite order.

2. Linearization of principal algebraic actions

Let Γ be a countably infinite discrete group, and let $\ell^\infty(\Gamma, \mathbb{R})$ be the space of all bounded maps $v\colon \Gamma \to \mathbb{R}$, furnished with the norm $\|v\|_\infty = \sup_{\gamma\in\Gamma} |v_\gamma|$. We write $\eta\colon \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{T}^\Gamma$ for the weak*-continuous map defined by

$$\eta(w)_\gamma = w_\gamma \pmod 1,\ \gamma \in \Gamma, \tag{2.1}$$

and define the shift-actions $\bar{\lambda}$ and $\bar{\rho}$ of Γ on $\ell^\infty(\Gamma, \mathbb{R})$ as in (1.1) by

$$(\bar{\lambda}^\gamma v)_\delta = v_{\gamma^{-1}\delta}, \quad (\bar{\rho}^\gamma v)_\delta = v_{\delta\gamma}, \tag{2.2}$$

for every $v = (v_\delta)_{\delta\in\Gamma} \in \ell^\infty(\Gamma, \mathbb{R})$ and $\gamma \in \Gamma$. Again we extend these Γ-actions to $\mathbb{Z}\Gamma$-actions on $\ell^\infty(\Gamma, \mathbb{R})$ by setting

$$\bar{\lambda}^h = \sum\nolimits_{\gamma\in\Gamma} h_\gamma \bar{\lambda}^\gamma \quad \text{and} \quad \bar{\rho}^h = \sum\nolimits_{\gamma\in\Gamma} h_\gamma \bar{\rho}^\gamma$$

for every $h = \sum_{\gamma\in\Gamma} h_\gamma \cdot \gamma \in \mathbb{Z}\Gamma$. These actions correspond to the usual convolutions

$$\bar{\lambda}^h v = h \cdot v, \qquad \bar{\rho}^h v = v \cdot h^*, \tag{2.3}$$

for $h \in \mathbb{Z}\Gamma$ and $v \in \mathbb{R}\Gamma$, and extend further to $h \in \ell^1(\Gamma, \mathbb{R})$ and $v \in \ell^\infty(\Gamma, \mathbb{R})$.

We fix a nonzero element $f \in \mathbb{Z}\Gamma$ and consider the left shift action λ_f on the compact group $X_f \subset \mathbb{T}^\Gamma$ defined in (1.1) – (1.2). The space

$$\begin{aligned} W_f \coloneqq \eta^{-1}(X_f) &= \{w \in \ell^\infty(\Gamma, \mathbb{R}) \mid \eta(w) \in X_f\} \\ &= \{w \in \ell^\infty(\Gamma, \mathbb{R}) \mid \bar{\rho}^f w = w \cdot f^* \in \ell^\infty(\Gamma, \mathbb{Z})\} \end{aligned} \tag{2.4}$$

[2] If τ is a measure-preserving action of a countably infinite group Γ on a standard probability space $(Y, \mathcal{B}_Y, \mu)$, a countable Borel partition $\mathcal{C}$ of Y is a *generator* for τ if the smallest τ-invariant sigma-algebra $\mathcal{T} \subset \mathcal{B}_Y$ containing $\mathcal{C}$ is equal to $\mathcal{B}_Y$ (mod μ), i.e. up to μ-null sets.

is the *linearization* of X_f, and the restriction of $\bar\lambda$ to W_f is the *linearization* of λ_f. We set

$$Y_f = W_f \cap [0,1)^\Gamma \subset \ell^\infty(\Gamma,\mathbb{R}), \qquad Z_f = \bar\rho^f(Y_f) \subset \ell^\infty(\Gamma,\mathbb{Z}), \tag{2.5}$$

write $\bar Y_f$ and $\bar Z_f$ for the weak* closures of Y_f and Z_f in $\ell^\infty(\Gamma,\mathbb{R})$. Since $0 \le y_\gamma < 1$ for every $y \in Y_f$ and $\gamma \in \Gamma$, it is clear that

$$c_f^- \le (\bar\rho^f y)_\gamma \le c_f^+$$

for every $y \in Y_f$ and $\gamma \in \Gamma$, where

$$c_f^- = \min\{0, 1 - \|f^-\|_1\} \quad \text{and} \quad c_f^+ = \max\{0, \|f^+\|_1 - 1\}.$$

It follows that

$$Z_f \subseteq \bar Z_f \subseteq \{c_f^-, \dots, c_f^+\}^\Gamma. \tag{2.6}$$

Finally we denote by

$$K_f = \{w \in \ell^\infty(\Gamma,\mathbb{R}) \mid \bar\rho^f w = 0\} \subset W_f \tag{2.7}$$

the kernel of $\bar\rho^f$ in $\ell^\infty(\Gamma,\mathbb{R})$. By [8, Theorem 3.2], the action λ_f on X_f is expansive if and only if $K_f = \{0\}$.

If there is any danger of confusion we denote the restrictions of $\bar\lambda$ to the $\bar\lambda$-invariant sets $\bar Y_f$, $\bar Z_f$, and K_f by $\bar\lambda_{\bar Y_f}$, $\bar\lambda_{\bar Z_f}$ and $\bar\lambda_{K_f}$, respectively. The map $\eta\colon \ell^\infty(\Gamma,\mathbb{R}) \to \mathbb{T}^\Gamma$ in (2.1) induces a left shift-equivariant, continuous, surjective map from $\bar Y_f$ to X_f whose restriction to Y_f is bijective, and $\bar\rho^f$ intertwines the Γ-actions $\bar\lambda_{\bar Y_f}$ and $\bar\lambda_{\bar Z_f}$.

Proposition 2.1. *Let Γ be a countably infinite discrete group, $0 \ne f \in \mathbb{Z}\Gamma$, and let $W_f, \bar Y_f, \bar Z_f, K_f$ be the closed, $\bar\lambda$-invariant subsets of $\ell^\infty(\Gamma,\mathbb{R})$ defined in (2.4) – (2.7). Put $\tilde Z_f = \bar Z_f \times K_f$, and denote by $\tilde\lambda = \bar\lambda_{\bar Z_f} \times \bar\lambda_{K_f}$ the product Γ-action on $\tilde Z_f$. Then there exists, for every $\bar\lambda$-invariant Borel probability measure ν on $\bar Y_f$, a $\tilde\lambda$-invariant Borel probability measure $\tilde\nu$ on $\tilde Z_f$ with the following properties:*

1. $\pi^{(1)}_*\tilde\nu = \bar\rho^f_*\nu =: \nu^\#$, *where $\pi^{(1)}\colon \tilde Z_f \to \bar Z_f$ is the first coordinate projection;*
2. $\tilde\nu(\bar Z_f \times B_2(K_f)) = 1$, *where $B_r(K_f) = \{w \in K_f \mid \|w\|_\infty \le r\}$ for every $r \ge 0$;*
3. *The Γ-actions $\bar\lambda$ on $(\bar Y_f, \nu)$ and $\tilde\lambda$ on $(\tilde Z_f, \tilde\nu)$ are measurably conjugate.*

Proof. Since $\bar\rho^f$ is continuous, [26, Theorem I.4.2] shows that there exists a Borel map $\zeta\colon \bar Z_f \to \bar Y_f$ with $\bar\rho^f \circ \zeta(z) = z$ for every $z \in \bar Z_f$. For every $z \in \bar Z_f$ and $\gamma \in \Gamma$ we set

$$c(\gamma, z) = \zeta \circ \bar\lambda^\gamma(z) - \bar\lambda^\gamma \circ \zeta(z) \in B_1(K_f). \tag{2.8}$$

Then

$$c(\gamma\delta, z) = c(\gamma, \bar\lambda^\delta z) + \bar\lambda^\gamma c(\delta, z)$$

for every $\gamma, \delta \in \Gamma$ and $z \in \bar Z_f$, i.e., the Borel map $c\colon \Gamma \times \bar Z_f \to K_f$ is a cocycle taking values in $B_1(K_f)$. We define a Borel action $\tilde\lambda_1$ of Γ on $\tilde Z_f$ by setting

$$\tilde\lambda_1^\gamma(z, v) = (\bar\lambda^\gamma z, \bar\lambda^\gamma v - c(\gamma, z))$$

for every $(z, v) \in \tilde Z_f$, and consider the injective Borel map $\theta_1\colon \bar Y_f \to \bar Z_f \times B_1(K_f) \subset \tilde Z_f$ given by

$$\theta_1(w) = (\bar\rho^f w, w - \zeta \circ \bar\rho^f(w)) \tag{2.9}$$

for every $w \in \bar Y_f$. Then

$$\theta_1 \circ \bar\lambda^\gamma = \tilde\lambda_1^\gamma \circ \theta_1 \tag{2.10}$$

for every $\gamma \in \Gamma$.

Let ν be a $\bar{\lambda}$-invariant Borel probability measure on $\bar{Y}_f$ and let $\nu^{\#} = \bar{\rho}^f_* \nu$. The probability measure $\tilde{\nu}^{(1)} = (\theta_1)_* \nu$ is $\tilde{\lambda}_1$-invariant by (2.10), and is supported in the weak*-compact and metrizable set $\bar{Z}_f \times B_1(K_f) \subset \tilde{Z}_f$. Furthermore, $\pi^{(1)}_* \tilde{\nu}^{(1)} = \nu^{\#}$, where $\pi^{(1)} \colon \tilde{Z}_f \to \bar{Z}_f$ is the first coordinate projection. We decompose $\tilde{\nu}^{(1)}$ over $\bar{Z}_f$ by choosing a Borel measurable family $\tilde{\nu}^{(1)}_z$, $z \in \bar{Z}_f$, of Borel probability measures on K_f with $\tilde{\nu}^{(1)}_z(B_1(K_f)) = 1$ for every $z \in \bar{Z}_f$, and with

$$\int g(z,v)\, d\tilde{\nu}^{(1)}(z,v) = \int_{\bar{Z}_f} \int_{K_f} g(z,v)\, d\tilde{\nu}^{(1)}_z(v)\, d\nu^{\#}(z)$$

for every bounded Borel map $g \colon \tilde{Z}_f \to \mathbb{R}$. Since $\tilde{\nu}^{(1)}$ is $\tilde{\lambda}_1$-invariant,

$$\int h(v)\, d\tilde{\nu}^{(1)}_z(v) = \int h(\bar{\lambda}^{\gamma} v - c(\gamma, \bar{\lambda}^{\gamma^{-1}} z))\, d\tilde{\nu}^{(1)}_{\bar{\lambda}^{\gamma^{-1}} z}(v) \tag{2.11}$$

for every bounded Borel map $h \colon K_f \to \mathbb{R}$, every $\gamma \in \Gamma$, and $\nu^{\#}$-*a.e.* $z \in \bar{Z}_f$.

Define a Borel map $b \colon \bar{Z}_f \to K_f$ by setting $b(z) = \int_{K_f} v\, d\tilde{\nu}^{(1)}_z(v) \in B_1(K_f)$ for every $z \in \bar{Z}_f$, where the integral is taken coordinate-wise (or, equivalently, in the weak*-topology) on $\ell^{\infty}(\Gamma, \mathbb{R})$. Equation (2.11) shows that

$$\begin{aligned} b(z) &= \int_{K_f} v\, d\tilde{\nu}^{(1)}_z(v) = \int_{K_f} (\bar{\lambda}^{\gamma} v - c(\gamma, \bar{\lambda}^{\gamma^{-1}} z))\, d\tilde{\nu}^{(1)}_{\bar{\lambda}^{\gamma^{-1}} z}(v) \\ &= \int_{K_f} \bar{\lambda}^{\gamma} v\, d\tilde{\nu}^{(1)}_{\bar{\lambda}^{\gamma^{-1}} z}(v) - c(\gamma, \bar{\lambda}^{\gamma^{-1}} z) = \bar{\lambda}^{\gamma} b(\bar{\lambda}^{\gamma^{-1}} z) - c(\gamma, \bar{\lambda}^{\gamma^{-1}} z) \end{aligned}$$

for $\nu^{\#}$-*a.e.* $z \in \bar{Z}_f$. If we replace z by $\bar{\lambda}^{\gamma} z$ in the last equation we see that

$$c(\gamma, \cdot) = \bar{\lambda}^{\gamma} \circ b - b \circ \bar{\lambda}^{\gamma} \quad \nu^{\#}\text{-}a.e., \ \text{for every } \ \gamma \in \Gamma. \tag{2.12}$$

In other words, the cocycle $c \colon \Gamma \times \bar{Z}_f \to K_f$ is a coboundary $(\operatorname{mod} \nu^{\#})$ with Borel cobounding function $b \colon \bar{Z}_f \to B_1(K_f)$.

Let $\theta_2 \colon \tilde{Z}_f \to \tilde{Z}_f$ be the bijection given by

$$\theta_2(z,v) = (z, v - b(z))$$

for every $(z,v) \in \tilde{Z}_f$, and put $\tilde{\nu} = (\theta_2)_* \tilde{\nu}^{(1)} = (\theta_2 \circ \theta_1)_* \nu$. Then $\pi^{(1)}_* \tilde{\nu} = \pi^{(1)}_* \tilde{\nu}^{(1)} = \nu^{\#}$. We set

$$\theta = \theta_2 \circ \theta_1 \colon \bar{Y}_f \to \tilde{Z}_f \tag{2.13}$$

and obtain that

$$\theta(w) = \bigl(\bar{\rho}^f w, w - \zeta \circ \bar{\rho}^f(w) - b \circ \bar{\rho}^f(w)\bigr) \ \text{ for every } \ w \in \bar{Y}_f.$$

For ν-*a.e.* $w \in \bar{Y}_f$ we have that

$$\begin{aligned} \tilde{\lambda}^{\gamma} \circ \theta(w) &= \bigl(\bar{\lambda}^{\gamma} \circ \bar{\rho}^f(w), \bar{\lambda}^{\gamma} w - \bar{\lambda}^{\gamma} \circ \zeta \circ \bar{\rho}^f(w) - \bar{\lambda}^{\gamma} \circ b \circ \bar{\rho}^f(w)\bigr) \\ &= \bigl(\bar{\rho}^f(\bar{\lambda}^{\gamma} w), \bar{\lambda}^{\gamma} w + c(\gamma, \bar{\rho}^f w) - \zeta \circ \bar{\rho}^f \circ \bar{\lambda}^{\gamma}(w) \\ &\qquad\qquad - c(\gamma, \bar{\rho}^f w) - b \circ \bar{\rho}^f \circ \bar{\lambda}^{\gamma}(w)\bigr) \qquad \text{(by (2.8) and (2.12))} \\ &= \bigl(\bar{\rho}^f(\bar{\lambda}^{\gamma} w), \bar{\lambda}^{\gamma} w - \zeta \circ \bar{\rho}^f(\bar{\lambda}^{\gamma} w) - b \circ \bar{\rho}^f(\bar{\lambda}^{\gamma} w)\bigr) = \theta \circ \bar{\lambda}^{\gamma}(w), \end{aligned} \tag{2.14}$$

which proves that the Γ-actions $\bar{\lambda}$ on $(\bar{Y}_f, \nu)$ and $\tilde{\lambda}$ on $(\tilde{Z}_f, \tilde{\nu})$ are measurably conjugate, as claimed in (3). □

In the next section we show that, if Γ is amenable and λ_f has finite and completely positive entropy, there exist unique $\bar{\lambda}$-invariant Borel probability measures ν_f on $\bar{Y}_f$ and $\nu_f^{\#}$ on $\bar{Z}_f$ such that the principal algebraic Γ-action λ_f on (X_f, μ_f) is measurably conjugate to the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Y}_f, \nu_f)$ and $(\bar{Z}_f, \nu_f^{\#})$, respectively (cf. Theorem 3.18).

3. Symbolic representation of intrinsically ergodic principal algebraic actions

Throughout this section we assume that Γ is a countably infinite discrete amenable group and that $f \in \mathbb{Z}\Gamma$ is nonzero. We denote by μ_f the normalized Haar measure on X_f and define $\bar{Y}_f \subset W_f \subset \ell^\infty(\Gamma, \mathbb{R})$ and $\bar{Z}_f = \bar{\rho}^f(\bar{Y}_f) \subset \ell^\infty(\Gamma, \mathbb{Z})$ as in (2.4) – (2.5).

Lemma 3.1. *The principal algebraic Γ-action λ_f on X_f has infinite topological entropy if and only if f is a left zero divisor in $\mathbb{R}\Gamma$, i.e., if and only if there exists a nonzero $g \in \mathbb{R}\Gamma$ with $fg = 0$.*

Proof. This is a special case of [4, Theorem 4.11]. For later reference we include here an explicit proof of the fact that $\mathrm{h}_{\mathrm{top}}(\lambda_f) = \infty$ if f is a left zero divisor in $\mathbb{R}\Gamma$.

We embed $\mathbb{R}\Gamma$ in $\ell^\infty(\Gamma, \mathbb{R})$ in the obvious manner by identifying each $h = \sum_{\gamma \in \Gamma} h_\gamma \cdot \gamma \in \mathbb{R}\Gamma$ with $(h_\gamma)_{\gamma \in \Gamma} \in \ell^\infty(\Gamma, \mathbb{R})$.

If $f \in \mathbb{Z}\Gamma$ is a left zero divisor in $\mathbb{R}\Gamma$ we choose $g = \sum_{\gamma \in \Gamma} g_\gamma \cdot \gamma \in \mathbb{R}\Gamma$ with $1_\Gamma \in \mathrm{supp}(g)$ and $fg = 0$. Then $\bar{\rho}^f g^* = g^* f^* = 0$ (cf. (2.3)), and hence $\bar{\rho}^f(cg^*) = cg^* f^* = 0$ for every $c \in \mathbb{R}$. This shows that $cg^* \in W_f$ and $\eta(cg^*) \in X_f$ for every $c \in \mathbb{R}$ (cf. (2.4)).

If $I \subset \mathbb{R}$ is the open interval $(-\frac{1}{2\|g\|_\infty}, \frac{1}{2\|g\|_\infty})$, then the elements $\eta(cg^*) \in X_f$, $0 \neq c \in I$, are all distinct with identical supports $E = \mathrm{supp}(g^*) = \mathrm{supp}(g)^{-1}$.

Choose a maximal set $D \subset \Gamma$ such that the translates $\{\delta E \mid \delta \in D\}$ are disjoint. We claim that $DEE^{-1} = \bigcup_{\delta \in D} \delta EE^{-1} = \Gamma$. Indeed, if $DEE^{-1} \neq \Gamma$, then there exists a $\gamma \in \Gamma$ which is not equal to $\delta\gamma'\gamma''^{-1}$ for any $\delta \in D$ and $\gamma', \gamma'' \in E$. Then $\gamma E \cap \delta E = \varnothing$ for every $\delta \in D$, which contradicts the maximality of D. This proves the last claim.

Since the sets δE, $\delta \in D$, are disjoint, we obtain for every $z = (z_\delta)_{\delta \in D} \in I^D$, a point $\tilde{z} \in W_f \cap (-\frac{1}{2}, \frac{1}{2})^\Gamma$ which coincides on each δE with $z_\delta \bar{\lambda}^\delta g^*$. This shows that the restriction of X_f to its coordinates in DE contains — in essence — a Cartesian product of the form I^D. Since $(DE)E^{-1} = \Gamma$ and E^{-1} is finite, λ_f must have infinite topological entropy on X_f. □

Proposition 3.2. *If the principal algebraic Γ-action λ_f on X_f has finite topological entropy, then the restriction $\bar{\lambda}_C$ of $\bar{\lambda}$ to every weak*-closed, bounded, $\bar{\lambda}$-invariant subset $C \subset K_f$ has topological entropy zero* (*cf.* (2.7)).

For the proof of Proposition 3.2 we need a lemma.

Lemma 3.3. *Let $\tau \colon \Gamma \to \mathrm{Aut}(X)$ be an action of a countably infinite discrete amenable group Γ by continuous automorphisms of a compact metrizable group X such that $\mathrm{h}_{\mathrm{top}}(\tau) < \infty$. Then there exists, for every $\varepsilon > 0$, a neighbourhood U of 1_X in X such that the topological entropy $\mathrm{h}_{\mathrm{top}}(\tau_C)$ of the restriction of τ to any closed τ-invariant subset $C \subset U$ is less than ε.*

Proof. Choose a compatible left translation invariant metric d on X (i.e., $\mathsf{d}(x, y) = \mathsf{d}(zx, zy)$ for all $x, y, z \in X$). For every nonempty finite subset $F \Subset \Gamma$, put

$$\mathsf{d}_F(x, y) = \max\nolimits_{\gamma \in F} \mathsf{d}(\tau^\gamma x, \tau^\gamma y), \quad x, y \in X.$$

For each $\zeta > 0$ we denote by $\mathrm{sep}(X, \mathsf{d}, \zeta)$ the maximal cardinality of subsets $Z \subset X$ which are (d, ζ)-separated in the sense that $\mathsf{d}(y, z) \geq \zeta$ for all distinct $y, z \in Z$.

Take a left Følner sequence $(F_n)_{n\geq 1}$ for Γ, i.e., a sequence of nonempty finite sets $F_n \subset \Gamma$ with $\lim_{n\to\infty} \frac{|\gamma F_n \cap F_n|}{|F_n|} = 0$ for every $\gamma \in \Gamma$. Then there exists, for every $\varepsilon > 0$, some $\zeta > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{sep}(X, \mathsf{d}_{F_n}, \zeta) \geq \mathrm{h}_{\mathrm{top}}(\tau) - \varepsilon/2.$$

For large enough n, take a $(\mathsf{d}_{F_n}, \zeta)$-separated set $X_n \subset X$ such that $\frac{1}{|F_n|} \log |X_n| \geq \mathrm{h}_{\mathrm{top}}(\tau) - \varepsilon$.

Put $U = \{x \in X \mid \mathsf{d}(x, 1_X) < \zeta/10\}$. Let $Y \subset U$ be closed and Γ-invariant, and let τ_Y be the restriction of τ to Y. In order to show that $\mathrm{h}_{\mathrm{top}}(\tau_Y) \leq \varepsilon$ it suffices to show that

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{sep}(Y, \mathsf{d}_{F_n}, \delta) \leq \varepsilon \tag{3.1}$$

whenever $0 < \delta < \zeta/10$. In order to verify (3.1) we choose, for each n, a $(\mathsf{d}_{F_n}, \delta)$-separated set $Y_n \subset Y$ of cardinality $|Y_n| = \mathrm{sep}(Y, \mathsf{d}_{F_n}, \delta)$. When n is large enough, then $|X_n Y_n| = |X_n| \cdot |Y_n|$ and $X_n Y_n$ is $(\mathsf{d}_{F_n}, \delta)$-separated: indeed,

$$\mathsf{d}_{F_n}(xy, xz) = \mathsf{d}_{F_n}(y, z) \geq \delta,$$

for $x \in X_n$ and distinct $y, z \in Y_n$, whereas

$$\begin{aligned} \mathsf{d}_{F_n}(x_1 y, x_2 z) &\geq \mathsf{d}_{F_n}(x_1, x_2) - \mathsf{d}_{F_n}(x_1 y, x_1) - \mathsf{d}_{F_n}(x_2 z, x_2) \\ &= \mathsf{d}_{F_n}(x_1, x_2) - \mathsf{d}_{F_n}(y, 1_X) - \mathsf{d}_{F_n}(z, 1_X) \geq \zeta - \zeta/10 - \zeta/10 \geq \delta \end{aligned}$$

for $y, z \in Y_n$ and distinct $x_1, x_2 \in X_n$.

Then

$$\begin{aligned} \mathrm{h}_{\mathrm{top}}(\tau) &\geq \limsup_{n\to\infty} \tfrac{1}{|F_n|} \log \mathrm{sep}(X, \mathsf{d}_{F_n}, \delta) \geq \limsup_{n\to\infty} \tfrac{1}{|F_n|} \log(|X_n| \cdot |Y_n|) \\ &\geq \mathrm{h}_{\mathrm{top}}(\tau) - \varepsilon + \limsup_{n\to\infty} \tfrac{1}{|F_n|} \log \mathrm{sep}(Y, \mathsf{d}_{F_n}, \delta), \end{aligned}$$

which implies (3.1). ☐

Proof of Proposition 3.2. Since the Γ-actions $\bar{\lambda}_{B_r(K_f)}$, $r > 0$, are all conjugate to each other, $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{B_r(K_f)})$ is the same for all $r > 0$.

For $0 < r < 1/2$, the map $\eta \colon \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{T}^\Gamma$ in (2.1) embeds $B_r(K_f)$ injectively as a closed Γ-invariant subset of X_f. If $U \subset X_f$ is any open neighbourhood of 1_{X_f}, then $\eta(B_r(K_f)) \subset U$ for all sufficiently small $r > 0$.

Let $C \subset K_f$ be a weak*-closed, bounded, $\bar{\lambda}$-invariant subset. Then $C \subset B_r(K_f)$ for some $r > 0$, and $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_C) \leq \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{B_r(K_f)}) = \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{B_{r'}(K_f)})$ for every $r' > 0$.

Let $\varepsilon > 0$. By Lemma 3.3 there is some neighbourhood U of 1_{X_f} in X_f such that for any closed Γ-invariant subset Y of X_f contained in U the restriction $(\lambda_f)_Y$ of λ_f to Y has entropy $\leq \varepsilon$. If $r' > 0$ is small enough, $\eta(B_{r'}(K_f)) \subset U$, so that

$$\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_C) \leq \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{B_r(K_f)}) = \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{B_{r'}(K_f)}) = \mathrm{h}_{\mathrm{top}}((\lambda_f)_{\eta(B_{r'}(K_f))}) \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_C) = 0$. ☐

Proposition 3.4 (cf. [30, Proposition 8.7]). *Let Y_1, Y_2 be compact metrizable spaces, and let τ_1, τ_2 be continuous actions of a countably infinite discrete amenable group Γ on Y_1 and Y_2 such that the topological entropy $\mathrm{h}_{\mathrm{top}}(\tau_2)$ of τ_2 is equal to zero. We write $\pi^{(i)}: Y_1 \times Y_2 \to Y_i$ for the two coordinate projections. If μ is a $(\tau_1 \times \tau_2)$-invariant Borel probability measure on $Y_1 \times Y_2$ we set $\mu_i = \pi^{(i)}_* \mu$. Then $h_\mu(\tau_1 \times \tau_2) = h_{\mu_1}(\tau_1)$.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be finite Borel partitions of Y_1 and Y_2, respectively, and set $\tilde{\mathcal{P}} = \{P \times Y_2 \mid P \in \mathcal{P}\}$ and $\tilde{\mathcal{Q}} = \{Y_1 \times Q \mid Q \in \mathcal{Q}\}$. If (F_n) is a left Følner sequence in Γ, then

$$\begin{aligned}
h_\mu(\tau_1 \times \tau_2, \tilde{\mathcal{P}} \vee \tilde{\mathcal{Q}}) &= \lim_{n\to\infty} \frac{1}{|F_n|} H_\mu\bigl(\textstyle\bigvee_{\gamma \in F_n} (\tau_1 \times \tau_2)^{\gamma^{-1}} (\tilde{\mathcal{P}} \vee \tilde{\mathcal{Q}})\bigr) \\
&= \lim_{n\to\infty} \frac{1}{|F_n|} H_\mu\bigl(\textstyle\bigvee_{\gamma \in F_n} \tau_1^{\gamma^{-1}} (\tilde{\mathcal{P}}) \vee \bigvee_{\gamma \in F_n} \tau_2^{\gamma^{-1}} (\tilde{\mathcal{Q}})\bigr) \\
&= \lim_{n\to\infty} \frac{1}{|F_n|} \Bigl(H_\mu\bigl(\textstyle\bigvee_{\gamma \in F_n} \tau_1^{\gamma^{-1}} (\tilde{\mathcal{P}})\bigr) \\
&\qquad\qquad + H_\mu\bigl(\textstyle\bigvee_{\gamma \in F_n} \tau_2^{\gamma^{-1}} (\tilde{\mathcal{Q}}) \mid \bigvee_{\gamma \in F_n} \tau_1^{\gamma^{-1}} (\tilde{\mathcal{P}})\bigr)\Bigr) \\
&\le \lim_{n\to\infty} \frac{1}{|F_n|} \Bigl(H_\mu\bigl(\textstyle\bigvee_{\gamma \in F_n} \tau_1^{\gamma^{-1}} (\tilde{\mathcal{P}})\bigr) + H_\mu\bigl(\bigvee_{\gamma \in F_n} \tau_2^{\gamma^{-1}} (\tilde{\mathcal{Q}})\bigr)\Bigr) \\
&= \lim_{n\to\infty} \frac{1}{|F_n|} \Bigl(H_{\mu_1}\bigl(\textstyle\bigvee_{\gamma \in F_n} \tau_1^{\gamma^{-1}} (\mathcal{P})\bigr) + H_{\mu_2}\bigl(\bigvee_{\gamma \in F_n} \tau_2^{\gamma^{-1}} (\mathcal{Q})\bigr)\Bigr) \\
&\le h_{\mu_1}(\tau_1, \mathcal{P}) + \mathrm{h}_{\mathrm{top}}(\tau_2) = h_{\mu_1}(\tau_1, \mathcal{P})
\end{aligned}$$

by the variational principle [16, Theorem 9.48]. By taking the supremum over all finite partitions $\mathcal{P}$ and $\mathcal{Q}$ we obtain that $h_\mu(\tau_1 \times \tau_2) \le h_{\mu_1}(\tau_1)$. The reverse inequality $h_{\mu_1}(\tau_1) \le h_\mu(\tau_1 \times \tau_2)$ is obvious. □

Proposition 3.5 (cf. [30, Corollary 8.9]). *Suppose that the principal algebraic Γ-action λ_f on X_f has finite topological entropy. Then the following is true.*

(1) *For every $\bar{\lambda}$-invariant Borel probability measure ν on $\bar{Y}_f$, the probability measure $\nu^\# = \bar{\rho}^f_* \nu$ on $\bar{Z}_f$ is $\bar{\lambda}$-invariant, and $h_\nu(\bar{\lambda}_{\bar{Y}_f}) = h_{\nu^\#}(\bar{\lambda}_{\bar{Z}_f})$.*
(2) $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) = \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Z}_f})$.

Proof. Since $\bar{\rho}^f$ induces a continuous surjective $\bar{\lambda}$-equivariant map from $\bar{Y}_f$ to $\bar{Z}_f$, $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Z}_f}) \le \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f})$ and $h_{\nu^\#}(\bar{\lambda}_{\bar{Z}_f}) \le h_\nu(\bar{\lambda}_{\bar{Y}_f})$ for every $\bar{\lambda}$-invariant Borel probability measure ν on $\bar{Y}_f$.

By applying the Propositions 3.2 and 3.4 with $Y_1 = \bar{Z}_f$, $Y_2 = B_2(K_f)$, $\tau_1 = \bar{\lambda}_{\bar{Z}_f}$, $\tau_2 = \bar{\lambda}_{B_2(K_f)}$, $\mu = \tilde{\nu}$, and $\mu_1 = \nu^\# = \bar{\rho}^f_* \nu$, we obtain that $h_\nu(\bar{\lambda}_{\bar{Y}_f}) = h_{\tilde{\nu}}(\tilde{\lambda}) = h_{\nu^\#}(\bar{\lambda}_{\bar{Z}_f})$. This proves (1).

(2): The variational principle [16, Theorem 9.48] implies that

$$\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) = \sup_\mu h_\mu(\bar{\lambda}_{\bar{Y}_f}) = \sup_\mu h_{\bar{\rho}^f_* \mu}(\bar{\lambda}_{\bar{Z}_f}) \le \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Z}_f}),$$

where the supremum is taken over the set of $\bar{\lambda}$-invariant Borel probability measures μ on $\bar{Y}_f$. Since the opposite inequality $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Z}_f}) \le \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f})$ is trivially satisfied, this completes the proof of the proposition. □

Proposition 3.2 yields a strengthening of Proposition 3.5 for $\bar{\lambda}$-invariant probability measures ν on $\bar{Y}_f$ with completely positive entropy. We use the same notation as in the Propositions 2.1 and 3.5.

Corollary 3.6. *Suppose that the principal algebraic Γ-action λ_f on X_f has finite topological entropy. Then the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ on $(\bar{Y}_f, \nu)$ and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Z}_f, \nu^{\#})$ are measurably conjugate for every $\bar{\lambda}$-invariant Borel probability measure ν on $\bar{Y}_f$ with completely positive entropy.*

Proof. As in the proof of Proposition 2.1 we define $\theta_1 \colon \bar{Y}_f \to \tilde{Z}_f$ by (2.9) and set $\tilde{\nu}^{(1)} = (\theta_1)_*\nu$ and $\tilde{\nu} = (\theta_2)_*\tilde{\nu}^{(1)} = \theta_*\nu$. Then $\tilde{\nu}$ is $\tilde{\lambda}$-invariant by (2.14), and $\pi_*^{(1)}\tilde{\nu} = \nu^{\#}$. We write $\pi^{(2)} \colon \tilde{Z}_f \to K_f$ for the second coordinate projection, denote by $\xi_f = \pi_*^{(2)}\tilde{\nu}$ the projection of $\tilde{\nu}$ onto K_f, and note that the Γ-action $\tilde{\lambda}$ on $(\tilde{Z}_f, \tilde{\nu})$ has the zero-entropy Γ-action $\bar{\lambda}$ on (K_f, ξ_f) as a factor (cf. Proposition 3.2). Since $\tilde{\lambda}$ on $(\tilde{Z}_f, \tilde{\nu})$ is measurably conjugate to $\bar{\lambda}$ on $(\bar{Y}_f, \nu)$ and thus has completely positive entropy, we obtain a contradiction unless ξ_f is concentrated in a single point.

Since ξ_f is a point mass, the first coordinate projection $\pi^{(1)} \colon (\tilde{Z}_f, \tilde{\nu}) \to (\bar{Z}_f, \nu^{\#})$ is injective (mod $\tilde{\nu}$), and the Γ-actions $\tilde{\lambda}$ on $(\tilde{Z}_f, \tilde{\nu})$ and $\bar{\lambda}$ on $(\bar{Z}_f, \nu^{\#})$ are conjugate. This proves that the Γ-actions $\bar{\lambda}$ on $(\bar{Y}_f, \nu)$ and on $(\bar{Z}_f, \nu^{\#})$ are measurably conjugate. □

Having discussed the relation between $\bar{\lambda}$-invariant probability measures on $\bar{Y}_f$ and $\bar{Z}_f$ we turn to the corresponding question for measures on $\bar{Y}_f$ and their images under η.

Lemma 3.7. *There exists a unique $\bar{\lambda}$-invariant Borel probability measure ν_f on $\bar{Y}_f$ with $\eta_*\nu_f = \mu_f$, and the map $\eta \colon \bar{Y}_f \to X_f$ induces a conjugacy of the Γ-actions $\bar{\lambda}$ on $(\bar{Y}_f, \nu_f)$ and λ_f on (X_f, μ_f).*

Proof. Let ν be a $\bar{\lambda}$-invariant Borel probability measure on $\bar{Y}_f$ such that $\eta_*\nu = \mu_f$. If $\nu(\bar{Y}_f \smallsetminus Y_f) > 0$, the set

$$V = \{y \in \bar{Y}_f \mid y_{1_\Gamma} = 1\}$$

must have positive ν-measure, which implies that the closed subgroup

$$H = \{x \in X_f \mid x_{1_\Gamma} = 0\} \supset \eta(V)$$

has positive μ_f-measure. We set

$$K = \pi_{1_\Gamma}(X_f) \subset \mathbb{T},$$

observe that K is a closed subgroup of $\mathbb{T}$, and denote by μ_K the normalized Haar measure of K. Since $\mu_K(\{t\}) = \mu_K(\{0\}) = \mu_f(H) > 0$ for every $t \in K$, the group K must be finite, which implies that $X_f \subset K^\Gamma$ and hence $Y_f = \bar{Y}_f$, contrary to our assumption that $\nu(\bar{Y}_f \smallsetminus Y_f) > 0$. It follows that $\nu(B) = \nu(B \cap Y_f) = \mu_f(\eta(B))$ for every Borel set $B \subset \bar{Y}_f$, as claimed. Hence the map $\eta \colon \bar{Y}_f \to X_f$ induces a measure space isomorphism from $(\bar{Y}_f, \nu_f)$ to (X_f, μ_f) which carries the Γ-action $\bar{\lambda}_f$ on $(\bar{Y}_f, \nu_f)$ to λ_f on (X_f, μ_f). □

Theorem 3.8. *Let Γ be a countably infinite discrete amenable group, $f \in \mathbb{Z}\Gamma$, and assume that the principal algebraic Γ-action λ_f on X_f has finite topological entropy. Then $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Z}_f}) = \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) = \mathrm{h}_{\mathrm{top}}(\lambda_f)$.*

We start the proof of Theorem 3.8 with four lemmas. For any finite subset $F \subset \Gamma$ containing 1_Γ and any $Q \subset \Gamma$ we put

$$\mathrm{Int}_F Q = \{\gamma \in \Gamma \mid \gamma F \subset Q\}. \tag{3.2}$$

Lemma 3.9 ([30, Lemma 6.4]). *Let V be a finite dimensional vector space over $\mathbb{R}$, and let $k > \dim V$. Let $\phi_1, \dots, \phi_k$ be affine functions on V and $b_1, \dots, b_k \in \mathbb{R}$. Then there exist $a_1, \dots, a_k \in \{0,1\}$ such that $\bigcap_{j=1}^k W_j(a_j) = \varnothing$, where*

$$W_j(a_j) = \begin{cases} \{x \in V \mid \phi_j(x) < b_j\} & \text{if } a_j = 0, \\ \{x \in V \mid \phi_j(x) \geq b_j\} & \text{if } a_j = 1. \end{cases}$$

Lemma 3.10. *Suppose that $f \in \mathbb{R}\Gamma$ satisfies that $\mathrm{h}_{\mathrm{top}}(\lambda_f) < \infty$, and that $1_\Gamma \in E = \mathrm{supp}(f)$. Let $Q \Subset \Gamma$. For every nonzero $v \in \mathbb{R}\Gamma$, the product $v \cdot f^*$ is nonzero (since f is not a left zero divisor), and the restriction of $v \cdot f^*$ to $\mathrm{Int}_E Q$ depends only on the restriction $\pi_Q(v)$ of v to Q: for every $v, v' \in \mathbb{R}\Gamma$ with $\pi_Q(v) = \pi_Q(v')$, $\pi_{\mathrm{Int}_E Q}(v \cdot f^*) = \pi_{\mathrm{Int}_E Q}(v' \cdot f^*)$. Since the map $v \mapsto \bar{\rho}^f v = v \cdot f^*$ from $\mathbb{R}\Gamma$ to $\mathbb{R}\Gamma$ in (2.3) induces an injective map from $\mathbb{R}^Q$ to $\mathbb{R}^{QE^{-1}}$, the linear space*

$$V_Q = \{v = (v_\gamma)_{\gamma \in Q} \in \mathbb{R}^Q \mid \pi_{\mathrm{Int}_E Q}(v \cdot f^*) = 0\}, \tag{3.3}$$

has dimension $\dim V_Q \leq |QE^{-1} \smallsetminus \mathrm{Int}_E Q|$ (cf. (3.2)).

Proof. Since $\dim(\{w \in \mathbb{R}^{QE^{-1}} \mid \pi_{\mathrm{Int}_E Q}(w) = 0\}) = |QE^{-1} \smallsetminus \mathrm{Int}_E Q|$ and the map $\mathbb{R}^Q \to \mathbb{R}^{QE^{-1}}$ induced by $\bar{\rho}^f$ is injective, $\dim V_Q \leq |QE^{-1} \smallsetminus \mathrm{Int}_E Q|$, as claimed. □

For the next lemma we recall that a family of subsets $\mathcal{Z}$ of a finite set Z is said to *scatter* a set $J \subset Z$ if $\mathcal{Z} \cap J = \{C \cap J \mid C \in \mathcal{Z}\} = \mathcal{P}(J)$, the set of all subsets of J.

Lemma 3.11 (Sauer-Perles-Shelah [25], [28, Theorem 1], [31]). *Let Z be a finite set with cardinality $n \geq 1$ and let $\mathcal{Z}$ be a collection of subsets of Z. If $|\mathcal{Z}| > \sum_{i=0}^{k-1} \binom{|Z|}{i}$ for some $k \in \{1, \dots, |Z|\}$, then $\mathcal{Z}$ scatters a subset $J \subset Z$ of size k.*

Proof. For a proof see, e.g., [36]. □

Lemma 3.12 ([3, Lemma A.1]). *Let $0 < \beta < 1/2$. Then there exist $\kappa = \kappa(\beta) > 0$ and $m_0 = m_0(\beta) \in \mathbb{N} = \{1, 2, \dots\}$ with*

$$\lim\nolimits_{\beta \to 0} \kappa(\beta) = 0,$$

such that

$$\sum\nolimits_{i=0}^{\lfloor \beta m \rfloor} \binom{m}{i} \leq e^{\kappa m}$$

for all $m \in \mathbb{N}$ with $m \geq m_0$.

Proof of Theorem 3.8. For $\delta \in \Gamma$, the spaces X_f, Y_f do not change if we replace $f = \sum_{\gamma \in \Gamma} f_\gamma \cdot \gamma$ by $\delta f = \sum_{\gamma \in \Gamma} f_\gamma \cdot \delta\gamma$, and $Z_{\delta f} = \bar{\rho}^\delta(Z_f)$. For the proof of this theorem we may therefore assume without loss of generality that $1_\Gamma \in \mathrm{supp}(f)$.

Since the continuous shift-equivariant map $\eta \colon \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{T}^\Gamma$ in (2.1) sends $\bar{Y}_f$ onto X_f, we know that $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) \geq \mathrm{h}_{\mathrm{top}}(\lambda_f)$. It will suffice to show that $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) \leq \mathrm{h}_{\mathrm{top}}(\lambda_f)$.

Denote by $\mathsf{d}_{\mathbb{I}}(s,t) = |s - t|$ the Euclidean metric on the closed unit interval $\mathbb{I} = [0,1]$ and by $\mathsf{d}_{\mathbb{T}}$ the metric on $\mathbb{T}$ given by

$$\mathsf{d}_{\mathbb{T}}(s + \mathbb{Z}, t + \mathbb{Z}) = \min_{k \in \mathbb{Z}} |s - t - k|.$$

For any $F \Subset \Gamma$ we define continuous pseudometrics $\mathsf{d}_{\mathbb{I}}^{(F)}$ and $\mathsf{d}_{\mathbb{T}}^{(F)}$ on $\bar{Y}_f$ and X_f, respectively, by

$$\mathsf{d}_{\mathbb{I}}^{(F)}(y, y') := \max_{\gamma \in F^{-1}} \mathsf{d}_{\mathbb{I}}(y_\gamma, y'_\gamma), \quad y, y' \in \bar{Y}_f,$$

$$\mathsf{d}_{\mathbb{T}}^{(F)}(x,x') := \max_{\gamma\in F^{-1}} \mathsf{d}_{\mathbb{T}}(x_\gamma, x'_\gamma), \quad x,x'\in X_f.$$

For every $\varepsilon > 0$ we denote by $\mathsf{sep}(\bar{Y}_f, \mathsf{d}_{\mathbb{I}}^{(F)}, \varepsilon)$ and $\mathsf{sep}(X_f, \mathsf{d}_{\mathbb{T}}^{(F)}, \varepsilon)$ the maximal cardinalities of $(\mathsf{d}_{\mathbb{I}}^{(F)}, \varepsilon)$-separated subsets of $\bar{Y}_f$ and $(\mathsf{d}_{\mathbb{T}}^{(F)}, \varepsilon)$-separated subsets of X_f, respectively.

Let (F_n) be a left Følner sequence of Γ. By [6, Proposition 2.3] we have

$$\mathsf{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) = \sup_{\varepsilon>0}\limsup_{n\to\infty}\frac{\log \mathsf{sep}(\bar{Y}_f, \mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon)}{|F_n|},$$
$$\mathsf{h}_{\mathrm{top}}(\lambda_f) = \sup_{\varepsilon>0}\limsup_{n\to\infty}\frac{\log \mathsf{sep}(X_f, \mathsf{d}_{\mathbb{T}}^{(F_n)}, \varepsilon)}{|F_n|}.$$

Assume that $\mathsf{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) > \mathsf{h}_{\mathrm{top}}(\lambda_f)$. Then we can find some $0<\varepsilon<\frac{1}{\max(10,2\|f\|_1)}$ and $c>0$ such that, passing to a subsequence of (F_n) if necessary, one has

$$\mathsf{sep}(\bar{Y}_f, \mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon) \ge \mathsf{sep}(X_f, \mathsf{d}_{\mathbb{T}}^{(F_n)}, \varepsilon/3)\exp(c|F_n|) \tag{3.4}$$

for all $n\ge 1$.

We fix $n\ge1$ for the moment and choose a $(\mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon)$-separated subset $\mathcal{W}_n\subset \bar{Y}_f$ with $|\mathcal{W}_n| = \mathsf{sep}(\bar{Y}_f, \mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon)$. Then $\mathcal{W}_n$ is $(\mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon)$-spanning in $\bar{Y}_f$. Since $\eta^{-1}(X_f)\cap[0,1)^\Gamma$ is dense in $\bar{Y}_f$, we may move some of the points in $\mathcal{W}_n$ by less than $\varepsilon/10$ in the pseudometric $\mathsf{d}_{\mathbb{I}}^{(F_n)}$, if necessary, and without changing notation, so that $\mathcal{W}_n\subset\bar{Y}_f\cap[0,1)^\Gamma$, while remaining $(\mathsf{d}_{\mathbb{I}}^{(F_n)}, \varepsilon)$-spanning and $(\mathsf{d}_{\mathbb{I}}^{(F_n)}, 4\varepsilon/5)$-separated in $\bar{Y}_f$. Similarly, if $\mathcal{V}_n\subset X_f$ is a maximal $(\mathsf{d}_{\mathbb{T}}^{(F_n)}, \varepsilon/3)$-separated subset in X_f, then

$$X_f\subset \bigcup\nolimits_{x\in\mathcal{V}_n} B_{\mathbb{T}}^{(F_n)}(x,\varepsilon/3),$$

where $B_{\mathbb{T}}^{(F_n)}(x,\varepsilon/3)$ is the open $\mathsf{d}_{\mathbb{T}}^{(F_n)}$-ball in X_f with centre x and radius $\varepsilon/3$, and we can find, for every $n\ge1$, a point $z^{(n)}\in\mathcal{V}_n$ such that $|\eta(\mathcal{W}_n)\cap B_{\mathbb{T}}^{(F_n)}(z^{(n)},\varepsilon/3)|\ge\exp(c|F_n|)$ (cf. (3.4)). For every $n\ge1$ we set $\mathcal{W}'_n=\{y\in\mathcal{W}_n\mid \eta(y)\in B_{\mathbb{T}}^{(F_n)}(z^{(n)},\varepsilon/3)\}$ and denote by $\tilde{z}^{(n)}\in[0,1)^{F_n^{-1}}$ the unique point with $z_\gamma^{(n)}=\tilde{z}_\gamma^{(n)} \pmod 1$ for every $\gamma\in F_n^{-1}$.

For every $y\in\mathcal{W}'_n$ there is a unique $\tilde{y}\in\{-1,0,1\}^{F_n^{-1}}$ such that $|y_\gamma-\tilde{y}_\gamma-\tilde{z}_\gamma^{(n)}|<\varepsilon/3$ for every $\gamma\in F_n^{-1}$. We set

$$G_y^+=\{\gamma\in F_n^{-1}\mid\tilde{y}_\gamma=1\},\qquad G_y^\circ=\{\gamma\in F_n^{-1}\mid\tilde{y}_\gamma=0\},\qquad G_y^-=\{\gamma\in F_n^{-1}\mid\tilde{y}_\gamma=-1\}.$$

Since $\mathcal{W}'_n$ is $(\mathsf{d}_{\mathbb{I}}^{(F_n)}, 4\varepsilon/5)$-separated and $G_y^+\cup G_y^\circ\cup G_y^-=F_n^{-1}$, it is clear that $\tilde{y}\ne\tilde{y}'$ and hence $(G_y^+,G_y^-)\ne(G_{y'}^+,G_{y'}^-)$ for any $y\ne y'$ in $\mathcal{W}'_n$.

We recall that $1_\Gamma\in E=\mathrm{supp}(f)$ and define $\mathrm{Int}_E F_n^{-1}$ as in (3.2). For any $y\in\mathcal{W}_n$, the restrictions of $y\cdot f^*$ and $y|_{F_n^{-1}}\cdot f^*$ to $\mathrm{Int}_E F_n^{-1}$ coincide; since $y\cdot f^*\in\ell^\infty(\Gamma,\mathbb{Z})$, this implies that $y|_{F_n^{-1}}\cdot f^*$ and $(y|_{F_n^{-1}}-\tilde{z}^{(n)})\cdot f^*$ have integral coordinates on $\mathrm{Int}_E F_n^{-1}$. Furthermore, since $|y_\gamma-\tilde{y}_\gamma-\tilde{z}_\gamma^{(n)}|<\varepsilon/3$ for every $\gamma\in F_n^{-1}$, we obtain that

$$\big\|\big((y|_{F_n^{-1}}-\tilde{y}-\tilde{z}^{(n)})\cdot f^*\big)\big|_{\mathrm{Int}_E F_n^{-1}}\big\|_\infty<\frac{\varepsilon}{3}\cdot\|f\|_1<1,$$

so that

$$v(y):=y|_{F_n^{-1}}-\tilde{y}-\tilde{z}^{(n)}\in V_{F_n^{-1}}$$

for every $y\in\mathcal{W}'_n$ (cf. (3.3)).

Put $\mathcal{W}_n'' = \{(G_y^+, G_y^-) \mid y \in \mathcal{W}_n'\}$, $\mathcal{W}_n^+ = \{G_y^+ \mid y \in \mathcal{W}_n'\}$ and $\mathcal{W}_n^- = \{G_y^- \mid y \in \mathcal{W}_n'\}$. Since $|\mathcal{W}_n''| = |\mathcal{W}_n'| \geq \exp(c|F_n|)$ it is clear that $\max(|\mathcal{W}_n^+|, |\mathcal{W}_n^-|) \geq \exp(c|F_n|/2)$.

Suppose that $|\mathcal{W}_n^+| \geq \exp(c|F_n|/2)$ for infinitely many $n \geq 1$ (if $|\mathcal{W}_n^-| \geq \exp(c|F_n|/2)$ for infinitely many n the proof is completely analogous). By passing to a subsequence we may assume that $|\mathcal{W}_n^+| \geq \exp(c|F_n|/2)$ for every $n \geq 1$. By Lemma 3.12 there exists $\beta > 0$ such that $\kappa = \kappa(\beta) < c/2$ and $\sum_{i=0}^{\lfloor \beta |F_n| \rfloor} \binom{|F_n|}{i} < \exp(\kappa |F_n|) < \exp(c|F_n|/2) \leq |\mathcal{W}_n^+|$ for every sufficiently large $n \geq 1$. Lemma 3.11 implies that $\mathcal{W}_n^+$ scatters a subset $J_n^+ \subset F_n^{-1}$ of size $\geq \beta |F_n^{-1}|$.

We are going to show that $\dim V_{F_n^{-1}} \geq |J_n^+|$ for infinitely many $n \geq 1$, thereby contradicting Lemma 3.10. For this we define, for every $\gamma \in F_n^{-1}$, a linear functional $\phi_\gamma \colon V_{F_n^{-1}} \to \mathbb{R}$ by setting $\phi_\gamma(v) = v_\gamma$ for every $v \in V_{F_n^{-1}}$. For every $y \in \mathcal{W}_n'$ and $\gamma \in F_n^{-1}$, we have the following possibilities:

$$\begin{aligned} &\gamma \in G_y^+ \ \text{ and } \ \phi_\gamma(v(y)) + \tilde{z}_\gamma^{(n)} = y_\gamma - 1 < 0, \\ &\gamma \in G_y^\circ \ \text{ and } \ 1 > \phi_\gamma(v(y)) + \tilde{z}_\gamma^{(n)} = y_\gamma \geq 0, \\ &\gamma \in G_y^- \ \text{ and } \ \phi_\gamma(v(y)) + \tilde{z}_\gamma^{(n)} = y_\gamma + 1 \geq 1. \end{aligned}$$

In particular, $\phi_\gamma(v(y)) < -\tilde{z}_\gamma^{(n)}$ if $\gamma \in G_y^+$, and $\phi_\gamma(v(y)) \geq -\tilde{z}_\gamma^{(n)}$ if $\gamma \in F_n^{-1} \smallsetminus G_y^+$.

We can thus find, for every subset $H \subset J_n^+$, a $y \in \mathcal{W}_n'$ for which

$$\phi_\gamma(v(y)) < -\tilde{z}_\gamma^{(n)} \quad \text{if} \quad \gamma \in H,$$

and

$$\phi_\gamma(v(y)) \geq -\tilde{z}_\gamma^{(n)} \quad \text{if} \quad \gamma \in J_n^+ \smallsetminus H.$$

According to Lemma 3.9 this means that $\dim V_{F_n^{-1}} \geq |J_n^+| \geq \beta |F_n|$. If we set $Q = F_n^{-1}$, where n is sufficiently large, we obtain a contradiction to Lemma 3.10. This contradiction shows that $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) \leq \mathrm{h}_{\mathrm{top}}(\lambda_f)$ and completes the proof of Theorem 3.8. □

Lemma 3.13. *Any $\bar{\lambda}$-invariant Borel probability measure ν on $\bar{Y}_f$ satisfies that $h_\nu(\bar{\lambda}_{\bar{Y}_f}) = h_{\eta_*\nu}(\lambda_f)$.*

Proof. For every $x \in X_f$ we denote by $\mathrm{h}_{\mathrm{top}}\big(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(x)\big)$ the fibre entropy of $\bar{\lambda}_{\bar{Y}_f}$, given x, defined in [18, Definition 6.7]. The proof of Theorem 3.8 shows that $\mathrm{h}_{\mathrm{top}}\big(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(x)\big) = 0$ for every $x \in X_f$. By [18, Lemmas 6.8 and 6.9], $h_\nu(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(\mathcal{B}_{X_f})) = 0$ for every $\bar{\lambda}$-invariant Borel probability measure ν on $\bar{Y}_f$, where $\mathcal{B}_{X_f}$ is the Borel σ-algebra of X_f. By [5, Theorem 0.2] or [16, Theorem 9.16], $h_\nu(\bar{\lambda}_{\bar{Y}_f}) = h_\nu(\bar{\lambda}_{\bar{Y}_f} | \mathcal{B}_{X_f}) + h_{\eta_*\nu}(\lambda_f) = h_{\eta_*\nu}(\lambda_f)$. □

The coincidence of topological entropies of the Γ-actions λ_f and $\bar{\lambda}_{\bar{Y}_f}$ in Theorem 3.8 is not quite as obvious as one might think. As noted in the proof of Lemma 3.13, the conditional fibre entropy $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(x))$ is equal to zero for every $x \in X_f$ whenever Γ is amenable and $f \in \mathbb{Z}\Gamma$ is not a left zero divisor. This is no longer true if f *is* a left zero divisor (in which case the topological entropies $\mathrm{h}_{\mathrm{top}}(\lambda_f)$ and $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f})$ are infinite by Lemma 3.1). A slight modification of the proof of Lemma 3.1 yields the following result.

Proposition 3.14. *Let Γ be a countably infinite amenable group, and let $f \in \mathbb{Z}\Gamma$ be a left zero divisor in $\mathbb{R}\Gamma$. Then the fibre entropy $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(0_{X_f}))$ is positive.*

Proof. Take a compatible metric d on $\bar{Y}_f$ such that $\mathsf{d}(y,z) \geq |y_{1_\Gamma} - z_{1_\Gamma}|$ for all $y, z \in \bar{Y}_f$.

If $f \in \mathbb{Z}\Gamma$ is a left zero divisor we choose $g = \sum_{\gamma\in\Gamma} g_\gamma \cdot \gamma \in \mathbb{R}\Gamma$ with $g_{1_\Gamma} = \|g\|_\infty > 0$ and $fg = 0$. Following the proof of Lemma 3.1 we note that $cg^* \in W_f$ and $\eta(cg^*) \in X_f$ for every $c \in \mathbb{R}$. Put $E = \mathrm{supp}(g^*)$ and choose a maximal set $D \subset \Gamma$ such that the translates $\{\delta E \mid \delta \in D\}$ are disjoint. Then $DEE^{-1} = \Gamma$ (cf. the proof of Lemma 3.1). Since the sets $\delta E, \delta \in D$, are disjoint, we obtain, for every $z = (z_\delta)_{\delta\in D} \in \{-1,1\}^D$ and every $c \in \mathbb{R}$ with $0 < c < \frac{1}{2\|g\|_\infty}$, a point $w^{(c,z)} = c \cdot \sum_{\delta\in D} z_\delta \bar{\lambda}^\delta g^* \in W_f$ with $\|w^{(c,z)}\|_\infty = c\|g\|_\infty$ and $w^{(c,z)}_\delta = cz_\delta\|g\|_\infty$ for every $\delta \in D$.

We set $x^{(c,z)} = \eta(w^{(c,z)}) \in X_f$ and denote by $y^{(c,z)}$ the unique point in Y_f satisfying $\eta(y^{(c,z)}) = x^{(c,z)} = \eta(w^{(c,z)})$. For every $\delta \in D$,

$$y^{(c,z)}_\delta = \begin{cases} c\|g\|_\infty & \text{if } \ z_\delta = 1, \\ 1 - c\|g\|_\infty & \text{if } \ z_\delta = -1. \end{cases}$$

As $c \searrow 0$, $y^{(c,z)}$ converges coordinate-wise to a point $y^{(z)} \in \bar{Y}_f$ with

$$y^{(z)}_\delta = \begin{cases} 0 & \text{if } \ z_\delta = 1, \\ 1 & \text{if } \ z_\delta = -1, \end{cases}$$

for $\delta \in D$. With the exception of the single point $z' = (z'_\delta)_{\delta\in D}$ with $z'_\delta = 1$ for every $\delta \in D$, all the points $y^{(z)}, z \in \{-1,1\}^D$ lie in $\bar{Y}_f \smallsetminus Y_f$ and satisfy that $\eta(y^{(z)}) = 0_{X_f}$. As in the proof of Lemma 3.1 we conclude that the fibre entropy $\mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f} | \eta^{-1}(0_{X_f}))$ is positive. □

Definition 3.15 ([35])**.** A continuous action τ of a countably infinite amenable group Γ on a compact metrizable space X is *intrinsically ergodic* if it has finite topological entropy and there exists a unique τ-invariant Borel probability measure μ on X with $h_\mu(\tau) = \mathrm{h}_{\mathrm{top}}(\tau)$.

If Γ is a countably infinite amenable group, and if $f \in \mathbb{Z}\Gamma$ satisfies that $\mathrm{h}_{\mathrm{top}}(\lambda_f) < \infty$, then the principal algebraic action λ_f on X_f is intrinsically ergodic (with unique maximal measure μ_f) if and only if λ_f has completely positive entropy w.r.t. μ_f ([4, Theorem 8.6]). If λ_f is intrinsically ergodic on X_f, the next result extends this property to the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$.

Proposition 3.16. *Suppose that Γ is a countably infinite discrete amenable group, $f \in \mathbb{Z}\Gamma$, and λ_f is intrinsically ergodic on X_f. Then the following are true.*

(1) *The Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$ are intrinsically ergodic;*
(2) *The maximal entropy measures of the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$ have completely positive entropy.*

The proof of Proposition 3.16 consists of three lemmas.

Lemma 3.17. *If λ_f is intrinsically ergodic on X_f, then the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ on $(\bar{Y}_f, \nu_f)$ and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Z}_f, \nu_f^\#)$ (with $\nu_f^\# := \bar{\rho}^f_* \nu_f$) have completely positive entropy.*

Proof. Since λ_f on (X_f, μ_f) is measurably conjugate to $\bar{\lambda}_{\bar{Y}_f}$ on $(\bar{Y}_f, \nu_f)$ by Lemma 3.7, and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Z}_f, \nu_f^\#)$ is a factor of $\bar{\lambda}_{\bar{Y}_f}$ on $(\bar{Y}_f, \nu_f)$, all these actions have completely positive entropy. □

Proof of Proposition 3.16. If ν is a $\bar{\lambda}$-invariant Borel probability measure on $\bar{Y}_f$ with entropy $h_\nu(\bar{\lambda}_{\bar{Y}_f}) = \mathrm{h}_{\mathrm{top}}(\bar{\lambda}_{\bar{Y}_f}) = \mathrm{h}_{\mathrm{top}}(\lambda_f)$ (cf. Theorem 3.8), then Lemma 3.13 implies that $\eta_*\nu = \mu_f$, the unique λ_f-invariant Borel probability measure on X_f with maximal entropy. By Lemma 3.7, $\nu = \nu_f$, and the Γ-actions λ_f and $\bar{\lambda}_{\bar{Y}_f}$ on (X_f, μ_f) and $(\bar{Y}_f, \nu_f)$ are conjugate. Lemma 3.17 completes the proof of Proposition 3.16. □

Theorem 3.18. *Suppose that Γ is a countably infinite amenable group, $f \in \mathbb{Z}\Gamma$, and the principal algebraic Γ-action λ_f on X_f is intrinsically ergodic. Then the principal algebraic Γ-action λ_f on (X_f, μ_f) is measurably conjugate to the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Y}_f, \nu_f)$ and $(\bar{Z}_f, \nu_f^{\#})$, respectively.*

Proof. If λ_f is intrinsically ergodic on X_f, then $\mathrm{h}_{\mathrm{top}}(\lambda_f) < \infty$ and μ_f has c.p.e. (cf. Definition 3.15). Lemma 3.7 shows that the Γ-actions λ_f on (X_f, μ_f) and $\bar{\lambda}$ on $(\bar{Y}_f, \nu_f)$ are measurably conjugate, and the Γ-actions $\bar{\lambda}_{\bar{Y}_f}$ and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Y}_f, \nu_f)$ and $(\bar{Z}_f, \nu_f^{\#})$ are measurably conjugate by Corollary 3.6. □

4. Generators of intrinsically ergodic principal algebraic actions

In this section we apply Theorem 3.18 to find generators of intrinsically ergodic principal algebraic actions of a countably infinite amenable group Γ.

Let $f \in \mathbb{Z}\Gamma$ be a nonzero element such that the principal algebraic Γ-action λ_f on X_f is intrinsically ergodic. We view $X_f \subset \mathbb{T}^\Gamma$ as a subset of $[0,1)^\Gamma$ as in (2.1) – (2.4) by identifying Y_f and X_f through η and set, for every $j \in \mathbb{Z}$,

$$B[j] = \bigl\{x \in X_f \mid \textstyle\sum_{\gamma \in \mathrm{supp}(f)} f_\gamma x_\gamma = j\bigr\} = \bigl\{x \in X_f \mid (\bar{\rho}^f x)_{1_\Gamma} = j\bigr\}. \tag{4.1}$$

The following corollaries are immediate consequences of Theorem 3.18.

Corollary 4.1. *Put*

$$\mathcal{B}_f = \begin{cases} \{B[j] \mid j = -\|f^-\|_1 + 1, \dots, \|f^+\|_1 - 1\} & \text{if both } f^+ \text{ and } f^- \text{ are nonzero,} \\ \{B[j] \mid j = 0, \dots, \|f^+\|_1 - 1\} & \text{if } f^+ \neq 0 \text{ and } f^- = 0, \\ \{B[j] \mid j = -\|f^-\|_1 + 1, \dots, 0\} & \text{if } f^+ = 0 \text{ and } f^- \neq 0. \end{cases} \tag{4.2}$$

Then $\mathcal{B}_f$ is a Borel partition of X_f which is a generator (mod μ_f) *for λ_f.*

Corollary 4.2. *The Borel partition $\mathcal{C}_f = \{C_j \mid j = 0, \dots, \|f\|_1 - 1\}$ of X_f with*

$$C_j = \bigl\{x \in X_f \mid j/\|f\|_1 \le x_{1_\Gamma} < (j+1)/\|f\|_1 \pmod 1\bigr\} \text{ for } j = 0, \dots, \|f\|_1 - 1,$$

is a generator (mod μ_f) *for λ_f.*

By imposing further conditions on Γ and f we can sometimes find slightly smaller generators (mod μ_f) for λ_f in Corollary 4.1.

Corollary 4.3. *Suppose that the group Γ in Theorem 3.18 is left* (*or, equivalently, right*) *orderable. If $f \in \mathbb{Z}\Gamma$ satisfies that $|\mathrm{supp}(f)| \ge 2$, then the collection of sets*

$$\mathcal{B}'_f = \bigl\{B[j] \mid j = -\|f^-\|_1 + 1, \dots, \|f^+\|_1 - 1\bigr\},$$

defined as in (4.1)*, is a generator* (mod μ_f) *for λ_f.*

For the proof of Corollary 4.3 we require an additional lemma. For notation we refer to Lemma 3.7.

Lemma 4.4. *Suppose that the group Γ in Theorem 3.18 is left (or, equivalently, right) orderable. If $f \in \mathbb{Z}\Gamma$ satisfies that $|\mathrm{supp}(f)| \geq 2$, then $K = \pi_{1_\Gamma}(X_f) = \mathbb{T}$ and hence $(\pi_{1_\Gamma})_* \mu_f = \mu_{\mathbb{T}}$, the Lebesgue measure on $\mathbb{T}$.*

Proof. Since $K \subset \mathbb{T}$ is a closed subgroup, it is either finite or equal to $\mathbb{T}$. If K is finite we choose $L \geq 1$ such that $LK = \{Lt \mid t \in K\} = \{0\}$ and conclude from (1.2) that L lies in the left ideal $(f) \subset \mathbb{Z}\Gamma$ generated by f. Then there exists $h \in \mathbb{Z}\Gamma$ such that $L = hf$ or, equivalently, $1 = h \cdot \frac{1}{L} f$. In other words, the rational group ring $\mathbb{Q}\Gamma$ contains the nontrivial unit $\frac{1}{L} f$, in violation of [27, Lemmas 13.1.7 and 13.1.10].

If $\pi_{1_\Gamma}(X_f) = \mathbb{T}$, then obviously $(\pi_{1_\Gamma})_* \mu_f = \mu_{\mathbb{T}}$. □

Proof of Corollary 4.3. If both f^+ and f^- are nonzero our assertion follows from Corollary 4.1. If $f^- = 0$, then

$$B[0] = \{x \in X_f \mid \textstyle\sum_{\gamma \in \mathrm{supp}(f)} f_\gamma x_\gamma = 0\} = \{x \in X_f \mid x_\gamma = 0 \ \text{ for every } \ \gamma \in \mathrm{supp}(f)\}.$$

Hence $\mu_f(B[0]) \leq \mu_f(\{x \in X_f \mid x_\gamma = 0\})$ for every $\gamma \in \Gamma$, so that $\mu_f(B[0]) = 0$ by Lemma 4.4. By Corollary 4.1, $\{B[j] \mid j = 1, \dots, \|f^+\|_1 - 1\}$ is a generator (mod μ_f) for λ_f.

If $f^+ = 0$ the proof is completely analogous. □

5. Examples

Let Γ be a countably infinite discrete amenable group, $f \in \mathbb{Z}\Gamma$, and let λ_f be the principal algebraic Γ-action on X_f in Definition 1.1. In order to apply Theorem 3.18 and its corollaries to λ_f we require the action λ_f to be intrinsically ergodic.

5.1. **Intrinsically ergodic principal algebraic $\mathbb{Z}^d$-actions.** If $\Gamma = \mathbb{Z}^d$ for some $d \geq 1$, the conditions for principal algebraic Γ-actions to be intrinsically ergodic are well understood: if f is nonzero and not divisible by a generalized cyclotomic polynomial, then λ_f is intrinsically ergodic [29, Theorem 11.2, Propositions 19.4 and 20.5].

Example 5.1. The matrix $M = \left[\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{smallmatrix}\right] \in \mathrm{SL}(4, \mathbb{Z})$ defines a nonhyperbolic ergodic automorphism α_M of $\mathbb{T}^4$. The question of finding 'nice' finite generating partitions for such automorphisms was discussed in [20, Theorem 1]. By observing that α_M is algebraically conjugate to the intrinsically ergodic algebraic $\mathbb{Z}$-action λ_f on (X_f, μ_f) for the characteristic polynomial $f = u^4 - u^3 - u^2 - u + 1$ of M and applying Corollary 4.2 we see that the sets

$$C_j = \left\{x \in X_f \mid \tfrac{j}{5} \leq x_0 < \tfrac{j+1}{5} \pmod 1\right\}, \ 0 \leq j \leq 4,$$

form a generating partition for λ_f w.r.t. μ_f on X_f. When translating this information back to α_M we obtain the generator

$$\mathcal{D} = \left\{D_j = \{t = (t_1, t_2, t_3, t_4) \in \mathbb{T}^4 \mid \tfrac{j}{5} \leq t_1 < \tfrac{j+1}{5} \pmod 1\} \,\Big|\, j = 0, \dots, 4\right\}$$

for α_M w.r.t. Lebesgue measure $\mu_{\mathbb{T}^4}$ on $\mathbb{T}^4$.

Corollary 4.1 shows that α_M on $(\mathbb{T}^4, \mu_{\mathbb{T}^4})$ also has the 4-element generator corresponding to $\{B[-2], B[-1], B[0], B[1]\}$ in (4.2).

Example 5.2. Let $\Gamma = \mathbb{Z}^2$, and let $f = 1 - u_1 - u_2 \in \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}] \cong \mathbb{Z}[\mathbb{Z}^2]$. By [29, Proposition 19.7], $\mathrm{h}_{\mathrm{top}}(\lambda_f) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3) > 0$, where $L(2, \chi_3)$ is the Dirichlet L-function defined there. Since λ_f is intrinsically ergodic, Corollary 4.1 shows that λ_f on (X_f, μ_f) has the 2-element generator $\{B[-1], B[0]\}$ (mod μ_f) defined as in (4.1).

5.2. **Intrinsically ergodic principal algebraic actions of amenable groups.** If Γ is a countably infinite discrete amenable group, establishing the intrinsic ergodicity of a principal algebraic action λ_f, $f \in \mathbb{Z}\Gamma$, is much more delicate than for $\Gamma = \mathbb{Z}^d$. A sufficient condition for intrinsic ergodicity of λ_f can be expressed in terms of homoclinic points: a point $x = (x_s) \in X_f$ is *summable homoclinic* if $\sum_{s \in \Gamma} \|x_s\| < \infty$, where $\|t\|$ denotes the distance from 0 of a point $t \in \mathbb{T}$ (cf. [24]). Clearly, every summable homoclinic point $x \in X_f$ is *homoclinic*, i.e., $\lim_{s \to \infty} \lambda_f^s x = 0$ (cf. e.g., [21, Definition 3.1] or [4]).

Proposition 5.3. *Let Γ be a countably infinite amenable discrete group, $f \in \mathbb{Z}\Gamma$, and let $\Delta^1(X_f) \subset X_f$ be the group of summable homoclinic points of the principal algebraic Γ-action λ_f on X_f. If $\Delta^1(X_f)$ is dense in X_f and $\mathrm{h}_{\mathrm{top}}(\lambda_f) < \infty$ then λ_f is intrinsically ergodic.*

The converse of Proposition 5.3 is clearly not true: the principal algebraic $\mathbb{Z}$-action λ_f on X_f (or, equivalently, the automorphism α_M of $\mathbb{T}^4$) in Example 5.1 is intrinsically ergodic, but has no nonzero homoclinic points (cf. [21, Example 3.4]).

Proof of Proposition 5.3. According to (1.2), the dual group $\widehat{X_f}$ is given by $\widehat{X_f} = \mathbb{Z}\Gamma/(f)$ and is, in particular, a finitely generated left $\mathbb{Z}\Gamma$-module. By [4, Theorem 7.8], $\Delta^1(X_f) \subset \mathrm{IE}(X_f)$, the closed subgroup of X_f defined in [4, Definition 7.2], and hence $\mathrm{IE}(X_f) = X_f$ by assumption. By [4, Corollary 8.4 and Theorem 8.6], λ_f is intrinsically ergodic. $\square$

5.2.1. *Expansive principal algebraic actions.* For every countably infinite discrete group Γ and every $f \in \mathbb{Z}\Gamma$, the principal algebraic Γ-action λ_f on X_f in Definition 1.1 is expansive if and only if the map $\bar{\rho}^f \colon \ell^\infty(\Gamma, \mathbb{R}) \to \ell^\infty(\Gamma, \mathbb{R})$ in (2.3) is injective or, equivalently, if f is invertible in $\ell^1(\Gamma, \mathbb{R})$ ([8, Theorem 3.2]). If this is the case, $w^\Delta := (f^*)^{-1} \in W_f$ since $\bar{\rho}^f(w^\Delta) = 1_\Gamma$. By [8, Proposition 4.2], the map $\bar{\rho}^{w^\Delta} \colon \ell^\infty(\Gamma, \mathbb{Z}) \to \ell^\infty(\Gamma, \mathbb{R})$ is continuous in the weak*-topology on closed, bounded subsets of $\ell^\infty(\Gamma, \mathbb{Z})$, and the map $\xi := \eta \circ \bar{\rho}^{w^\Delta} \colon \ell^\infty(\Gamma, \mathbb{Z}) \to \mathbb{T}^\Gamma$ satisfies that $\xi(\{v \in \ell^\infty(\Gamma, \mathbb{Z}) \mid \|v\|_\infty \leq \|f\|_1/2\}) = X_f$ (cf. [8, Lemma 4.5]). Since $\xi(\mathbb{Z}\Gamma) \subset \Delta^1(X_f)$, the continuity of ξ implies that $\Delta^1(X_f)$ is dense in X_f. If Γ is amenable, we conclude from Proposition 5.3 that every expansive principal algebraic Γ-action is intrinsically ergodic.

If Γ is amenable and λ_f is expansive, the partitions $\mathcal{B}_f$ and $\mathcal{C}_f$ in the Corollaries 4.1 and 4.2 are obviously generators (and not only generators (mod μ_f)) for λ_f.

We mention two examples, taken from [9]. Let $\mathbb{H} \subset \mathrm{SL}(3, \mathbb{Z})$ be the discrete Heisenberg group, generated by

$$u_1 = \begin{bmatrix} 1&0&0\\0&1&1\\0&0&1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1&1&0\\0&1&0\\0&0&1 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} 1&0&1\\0&1&0\\0&0&1 \end{bmatrix}. \tag{5.1}$$

Example 5.4 ([9, Example 8.4])**.** Let $f = |a_1| + |a_2| + |a_3| + a_1 \cdot u_1 + a_2 \cdot u_2 + a_3 \cdot u_3 \in \mathbb{Z}\mathbb{H}$. Then the principal algebraic $\mathbb{H}$-action λ_f is expansive if and only if $a_1 a_2 \neq 0$ and $a_3 > 0$. In these cases the partitions $\mathcal{B}_f$ and $\mathcal{C}_f$ in the Corollaries 4.1 – 4.2 are generators for λ_f.

Example 5.5. Let $f = 1 - u_1 - u_2 \in \mathbb{Z}\mathbb{H}$. The principal algebraic $\mathbb{H}$-action λ_f on (X_f, μ_f) has zero entropy by [7, Theorem 11] or [22, Theorem 9.2], so that the hypotheses of Theorem 3.18

are not satisfied. We do not know whether the $\mathbb{H}$-actions λ_f on (X_f, μ_f) and $\bar{\lambda}_{\bar{Z}_f}$ on $(\bar{Z}_f, \nu_f^{\#})$ are measurably conjugate and whether $\mathcal{B}_f = \{B[-1], B[0]\}$ is a generator $(\mathrm{mod}\ \mu_f)$ for λ_f.

6. Summable homoclinic points of nonexpansive principal algebraic actions

The existence of a nonzero summable homoclinic point for a nonexpansive principal algebraic action in the second part of Example 5.5 is an interesting fact. For $\Gamma = \mathbb{Z}^d$ this phenomenon is well understood: for $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \in \mathbb{Z}[\mathbb{Z}^d]$ we denote by

$$\mathsf{U}(f) = \{\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{S}^d \mid f(\mathbf{z}) = 0\} \tag{6.1}$$

the *unitary variety* of f, where $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$. According to [29, Theorem 6.5], λ_f is expansive if and only if $\mathsf{U}(f) = \varnothing$. If f is nonzero and irreducible, then [24, Theorem 3.2] shows that $\Delta^1(X_f) \neq \{0\}$ if and only if the dimension of $\mathsf{U}(f) \subset \mathbb{S}^d$ is $\leq d-2$. In this case $\Delta^1(X_f)$ is dense in X_f, λ_f is intrinsically ergodic, and the partitions $\mathcal{B}_f$ and $\mathcal{C}_f$ are generators $(\mathrm{mod}\ \mu_f)$ for λ_f.

For nonabelian groups Γ, examples of nonexpansive principal algebraic actions with summable homoclinic points are much harder to come by. In order to present a class of such actions we assume for the remainder of this section that Γ is a countably infinite discrete group with center H. We say $f \in \mathbb{R}\Gamma$ is *well-balanced* ([1, Definition 1.2]) if

(1) $\sum_{s\in\Gamma} f_s = 0$,
(2) $f_s \leq 0$ for every $s \in \Gamma \smallsetminus \{1_\Gamma\}$,
(3) $f = f^*$,
(4) $\mathrm{supp}(f)$ generates Γ.

We shall prove the following theorem.

Theorem 6.1. *Assume that for any finite $F \subseteq \Gamma$ there is some s in the center H of Γ such that none of s, s^2, s^3 is in F (this happens for example when $H^6 = \{s^6 \mid s \in H\}$ is infinite). Also assume that Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Let $f \in \mathbb{Z}\Gamma$ be well-balanced. Then $\Delta^1(X_f)$ is dense in X_f.*

Corollary 6.2. *Assume that the group Γ in Theorem 6.1 is amenable. If $f \in \mathbb{Z}\Gamma$ is well-balanced, then the principal algebraic Γ-action λ_f on X_f is intrinsically ergodic.*

Proof. The proof of Corollary 6.11 implies that f is not a left zero divisor in $\mathbb{Z}\Gamma$, so that $\mathrm{h}_{\mathrm{top}}(\lambda_f) < \infty$ by Lemma 3.1. Now apply Theorem 6.1 and Proposition 5.3. □

The proof of Theorem 6.1 requires a brief excursion into Banach algebras.

Lemma 6.3. *Let $\mathcal{A}$ be a unital Banach algebra such that $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. Let $0 < c < 1$ and $q, r \in \mathcal{A}$ such that $\|r\| \leq 1 - c$ and $\|q\| \leq c$. Then*

$$\sum_{k=0}^{\infty} \|r^k (c-q)^3 (1-q)^{-(k+1)}\| < \infty.$$

The proof of Lemma 6.3 requires some auxiliary results.

Lemma 6.4. *Let $0 < c < 1$. For any $x, y > 0$ one has*

$$\frac{c^x (1-c)^y (x+y)^{x+y}}{x^x y^y} \leq 1.$$

Proof. Fix $y > 0$ and put

$$\phi(x) = \log \frac{c^x(1-c)^y(x+y)^{x+y}}{x^x y^y} = x\log c + y\log(1-c) + (x+y)\log(x+y) - x\log x - y\log y$$

for $x > 0$. Then

$$\phi'(x) = \log c + \log(x+y) - \log x = \log \frac{c(x+y)}{x}.$$

Clearly $\phi' > 0$ on the interval $(0, \frac{cy}{1-c})$, $\phi' = 0$ at $\frac{cy}{1-c}$, and $\phi' < 0$ on $(\frac{cy}{1-c}, \infty)$. Thus ϕ attains its maximum value at $\frac{cy}{1-c}$. Since $\phi(\frac{cy}{1-c}) = 0$, one concludes that $\phi(x) \le 0$ for all $x > 0$. ☐

Lemma 6.5. *Let $0 < c < 1$ and $k \in \mathbb{N}$. Let g_k be the cubic polynomial given by*

$$\begin{aligned} g_k(x) &= -(x+1)(x+2)(x+3) + 3c(x+k+1)(x+2)(x+3) \\ &\quad - 3c^2(x+k+1)(x+k+2)(x+3) + c^3(x+k+1)(x+k+2)(x+k+3) \\ &= (c-1)^3x^3 + (3c(c-1)^2k + 6(c-1)^3)x^2 \\ &\quad + (3c^2(c-1)k^2 + 3c(c-1)(4c-5)k + 11(c-1)^3)x \\ &\quad + c^3k^3 + (6c^3 - 9c^2)k^2 + (11c^3 - 27c^2 + 18c)k + 6(c-1)^3. \end{aligned} \tag{6.2}$$

For $\eta > 0$, put

$$y_{k,\eta,\pm} = \frac{ck}{1-c} - 2 \pm \frac{1}{1-c}\sqrt{\eta k + \tfrac{1}{3}(1-c)^2}. \tag{6.3}$$

Then

$$g_k(y_{k,\eta,\pm}) = (c^2+c)k \pm \big((3c-\eta)k + \tfrac{2}{3}(c-1)^2\big)\sqrt{\eta k + \tfrac{1}{3}(1-c)^2}.$$

Proof. This is a direct computation:

$$\begin{aligned} g_k(y_{k,\eta,\pm}) &= (c-1)^3 y_{k,\eta,\pm}^3 + (3c(c-1)^2k + 6(c-1)^3)y_{k,\eta,\pm}^2 \\ &\quad + (3c^2(c-1)k^2 + 3c(c-1)(4c-5)k + 11(c-1)^3)y_{k,\eta,\pm} \\ &\quad + c^3k^3 + (6c^3 - 9c^2)k^2 + (11c^3 - 27c^2 + 18c)k + 6(c-1)^3 \\ &= (c-1)^3(\tfrac{ck}{1-c} - 2)((\tfrac{ck}{1-c} - 2)^2 + 3\tfrac{1}{(1-c)^2}(\eta k + \tfrac{1}{3}(1-c)^2)) \\ &\quad + (3c(c-1)^2k + 6(c-1)^3)((\tfrac{ck}{1-c} - 2)^2 + \tfrac{1}{(1-c)^2}(\eta k + \tfrac{1}{3}(1-c)^2)) \\ &\quad + (3c^2(c-1)k^2 + 3c(c-1)(4c-5)k + 11(c-1)^3)(\tfrac{ck}{1-c} - 2) \\ &\quad + c^3k^3 + (6c^3 - 9c^2)k^2 + (11c^3 - 27c^2 + 18c)k + 6(c-1)^3 \\ &\quad \pm (c-1)^3(\tfrac{ck}{1-c} - 2)^2\tfrac{3}{1-c}\sqrt{\eta k + \tfrac{1}{3}(1-c)^2} \\ &\quad \pm (c-1)^3\tfrac{1}{(1-c)^3}(\eta k + \tfrac{1}{3}(1-c)^2)\sqrt{\eta k + \tfrac{1}{3}(1-c)^2} \\ &\quad \pm 2(3c(c-1)^2k + 6(c-1)^3)(\tfrac{ck}{1-c} - 2)\tfrac{1}{1-c}\sqrt{\eta k + \tfrac{1}{3}(1-c)^2} \\ &\quad \pm (3c^2(c-1)k^2 + 3c(c-1)(4c-5)k + 11(c-1)^3)\tfrac{1}{1-c}\sqrt{\eta k + \tfrac{1}{3}(1-c)^2} \\ &= -(ck + 2(c-1))((ck + 2(c-1))^2 + (3\eta k + (1-c)^2)) \\ &\quad + (ck + 2(c-1))(3(ck + 2(c-1))^2 + (3\eta k + (1-c)^2)) \\ &\quad - (3c^2k^2 + 3c(4c-5)k + 11(c-1)^2)(ck + 2(c-1)) \\ &\quad + c^3k^3 + (6c^3 - 9c^2)k^2 + (11c^3 - 27c^2 + 18c)k + 6(c-1)^3 \end{aligned}$$

$$\begin{aligned}
&\mp 3(ck+2(c-1))^2\sqrt{\eta k+\tfrac{1}{3}(1-c)^2}\\
&\mp (\eta k+\tfrac{1}{3}(1-c)^2)\sqrt{\eta k+\tfrac{1}{3}(1-c)^2}\\
&\pm 6(ck+2(c-1))^2\sqrt{\eta k+\tfrac{1}{3}(1-c)^2}\\
&\mp (3c^2k^2+3c(4c-5)k+11(c-1)^2)\sqrt{\eta k+\tfrac{1}{3}(1-c)^2}\\
=\ &(c^2+c)k\pm((3c-\eta)k+\tfrac{2}{3}(c-1)^2)\sqrt{\eta k+\tfrac{1}{3}(1-c)^2}.
\end{aligned}$$ □

Lemma 6.6. *Fix $0<c<1$. Then there is some $k_c\in\mathbb{N}$ such that for each $k\in\mathbb{N}$ with $k\geq k_c$ the following hold:*

(1) *the polynomial g_k given by (6.2) has 3 roots $t_{k,1}<t_{k,2}<t_{k,3}$ such that*
$$1<y_{k,4c,-}<t_{k,1}<y_{k,c,-}<t_{k,2}<y_{k,c,+}<t_{k,3}<y_{k,4c,+},$$
where $y_{k,\eta,\pm}$ is given in (6.3);

(2) *$g_k>0$ on $(-\infty,t_{k,1})\cup(t_{k,2},t_{k,3})$ and $g_k<0$ on $(t_{k,1},t_{k,2})\cup(t_{k,3},\infty)$.*

Proof. Take $k_c\in\mathbb{N}$ such that for each $k\geq k_c$ one has
$$\tfrac{ck}{1-c}-2-\tfrac{1}{1-c}\sqrt{4ck+\tfrac{1}{3}(1-c)^2}>1,$$
and
$$(c^2+c)k-(-ck+\tfrac{2}{3}(c-1)^2)\sqrt{4ck+\tfrac{1}{3}(1-c)^2}>0,$$
and
$$(c^2+c)k-(2ck+\tfrac{2}{3}(c-1)^2)\sqrt{ck+\tfrac{1}{3}(1-c)^2}<0,$$
and
$$(c^2+c)k+(-ck+\tfrac{2}{3}(c-1)^2)\sqrt{4ck+\tfrac{1}{3}(1-c)^2}<0.$$
Then for each $k\in\mathbb{N}$ with $k\geq k_c$, one has $y_{k,4c,-}>1$ and by Lemma 6.5 one has $g_k(y_{k,4c,-})>0$, $g_k(y_{k,c,-})<0$, $g_k(y_{k,c,+})>0$, and $g_k(y_{k,4c,+})<0$. Since $y_{k,4c,-}<y_{k,c,-}<y_{k,c,+}<y_{k,4c,+}$, it follows that (1) holds. As g_k is a cubic polynomial, (2) must also hold. □

Lemma 6.7. *Fix $0<c<1$. For $k\in\mathbb{N}$ set*
$$f_k(m)=c^m\binom{m+k}{k}-2c^{m+1}\binom{m+k+1}{k}+c^{m+2}\binom{m+k+2}{k} \tag{6.4}$$
and
$$h_k(m)=(1-c)^kf_k(m) \tag{6.5}$$
for $m\in\mathbb{Z}_{\geq 0}$. Let $\xi:\mathbb{N}\to\mathbb{N}$ such that $\xi(k)=\frac{ck}{1-c}+\mathcal{O}(k^{1/2})$ as $k\to\infty$. Then
$$h_k(\xi(k))=\mathcal{O}(k^{-3/2})$$
as $k\to\infty$.

Proof. For $k\in\mathbb{N}$ define $\varphi_k,\psi_k:\mathbb{N}\to\mathbb{R}$ by
$$\varphi_k(m)=(1-c)^kc^m\binom{m+k}{k}$$
and
$$\psi_k(m)=\tfrac{(m+1)(m+2)-2c(m+k+1)(m+2)+c^2(m+k+1)(m+k+2)}{(m+1)(m+2)}.$$

For each $m \in \mathbb{N}$, one has

$$f_k(m) = c^m \binom{m+k}{k}\left(1 - 2c\tfrac{m+k+1}{m+1} + c^2 \tfrac{(m+k+1)(m+k+2)}{(m+1)(m+2)}\right) = c^m \binom{m+k}{k}\psi_k(m),$$

whence

$$h_k(m) = \varphi_k(m)\psi_k(m).$$

Therefore it suffices to show that $\varphi_k(\xi(k)) = \mathcal{O}(k^{-1/2})$ and $\psi_k(\xi(k)) = \mathcal{O}(k^{-1})$.

By Stirling's approximation formula there are constants $C_1, C_2 > 0$ such that

$$C_1\sqrt{2\pi n}(\tfrac{n}{e})^n \leq n! \leq C_2\sqrt{2\pi n}(\tfrac{n}{e})^n$$

for all $n \in \mathbb{N}$. Then

$$\varphi_k(m) = (1-c)^k c^m \tfrac{(m+k)!}{m!k!} \leq (1-c)^k c^m \tfrac{C_2\sqrt{2\pi(m+k)}(m+k)^{m+k}}{C_1^2 2\pi\sqrt{mk}m^m k^k} \leq \tfrac{C_2\sqrt{2\pi(m+k)}}{C_1^2 2\pi\sqrt{mk}},$$

where the 2nd inequality comes from taking $x = m$ and $y = k$ in Lemma 6.4. Thus

$$\varphi_k(\xi(k)) \leq \tfrac{C_2\sqrt{2\pi(\xi(k)+k)}}{C_1^2 2\pi\sqrt{\xi(k)k}} = \mathcal{O}(k^{-1/2}),$$

whence $\varphi_k(\xi(k)) = \mathcal{O}(k^{-1/2})$.

Write $\xi(k)$ as $\frac{ck}{1-c} + \lambda_k k^{1/2}$ with $\lambda_k = \mathcal{O}(1)$ as $k \to \infty$. Then

$$\begin{aligned}
(\xi(k)+1)(\xi(k)+2) &= \tfrac{c^2k^2}{(1-c)^2} + \tfrac{2c\lambda_k}{1-c}k^{3/2} + \mathcal{O}(k),\\
2c(\xi(k)+k+1)(\xi(k)+2) &= \tfrac{2c^2k^2}{(1-c)^2} + \tfrac{2c(1+c)\lambda_k}{1-c}k^{3/2} + \mathcal{O}(k),\\
c^2(\xi(k)+k+1)(\xi(k)+k+2) &= \tfrac{c^2k^2}{(1-c)^2} + \tfrac{2c^2\lambda_k}{1-c}k^{3/2} + \mathcal{O}(k),
\end{aligned}$$

whence

$$(\xi(k)+1)(\xi(k)+2) - 2c(\xi(k)+k+1)(\xi(k)+2) + c^2(\xi(k)+k+1)(\xi(k)+k+2) = \mathcal{O}(k).$$

It follows that $\psi_k(\xi(k)) = \mathcal{O}(k^{-1})$. □

For a power series $\phi(x) = \sum_{m=0}^{\infty} a_m x^m \in \mathbb{C}[[x]]$, we set $|\phi|$ to be the power series

$$|\phi|(x) = \sum_{m=0}^{\infty} |a_m| x^m.$$

Lemma 6.8. *Fix $0 < c < 1$. For each $k \in \mathbb{Z}_{\geq 0}$ let ϕ_k be the power series given by*

$$\phi_k(x) = (c-x)^3 \sum_{m=0}^{\infty} \binom{m+k}{k} x^m. \tag{6.6}$$

Then $(1-c)^k|\phi_k|(c) = \mathcal{O}(k^{-3/2})$ as $k \to \infty$.

Proof. Let $k \in \mathbb{Z}_{\geq 0}$. For each $m \in \mathbb{Z}_{\geq 0}$, put

$$\begin{aligned}
b_{k,m} &= -\binom{m+k}{k} + 3c\binom{m+k+1}{k} - 3c^2\binom{m+k+2}{k} + c^3\binom{m+k+3}{k}\\
&= \binom{m+k}{k}\left(-1 + \tfrac{3c(m+k+1)}{m+1} - \tfrac{3c^2(m+k+1)(m+k+2)}{(m+1)(m+2)} + \tfrac{c^3(m+k+1)(m+k+2)(m+k+3)}{(m+1)(m+2)(m+3)}\right)\\
&= \binom{m+k}{k}\tfrac{g_k(m)}{(m+1)(m+2)(m+3)},
\end{aligned}$$

where g_k is given by (6.2). Then

$$\phi_k(x) = c^3 + x(c^3(k+1) - 3c^2) + x^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) + \sum_{m=0}^{\infty} x^{m+3} b_{k,m}. \tag{6.7}$$

Let $k_c \in \mathbb{N}$ be given by Lemma 6.6. Let $k \in \mathbb{N}$ with $k \geq k_c$. Then g_k has the roots $t_{k,i}$ for $i = 1,2,3$ described in Lemma 6.6. Put $m_{k,i} = \lfloor t_{k,i} \rfloor$ for $i = 1,3$, and $m_{k,2} = \lceil t_{k,2} \rceil$. Then $b_{k,m} \geq 0$ exactly when $0 \leq m \leq m_{k,1}$ or $m_{k,2} \leq m \leq m_{k,3}$. Increasing k_c if necessary, we may assume that $c^3(k+1) - 3c^2, c^3\frac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c > 0$. Then

$$\begin{aligned}|\phi_k|(x) &= |c^3| + x|c^3(k+1) - 3c^2| + x^2\left|c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right| + \sum_{m=0}^{\infty} x^{m+3}|b_{k,m}| \\ &= c^3 + x(c^3(k+1) - 3c^2) + x^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) \\ &\quad + \sum_{0\leq m\leq m_{k,1} \text{ or } m_{k,2}\leq m\leq m_{k,3}} x^{m+3}b_{k,m} - \sum_{m_{k,1}<m<m_{k,2} \text{ or } m>m_{k,3}} x^{m+3}b_{k,m}.\end{aligned}$$

Note that ϕ_k converges absolutely on the open interval $(-1,1)$. Taking $x = c$ in (6.7) we get

$$0 = c^3 + c(c^3(k+1) - 3c^2) + c^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) + \sum_{m=0}^{\infty} c^{m+3}b_{k,m}.$$

Thus

$$\begin{aligned}|\phi_k|(c) = 2\Big(&c^3 + c(c^3(k+1) - 3c^2) + c^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) \\ &+ \sum_{0\leq m\leq m_{k,1} \text{ or } m_{k,2}\leq m\leq m_{k,3}} c^{m+3}b_{k,m}\Big).\end{aligned} \tag{6.8}$$

Similar to (6.7), we have

$$\begin{aligned}(c-x)^3 \sum_{0\leq m\leq m_{k,1}} x^m\tbinom{m+k}{k} &= c^3 + x(c^3(k+1) - 3c^2) + x^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) \\ &\quad + \sum_{0\leq m\leq m_{k,1}} x^{m+3}b_{k,m} - (c^3 - 3c^2x + 3cx^2)x^{m_{k,1}+1}\tbinom{m_{k,1}+k+1}{k} \\ &\quad - (c^3 - 3c^2x)x^{m_{k,1}+2}\tbinom{m_{k,1}+k+2}{k} - c^3x^{m_{k,1}+3}\tbinom{m_{k,1}+k+3}{k},\end{aligned}$$

and

$$\begin{aligned}(c-x)^3 \sum_{m_{k,2}\leq m\leq m_{k,3}} x^m\tbinom{m+k}{k} &= (c^3 - 3c^2x + 3cx^2)x^{m_{k,2}}\tbinom{m_{k,2}+k}{k} \\ &\quad + (c^3 - 3c^2x)x^{m_{k,2}+1}\tbinom{m_{k,2}+k+1}{k} + c^3x^{m_{k,2}+2}\tbinom{m_{k,2}+k+2}{k} \\ &\quad + \sum_{m_{k,2}\leq m\leq m_{k,3}} x^{m+3}b_{k,m} - (c^3 - 3c^2x + 3cx^2)x^{m_{k,3}+1}\tbinom{m_{k,3}+k+1}{k} \\ &\quad - (c^3 - 3c^2x)x^{m_{k,3}+2}\tbinom{m_{k,3}+k+2}{k} - c^3x^{m_{k,3}+3}\tbinom{m_{k,3}+k+3}{k}.\end{aligned}$$

Taking $x = c$ in the two identities above, we get

$$\begin{aligned}0 &= c^3 + c(c^3(k+1) - 3c^2) + c^2\left(c^3\tfrac{(k+1)(k+2)}{2} - 3c^2(k+1) + 3c\right) \\ &\quad + \sum_{0\leq m\leq m_{k,1}} c^{m+3}b_{k,m} - c^3f_k(m_{k,1}+1),\end{aligned}$$

and

$$0 = c^3f_k(m_{k,2}) + \sum_{m_{k,2}\leq m\leq m_{k,3}} c^{m+3}b_{k,m} - c^3f_k(m_{k,3}+1),$$

where f_k is defined in (6.4). Then (6.8) becomes

$$|\phi_k|(c) = 2c^3(f_k(m_{k,1}+1) - f_k(m_{k,2}) + f_k(m_{k,3}+1)),$$

whence

$$(1-c)^k|\phi_k|(c) = 2c^3(h_k(m_{k,1}+1) - h_k(m_{k,2}) + h_k(m_{k,3}+1)),$$

where h_k is defined in (6.5). Note that the two sequences $\{y_{k,4c,-}\}$ and $\{y_{k,4c,+}\}$ are both $\frac{ck}{1-c} + \mathcal{O}(k^{1/2})$. It follows from Lemma 6.6.(1) that the three sequences $\{m_{k,1}+1\}$, $\{m_{k,2}\}$ and $\{m_{k,3}+1\}$ are all $\frac{ck}{1-c}+\mathcal{O}(k^{1/2})$. Thus from Lemma 6.7 we conclude that the three sequences $\{h_k(m_{k,1}+1)\}$, $\{h_k(m_{k,2})\}$ and $\{h_k(m_{k,3}+1)\}$ are all $\mathcal{O}(k^{-3/2})$. Therefore $(1-c)^k|\phi_k|(c) = \mathcal{O}(k^{-3/2})$. □

We are ready to prove Lemma 6.3.

Proof of Lemma 6.3. Let $k \in \mathbb{N}$. One has

$$(1-q)^{-(k+1)} = \Big(\sum_{j=0}^{\infty} q^j\Big)^{k+1} = \sum_{m=0}^{\infty} q^m \sum_{\substack{j_1+\cdots+j_{k+1}=m \\ j_1,\dots,j_{k+1}\geq 0}} 1 = \sum_{m=0}^{\infty} q^m \binom{m+k}{k},$$

whence

$$(c-q)^3(1-q)^{-(k+1)} = \phi_k(q),$$

where ϕ_k is defined in (6.6). Write ϕ_k as $\sum_{m=0}^{\infty}\lambda_m x^m$ with $\lambda_m \in \mathbb{C}$. Then

$$\begin{aligned}\|r^k(c-q)^3(1-q)^{-(k+1)}\| &= \Big\|\sum_{m=0}^{\infty}\lambda_m r^k q^m\Big\| \leq \sum_{m=0}^{\infty}|\lambda_m|\cdot\|r\|^k\|q\|^m \\ &\leq \sum_{m=0}^{\infty}|\lambda_m|(1-c)^k c^m = (1-c)^k|\phi_k|(c).\end{aligned}$$

Now the assertion follows from Lemma 6.8. □

Lemma 6.3 implies the following proposition.

Proposition 6.9. *Let $\mathcal{A}$ be a unital Banach algebra such that $\|ab\| \leq \|a\|\cdot\|b\|$ for all $a, b \in \mathcal{A}$. Let $0 < c < 1$ and $q, r \in \mathcal{A}$ such that $\|r\| \leq 1-c$, $\|q\| \leq c$ and $qr = rq$. Then there is some $a \in \mathcal{A}$ such that*

$$a(1-(q+r)) = (1-(q+r))a = (c-q)^3. \tag{6.9}$$

Proof. Formally, the element a in (6.9) is given by

$$\begin{aligned} a &= (c-q)^3(1-(q+r))^{-1} = (c-q)^3((1-q)-r)^{-1} \\ &= (c-q)^3(1-q)^{-1}(1-r(1-q)^{-1})^{-1} = (c-q)^3(1-q)^{-1}\sum_{k=0}^{\infty} r^k(1-q)^{-k} \\ &= \sum_{k=0}^{\infty} r^k(c-q)^3(1-q)^{-(k+1)}.\end{aligned}$$

By Lemma 6.3, the last series in this expression for a converges in norm, so that $a \in \mathcal{A}$ is well defined. Since $qr = rq$, one has $qa = aq$, $ra = ar$, and $(1-q)^{-1}r = r(1-q)^{-1}$. Thus

$$\begin{aligned} a(1-(q+r)) &= a(1-q) - ar \\ &= \sum_{k=0}^{\infty} r^k(c-q)^3(1-q)^{-k} - \sum_{k=0}^{\infty} r^{k+1}(c-q)^3(1-q)^{-(k+1)} = (c-q)^3,\end{aligned}$$

and similarly $(1-(q+r))a=(c-q)^3$. □

Example 6.10. Let $\mathbb{H}$ be the discrete Heisenberg group with canonical generators u_1,u_2,u_3 defined in (5.1), and let $f=4-u_1-u_1^{-1}-u_2-u_2^{-1}\in\mathbb{ZH}$. Then $f=4(1-p)$, where $p=\frac{1}{4}(u_1+u_2+u_1^{-1}+u_2^{-1})\in\mathbb{QH}$ can be viewed as a symmetric probability measure on $\mathbb{H}$. The polynomial $p^4\in\mathbb{QH}$ can again be viewed as a probability measure on $\mathbb{H}$, and the coefficient $c:=p^4_{u_3}$ of p^4 at u_3 is strictly positive. If we set $q=c\cdot u_3$ and $r=p^4-q$, then $qr=rq$ and Proposition 6.9 yields an element $a\in\ell^1(\mathbb{H},\mathbb{R})$ such that

$$a(1-p^4)=a(1-(q+r))=c^3(1-u_3)^3.$$

Then $b=a(1+p+p^2+p^3)/4c^3\in\ell^1(\mathbb{H},\mathbb{R})$ satisfies that

$$fb=bf=bf^*=4(1-p)b=(1-u_3)^3. \tag{6.10}$$

It follows that $b\in W_f$ and $\eta(b)\in\Delta^1(X_f)$ (cf. (2.4) and Proposition 5.3).

We remark in passing that the existence of such a homoclinic point b was conjectured in [12, page 130]; in [12, Theorem 4.1.2] it was shown that there is some $b'\in\ell^1(\mathbb{H},\mathbb{R})$ satisfying $b'f^*=(1-u_3)^9$.

Having found a nonzero element of $\Delta^1(X_f)$ we claim that $\Delta^1(X_f)$ is actually dense in X_f. To verify this we give an ad-hoc proof based on [13, Theorem 5.1]: for every $x\in X_f$ there exists a $y\in Y_f$ with $\eta(y)=x$ (for notation we refer to (2.1)). Then $v:=\bar\rho^f y\in\{-3,\dots,3\}^{\mathbb{H}}\subset\ell^\infty(\mathbb{H},\mathbb{Z})$ (cf. (2.4)) and $(\bar\rho^f\circ\bar\rho^b)(v)=\bar\rho^{fb}v=\bar\rho^{(1-u_3)^3}v\in\ell^\infty(\mathbb{H},\mathbb{Z})$. It follows that $\bar\rho^b v\in W_f$ and $(\eta\circ\bar\rho^b)(v)=(\eta\circ\bar\rho^{bf})(y)=\rho^{bf}(\eta(y))=\rho^{(1-u_3)^3}x$ by (6.10). We set $\mathcal{V}=\{-3,\dots,3\}^{\mathbb{H}}\subset\ell^\infty(\mathbb{H},\mathbb{Z})$ and conclude that $(\eta\circ\bar\rho^b)(\mathcal{V})\supseteq\rho^{(1-u_3)^3}(X_f)$.

We recall that $X_f=\widehat{\mathbb{ZH}/(f)}=(f)^\perp\subset\widehat{\mathbb{ZH}}$ (cf. (1.2)). If $\rho^{1-u_3}(X_f)\subsetneq X_f$ there exists an element $h\in\mathbb{ZH}$ such that $h\notin(f)$ and $\langle h,\rho^{1-u_3}x\rangle=\langle h(1-u_3),x\rangle=1$ for every $x\in X_f$. Hence $h(1-u_3)=(1-u_3)h\in(f)$, i.e. $(1-u_3)h=gf$ for some $g\in\mathbb{ZH}$.

We denote by $\langle u_3\rangle$ the subgroup of $\mathbb{H}$ generated by u_3, set $\mathbb{H}'=\mathbb{H}/\langle u_3\rangle\cong\mathbb{Z}^2$, and denote by $\pi\colon\mathbb{ZH}\to\mathbb{ZH}'\cong\mathbb{ZH}/(z_3-1)\mathbb{ZH}$ the group ring homomorphism corresponding to the quotient map $\mathbb{H}\to\mathbb{H}'$. As f is not divisible by $1-u_3$, $\pi(f)\neq0$, but $\pi(gf)=\pi(g)\pi(f)=0$. Since $\mathbb{ZH}'\cong\mathbb{Z}\mathbb{Z}^2$ is an integral domain we obtain that $\pi(g)=0$, i.e. that $g=(1-u_3)g'$ for some $g'\in\mathbb{ZH}$. Since $\mathbb{ZH}$ has no nontrivial zero divisors (cf. e.g. [27, Theorem 13.1.11]) we obtain that $h=g'f$, contrary to our hypothesis that $h\notin\mathbb{ZH}f$.

This contradiction implies that $X_f=\rho^{1-u_3}(X_f)=\rho^{(1-u_3)^3}(X_f)=(\eta\circ\bar\rho^b)(\mathcal{V})$. Since $\eta\circ\bar\rho^b\colon\mathcal{V}\to X_f$ is continuous and $\mathbb{ZH}\cap\mathcal{V}$ is dense in $\mathcal{V}$ (both in the product topology on $\mathcal{V}$) we conclude that $(\eta\circ\bar\rho^b)(\mathbb{ZH})$ is dense in X_f. Finally we note that $(\eta\circ\bar\rho^b)(\mathbb{ZH})\subset\Delta^1(X_f)$, so that $\Delta^1(X_f)$ is dense in X_f, as promised.

According to Proposition 5.3 this shows that λ_f is intrinsically ergodic.

The following corollary of Proposition 6.9 will allow us to extend the argument in Example 6.10 to the much more general setting of Theorem 6.1.

Corollary 6.11. *Assume that Γ is infinite and not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Let p be a finitely supported symmetric probability measure on Γ such that* $\mathrm{supp}(p)$ *generates Γ. By a result of Varopoulos (cf.* [33], [11, Theorem 2.1], [37, Theorem 3.24]), *$\sum_{j=0}^\infty p^j$ converges in $\|\cdot\|_\infty$ to some ω in $C_0(\Gamma,\mathbb{R})$. Then $(1-s)^3\omega\in\ell^1(\Gamma,\mathbb{R})$ for every s in the center of Γ.*

Proof. It is easily checked that

$$\omega(1-p)=1.$$

Let $s \neq 1_\Gamma$ be a central element of Γ. Then $s \in \operatorname{supp}(p^k) \setminus \{1_\Gamma\}$ for some $k \in \mathbb{N}$. As in Example 6.10 there is some a in $\ell^1(\Gamma,\mathbb{R})$ such that

$$a(1-p^k)=(1-s)^3.$$

Then $b = a\sum_{j=0}^{k-1} p^j$ is in $\ell^1(\Gamma,\mathbb{R})$ and

$$b(1-p)=(1-s)^3.$$

Note that $(1-s)^3\omega \in C_0(\Gamma,\mathbb{R})$ and

$$(1-s)^3\omega(1-p)=(1-s)^3=b(1-p).$$

It is well known that if $x \in C_0(\Gamma,\mathbb{R})$ satisfies $x(1-p)=0$, then $x=0$. Thus

$$(1-s)^3\omega = b \in \ell^1(\Gamma,\mathbb{R}).$$

□

Proof of Theorem 6.1. Since f is well-balanced, one has $f = f_{1_\Gamma}(1-p)$ for some symmetric probability measure p on Γ such that $\operatorname{supp}(p)$ generates Γ.

Since Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, $\omega = \sum_{j=0}^{\infty} p^j$ is in $C_0(\Gamma,\mathbb{R})$. Then

$$(f_{1_\Gamma}^{-1}\omega)f=1.$$

By [1, Theorem 4.1 and Lemma 4.10], the group $\Delta(X_f)$ of homoclinic points of λ_f is dense in X_f and is the Γ-invariant subgroup of X_f generated by $\eta(f_{1_\Gamma}^{-1}\omega)$. Thus it suffices to show that $\eta(f_{1_\Gamma}^{-1}\omega)$ is in the closure of $\Delta^1(X_f)$.

By assumption we can find a sequence $(s_n)_{n\geq 1}$ in the center H of Γ such that for any finite subset F of Γ one has $s_n, s_n^2, s_n^3 \notin F$ for all large enough n. Then $\eta((1-s_n)^3 f_{1_\Gamma}^{-1}\omega)$ converges to $\eta(f_{1_\Gamma}^{-1}\omega)$ as $n\to\infty$. By Corollary 6.11 one has $(1-s_n)^3 f_{1_\Gamma}^{-1}\omega \in \ell^1(\Gamma,\mathbb{R})$ and hence $\eta((1-s_n)^3 f_{1_\Gamma}^{-1}\omega) \in \Delta^1(X_f)$ for each n. Therefore $\eta(f_{1_\Gamma}^{-1}\omega)$ lies in the closure of $\Delta^1(X_f)$, as desired. □

Remarks 6.12. (1) For $\Gamma=\mathbb{Z}^d$ with $d\geq 1$, [24, Corollary 3.4] shows that any atoral polynomial $f\in\mathbb{Z}[\mathbb{Z}^d]$ which is not a unit in $\mathbb{Z}[\mathbb{Z}^d]$ satisfies that $\Delta^1(X_f)$ is dense in X_f (for the definition of atorality we refer to [24, Definition 2.1 and Proposition 2.2]). In particular, Theorem 6.1 also holds for $\Gamma=\mathbb{Z}^2$. Does Theorem 6.1 hold for virtually $\mathbb{Z}^2$?

(2) For the polynomial $h = 2-u_1-u_2 \in \mathbb{Z}\mathbb{H}$ there exists a nonzero element $w\in\ell^1(\mathbb{H},\mathbb{R})$ such that $wh = hw = (1-u_3)^2$ (cf. [13, Theorem 4.2]). As in Subsection 5.2.1, the map $\bar\rho^w\colon \ell^\infty(\mathbb{H},\mathbb{Z}) \to \ell^\infty(\mathbb{H},\mathbb{R})$ is continuous in the weak*-topology on closed, bounded subsets of $\ell^\infty(\mathbb{H},\mathbb{Z})$, and the map $\xi := \eta\circ\bar\rho^w\colon \ell^\infty(\mathbb{H},\mathbb{Z}) \to \mathbb{T}^{\mathbb{H}}$ satisfies that $\xi(\{-1,0,1\}^{\mathbb{H}}) = X_h$ (cf. [13, Theorem 5.1]). It follows that $\xi(\mathbb{Z}\mathbb{H}) \subset \Delta^1(X_h)$. Hence $\Delta^1(X_h)$ is dense in X_h and λ_h is intrinsically ergodic.

Is there any way to use an argument similar to Proposition 6.9 to prove the existence of summable homoclinic points for this and other 'asymmetric' elements of $h\in\mathbb{Z}\mathbb{H}$?

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