STRONGLY INDEPENDENT MATRICES AND RIGIDITY OF $\times A$-INVARIANT MEASURES ON n-TORUS

HUICHI HUANG, HANFENG LI, ENHUI SHI, AND HUI XU

ABSTRACT. We introduce the concept of strongly independent matrices over any field, and prove the existence of such matrices for certain fields and the non-existence for algebraically closed fields. Then we apply strongly independent matrices over rational numbers to obtain measure rigidity result for endomorphisms on n-torus.

1. Introduction

For an integer m, the $\times m$ map T_m on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is given by $T_m(z) = z^m$ for all $z \in \mathbb{T}$.

H. Furstenberg proved that under the action of a non-lacunary multiplicative semigroup of positive integers on $\mathbb{T}$, a nonempty closed invariant subset of $\mathbb{T}$ containing a dense orbit is either finite or the whole $\mathbb{T}$ [F, Theorem IV.1]. Here a multiplicative semigroup of positive integers is called **non-lacunary** if it is not contained in any singly generated multiplicative semigroup. In other words a non-lacunary multiplicative semigroup of positive integers always contains two positive integers p and q with $\frac{\log p}{\log q}$ irrational (we say that p, q are non-lacunary).

Furthermore, Furstenberg conjectured the following.

Conjecture 1.1 (Furstenberg's Conjecture)**.**

An ergodic invariant Borel probability measure on $\mathbb{T}$ under the action of a non-lacunary multiplicative semigroup of positive integers is either finitely supported or the Lebesgue measure.

Date: January 2, 2022.

2010 *Mathematics Subject Classification.* Primary 37A05, 37A25, 37A46, 43A05, 28C10, 12E05.

Key words and phrases. Strongly independent, ergodicity, mixing, Fourier coefficient, measure rigidity.

Huichi Huang was partially supported by NSFC no. 11871119 and Chongqing Municipal Science and Technology Commission fund no. cstc2018jcyjAX0146.

Hanfeng Li was partially supported by NSF grant DMS-1900746.

Enhui Shi was partially supported by NSFC no. 11771318 and no. 11790274.

The first breakthrough of Furstenberg's conjecture was achieved by R. Lyons.

Theorem 1.2. [L, Theorem 1]
Suppose $p, q \geq 2$ are two relatively prime integers. If a non-atomic $\times p, \times q$-invariant Borel probability measure μ on $\mathbb{T}$ is T_p-exact, then it is the Lebesgue measure. Here μ is T_p-exact means that $(\mathbb{T}, \mathcal{B}, \mu, T_p)$ has no nontrivial zero entropy factor.

This result was improved by D. J. Rudolph under the assumption that p and q are coprime and an extra positive entropy condition [R, Theorem 4.9] and later by A. S. A. Johnson [J, Theorem A] under the assumption that p, q are non-lacunary and the positive entropy condition.

Theorem 1.3 (Rudolph-Johnson's Theorem)**.**
Suppose p and q are non-lacunary positive integers greater than 1. If μ is an ergodic $\times p, \times q$-invariant Borel probability measure on $\mathbb{T}$ such that T_p or T_q has positive measure entropy with respect to μ, then μ is the Lebesgue measure.

One may consult [KK, KS1, KS2] for the extensions of above results to automorphisms on n-torus with $n \geq 2$.

Recently, the first named author obtained the following rigidity theorem.

Theorem 1.4. [H, Theorem 1.5]
Let p be a nonzero integer. The Lebesgue measure is the unique non-atomic $\times p$-invariant Borel probability measure on $\mathbb{T}$ satisfying one of the following:

(1) it is ergodic and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ in $\mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in some $E \subseteq \mathbb{N}$ with upper density $\overline{D}_\Sigma(E)$ (see Definition 2.2) equal to 1;
(2) it is weakly mixing and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ in $\mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in some $E \subseteq \mathbb{N}$ with $\overline{D}_\Sigma(E) > 0$;
(3) it is strongly mixing and there exist a nonzero integer l and an infinite set $E \subseteq \mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in E.

Moreover, a $\times p$-invariant Borel probability measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

In this paper, we introduce so-called strongly independent matrices over a field $\mathbb{F}$, and use strongly independent matrices over the rational field $\mathbb{Q}$ to extend the above measure rigidity results to endomorphisms on $\mathbb{T}^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1| = \cdots = |z_n| = 1\}$.

We say that an n-tuple $(B_1, B_2, \cdots, B_n)$ of $n \times n$ matrices over a field $\mathbb{F}$ is **strongly independent over** $\mathbb{F}$ if for any nonzero column vector v in $\mathbb{F}^{n\times 1}$, the vectors $B_1v, B_2v, \cdots, B_nv$ are linearly independent over $\mathbb{F}$. A nonzero matrix B in $M_n(\mathbb{F})$ is called strongly independent over $\mathbb{F}$ if the n-tuple $(I_n, B, \cdots, B^{n-1})$ is strongly independent over $\mathbb{F}$.

The next main theorem illustrates the existence of an abundance of strongly independent matrices.

Theorem 1.5. A nonzero matrix B in $M_n(\mathbb{F})$ is strongly independent over $\mathbb{F}$ iff the characteristic polynomial of B is irreducible in $\mathbb{F}[t]$.

The above shows existence of strongly independent matrices over certain fields, say, the field of rational numbers $\mathbb{Q}$. However over some fields, there are no strongly independent matrices.

Theorem 1.6. If $\mathbb{F}$ is an algebraically closed field, then there are no strongly independent n-tuples in $M_n(\mathbb{F})$ for $n \geq 2$.

We shall identify $\mathbb{R}^n/\mathbb{Z}^n$ with the n-torus $\mathbb{T}^n$ naturally via

$$\mathbb{R}^n/\mathbb{Z}^n \ni (x_1, x_2, \cdots, x_n) + \mathbb{Z}^n \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2}, \cdots, e^{2\pi i x_n}) \in \mathbb{T}^n$$

for $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Let A be a matrix in $M_n(\mathbb{Z})$. The $\times A$ map on $\mathbb{T}^n$ is defined by $T_A : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$

$$T_A((x_1, x_2, \cdots, x_n) + \mathbb{Z}^n) = (x_1, x_2, \cdots, x_n)A + \mathbb{Z}^n,$$

for $(x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$.

Theorem 1.7. Let A be in $M_n(\mathbb{Z})$. Suppose that μ is a $\times A$-invariant Borel probability measure on $\mathbb{T}^n$ satisfying one of the following:

(1) it is ergodic and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ strongly independent over $\mathbb{Q}$ and a Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ in $\mathbb{N}$ such that μ is $\times(A^j + B_i)$-invariant for all j in some $E \subseteq \mathbb{N}$ with upper density $\overline{D}_\Sigma(E) = 1$ and all $i = 1, 2, \cdots, n$;

(2) it is weakly mixing and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ strongly independent over $\mathbb{Q}$ and a Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ in $\mathbb{N}$ such that μ is $\times(A^j + B_i)$-invariant for all j in some $E \subseteq \mathbb{N}$ with $\overline{D}_\Sigma(E) > 0$ and all $i = 1, 2, \cdots, n$;

(3) it is strongly mixing and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ strongly independent over $\mathbb{Q}$ and an infinite set $E \subseteq \mathbb{N}$ such that μ is $\times(A^j + B_i)$-invariant for all j in E and all $i = 1, 2, \cdots, n$.

Then μ is either finitely supported or the Lebesgue measure.

Moreover, a $\times A$-invariant Borel probability measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

Sataev [S] and Einsiedler-Fish [EF] independently proved that a multiplicative semigroup of positive integers with positive lower logarithmic density acting on the circle has measure rigidity, whereas Theorem 1.4 implies that there exists a multiplicative semigroup of positive integers with zero logarithmic density acting on the circle which also has measure rigidity [H, Theorem 5.2]. Analogously, we conclude that there exist "very small" semigroups acting on $\mathbb{T}^n$ such that the Lebesgue measure is the unique non-atomic invariant measure.

Corollary 1.8. There exist an abelian multiplicative semigroup $S \subseteq M_n(\mathbb{Z})$ and a matrix B in S such that the Lebesgue measure is the unique non-atomic Borel probability measure on $\mathbb{T}^n$ which is both invariant under $\times A$ for all A in S and ergodic under $\times B$.

The paper is organized as follows. We lay down some definitions and notations in Section 2. Theorem 1.5 and Theorem 1.6 are proved in Section 3. In Section 4, we characterize mixing properties of Borel probability measures on $\mathbb{T}^n$ in terms of their Fourier coefficients. Finally we establish Theorem 1.7 in Section 5.

2. Preliminaries

Denote the set of nonnegative integers by $\mathbb{N}$, and the cardinality of a set E by $|E|$.

For a ring R, denote by $M_n(R)$ the ring of $n \times n$ square matrices with entries in R. Denote by $GL_n(R)$ the group of invertible elements in $M_n(R)$. For a field $\mathbb{F}$, denote by $\overline{\mathbb{F}}$ its algebraic closure. For any $A \in M_n(\mathbb{F})$, denote by $P_A(t)$ the characteristic polynomial $\det(tI_n - A)$ of A in $\mathbb{F}[t]$.

For a nonempty set Z, denote by Z^n the set of row vectors of length n with coordinates in Z, and by $Z^{n\times 1}$ the set of column vectors of length n with coordinates in Z.

Within this paper, a measure on a compact metrizable space X always means a Borel probability measure. Denote by $C(X)$ the space of complex-valued continuous functions on X.

Definition 2.1. A **Følner sequence** in $\mathbb{N}$ is a sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ of nonempty finite subsets of $\mathbb{N}$ satisfying

$$\lim_{m\to\infty} \frac{|(F_m + m')\Delta F_m|}{|F_m|} = 0$$

for every m' in $\mathbb{N}$. Here Δ stands for the symmetric difference.

Definition 2.2. Let $\Sigma = \{F_m\}_{m=1}^{\infty}$ be a sequence of nonempty finite subsets of $\mathbb{N}$. For a subset E of $\mathbb{N}$, the **upper density** $\overline{D}_{\Sigma}(E)$ is given by

$$\overline{D}_{\Sigma}(E) := \limsup_{m\to\infty} \frac{|E \cap F_m|}{|F_m|}.$$

Definition 2.3. For $k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \in \mathbb{Z}^{n\times 1}$ and $z = (z_1, z_2, \cdots, z_n) \in \mathbb{T}^n$, use z^k to denote $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$, and the **Fourier coefficient** $\hat{\mu}(k)$ of a measure μ on $\mathbb{T}^n$ is defined by

$$\hat{\mu}(k) = \int_{\mathbb{T}^n} z^k \, d\mu(z).$$

For a measure μ on a compact metrizable space X, if $\mu(\{x\}) > 0$ for some x in X, then x is called an **atom** for μ. A measure with no atoms is called **non-atomic**.

For a continuous map $T : X \to X$, a measure μ on X is called T-**invariant** if $\mu(E) = \mu(T^{-1}E)$ for every Borel subset E of X. For A in $M_n(\mathbb{Z})$, we call a measure μ on $\mathbb{T}^n$ $\times A$-invariant if μ is T_A-invariant.

A T-invariant measure μ is called **ergodic** if every Borel subset E with $T^{-1}E = E$ satisfies $\mu(E) = 0$ or 1. A measure μ is called **weakly mixing** if $\mu \times \mu$ is an ergodic $T \times T$-invariant measure on $X \times X$, and it is called **strongly mixing** if $\lim_{j\to\infty} \mu(T^{-j}E \cap F) = \mu(E)\mu(F)$ for all Borel subsets E, F of X.

3. Existence and non-existence of Strongly Independent Matrices over certain fields

In this section, we prove Theorems 1.5 and 1.6, which illustrate that the existence of strongly independent matrices over a field $\mathbb{F}$ depends on algebraic properties of $\mathbb{F}$.

Definition 3.1. For a field $\mathbb{F}$, we call an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{F})$ **strongly independent over** $\mathbb{F}$ if for any nonzero v in $\mathbb{F}^{n\times 1}$, the vectors $B_1v, B_2v, \cdots, B_nv$ are linearly independent over $\mathbb{F}$. We call a nonzero matrix B in $M_n(\mathbb{F})$ **strongly independent over** $\mathbb{F}$ if the n-tuple $(I_n, B, \cdots, B^{n-1})$ is strongly independent over $\mathbb{F}$.

Lemma 3.2. Let $B_1, \cdots, B_n \in M_n(\mathbb{F})$. The tuple $(B_1, \cdots, B_n)$ is strongly independent over $\mathbb{F}$ iff for any nonzero $(u_1, \cdots, u_n) \in \mathbb{F}^n$ the matrix $\sum_{j=1}^n u_j B_j$ is invertible.

Proof. The tuple $(B_1, \cdots, B_n)$ is strongly independent over $\mathbb{F}$ iff for any nonzero $v \in \mathbb{F}^{n\times 1}$ the vectors $B_1 v, \cdots, B_n v$ are linearly independent, iff for any nonzero $v \in \mathbb{F}^{n\times 1}$ and any nonzero $(u_1, \cdots, u_n) \in \mathbb{F}^n$ the vector $\sum_{j=1}^n u_j B_j v$ is nonzero, iff for any nonzero $(u_1, \cdots, u_n) \in \mathbb{F}^n$ the matrix $\sum_{j=1}^n u_j B_j$ is invertible. $\square$

Proof of Theorem 1.5.

Suppose that $P_B(t)$ is not irreducible in $\mathbb{F}[t]$. We have $P_B(t) = f(t)g(t)$ for some $f, g \in \mathbb{F}[t]$ with $1 \leq \deg(f), \deg(g) \leq n-1$. Then $0 = P_B(B) = f(B)g(B)$ by Hamilton-Cayley Theorem [La, Theorem XIV.3.1], whence at least one of $f(B)$ and $g(B)$ is not invertible. By Lemma 3.2 we conclude that B is not strongly independent over $\mathbb{F}$.

Now assume that $P_B(t)$ is irreducible in $\mathbb{F}[t]$. Denote by D the Jordan canonical form of B. That is, $D \in M_n(\overline{\mathbb{F}})$ and there is some invertible $W \in M_n(\overline{\mathbb{F}})$ satisfying $WB = DW$ and

$$D = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{bmatrix}$$

for some positive integer k such that each D_i is in $M_{m_i}(\overline{\mathbb{F}})$ of the form

$$\begin{bmatrix} \lambda_i & & & & \\ 1 & \lambda_i & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda_i & \\ & & & 1 & \lambda_i \end{bmatrix}$$

for some $\lambda_i \in \overline{\mathbb{F}}$ and positive integer m_i. Then $P_B(t) = \prod_{i=1}^k (t-\lambda_i)^{m_i}$, whence $P_B(\lambda_i) = 0$ for every $1 \leq i \leq k$. Since $P_B(t)$ is irreducible in $\mathbb{F}[t]$, it follows that for any nonzero $f(t) \in \mathbb{F}[t]$ of degree at most $n-1$, one has $f(\lambda_i) \neq 0$ for every $1 \leq i \leq k$.

Let $(u_1, \cdots, u_n)$ be a nonzero vector in $\mathbb{F}^n$. Then $f(t) = \sum_{j=1}^n u_j t^{j-1} \in \mathbb{F}[t]$ is nonzero and has degree at most $n-1$. Thus $f(\lambda_i) \neq 0$ for every $1 \leq i \leq k$. It follows that $f(D)$ is invertible, whence $u_1 I_n + u_2 B + \cdots + u_n B^{n-1} = f(B) = W^{-1} f(D) W$ is invertible. By Lemma 3.2 we conclude that B is strongly independent over $\mathbb{F}$. $\square$

Remark 3.3. Suppose $f(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$ is an irreducible polynomial in $\mathbb{F}[t]$. Define $B \in M_n(\mathbb{F})$ as

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}.$$

Then $P_B(t) = f(t)$ [Ro, Definition on page 173 and Lemma 7.17]. By Theorem 1.5, the matrix B is strongly independent over $\mathbb{F}$.

For any $n \geq 1$, by Eisenstein's criterion [La, Theorem IV.3.1], there exist infinitely many monic polynomials of degree n in $\mathbb{Z}[t]$, which are irreducible in $\mathbb{Q}[t]$ (for example $t^n + p$ for any prime number p in $\mathbb{Z}$). Theorem 1.5 illustrates that for $n \geq 2$ there are infinitely many n-tuples of the form $(I_n, B, \cdots, B^{n-1})$ in $M_n(\mathbb{Z})$ strongly independent over $\mathbb{Q}$.

Next we prove Theorem 1.6 which gives the non-existence of strongly independent matrices over algebraically closed fields.

Proof of Theorem 1.6.

For any matrices $B_1, B_2, \cdots, B_n$ in $M_n(\mathbb{F})$, taking $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, the polynomial $f(z_1, z_2, \cdots, z_n) = \det \begin{bmatrix} B_1 z & B_2 z & \cdots & B_n z \end{bmatrix}$ is in $\mathbb{F}[z_1, z_2, \cdots, z_n]$. Now $f(z_1, z_2, \cdots, z_n) = 0$ always has a nonzero solution $\tilde{z}$ in $\mathbb{F}^{n\times 1}$ since $\mathbb{F}$ is algebraically closed and $n \geq 2$. $\square$

4. Fourier Coefficients of Ergodic, Weakly Mixing and Strongly Mixing measures on $\mathbb{T}^n$

In this section we prove Theorem 4.1, characterizing the mixing properties of measures on $\mathbb{T}^n$ under $\times A$ map via their Fourier coefficients.

Theorem 4.1. Let $A \in M_n(\mathbb{Z})$ and let $\Sigma = \{F_m\}_{m=1}^{\infty}$ be a Følner sequence in $\mathbb{N}$. The following are true.

(1) A measure μ on $\mathbb{T}^n$ is an ergodic $\times A$-invariant measure iff

$$\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \hat{\mu}(A^j k + l) = \hat{\mu}(k)\hat{\mu}(l) \tag{4.1}$$

for all k, l in $\mathbb{Z}^{n\times 1}$.

(2) A measure μ on $\mathbb{T}^n$ is a weakly mixing $\times A$-invariant measure iff

$$\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} |\hat{\mu}(A^j k + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 = 0 \tag{4.2}$$

for all k, l in $\mathbb{Z}^{n\times 1}$.

(3) A measure μ on $\mathbb{T}^n$ is a strongly mixing $\times A$-invariant measure iff

$$\lim_{j\to\infty} \hat{\mu}(A^j k + l) = \hat{\mu}(k)\hat{\mu}(l) \tag{4.3}$$

for all k, l in $\mathbb{Z}^{n\times 1}$.

To prove Theorem 4.1 we need to make some preparations.

Lemma 4.2. Let $A \in M_n(\mathbb{Z})$. A measure μ on $\mathbb{T}^n$ is $\times A$-invariant iff $\hat{\mu}(k) = \hat{\mu}(Ak)$ for all k in $\mathbb{Z}^{n\times 1}$.

Proof. A measure μ on $\mathbb{T}^n$ is $\times A$-invariant iff $\int_{\mathbb{T}^n} f(T_A z)\, d\mu(z) = \int_{\mathbb{T}^n} f(z)\, d\mu(z)$ for all f in $C(\mathbb{T}^n)$ [W, Theorem 6.8] iff $\int_{\mathbb{T}^n} f(T_A z)\, d\mu(z) = \int_{\mathbb{T}^n} f(z)\, d\mu(z)$ for all f in a dense subset of $C(\mathbb{T}^n)$ iff $\int_{\mathbb{T}^n} f(T_A z)\, d\mu(z) = \int_{\mathbb{T}^n} f(z)\, d\mu(z)$ for $f(z) = z^k$ for all k in $\mathbb{Z}^{n\times 1}$ since the linear span of z^k's is dense in $C(\mathbb{T}^n)$. Note that $\int_{\mathbb{T}^n} (T_A z)^k\, d\mu(z) = \hat{\mu}(Ak)$ for all k in $\mathbb{Z}^{n\times 1}$. □

Lemma 4.3. Let μ be a measure on $\mathbb{T}^n$. For any k in $\mathbb{Z}^{n\times 1}$, if $\hat{\mu}(k) = 1$ then the support of μ

$$\mathrm{supp}(\mu) \subseteq \{z \in \mathbb{T}^n : \ z^k = 1\}.$$

Proof. Since $\hat{\mu}(k) = 1$, by the definition of $\hat{\mu}(k)$, we have

$$\int_{\mathbb{T}^n} z^k d\mu(z) = 1.$$

Thus $\int_{\mathbb{T}^n} \mathrm{Re}(z^k) d\mu(z) = 1$. Therefore,

$$\int_{\mathbb{T}^n} |z^k - 1|^2 d\mu(z) = \int_{\mathbb{T}^n} (2 - 2\mathrm{Re}(z^k)) d\mu(z) = 0.$$

Hence, $\mathrm{supp}(\mu) \subseteq \{z \in \mathbb{T}^n : \ z^k = 1\}$. □

Lemma 4.4. Let μ be a measure on $\mathbb{T}^n$. Let an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ be strongly independent over $\mathbb{Q}$. If there is some nonzero k in $\mathbb{Z}^{n\times 1}$ such that $\hat{\mu}(B_i k) = 1$ for every $1 \le i \le n$, then μ is finitely supported.

Proof. Let $L = \begin{bmatrix} B_1 k & \cdots & B_n k \end{bmatrix} \in M_n(\mathbb{Z})$. Since $B_1 k, B_2 k, \cdots, B_n k$ are linearly independent over $\mathbb{Q}$, the matrix L is in $GL_n(\mathbb{Q})$. Write L as

$(L_{i,j})_{1\le i,j\le n}$ and put $M = \sum_{1\le i,j\le n} |L_{i,j}|$. By Lemma 4.3, the support of μ, $\mathrm{supp}(\mu)$, is a subset of $\bigcap_{i=1}^{n}\{z \in \mathbb{T}^n :\ z^{B_i k} = 1\}$. That is,

$$\begin{aligned}
\mathrm{supp}(\mu) &\subseteq \bigcap_{i=1}^{n}\{x + \mathbb{Z}^n :\ x \in [0,1)^n, xB_i k \in \mathbb{Z}\} \\
&= \{x + \mathbb{Z}^n :\ x \in [0,1)^n, xL \in \mathbb{Z}^n\} \\
&\subseteq \{wL^{-1} + \mathbb{Z}^n :\ w \in [-M, M]^n \cap \mathbb{Z}^n\}.
\end{aligned}$$

Note that $\{wL^{-1} :\ w \in [-M, M]^n \cap \mathbb{Z}^n\}$ is finite, so is $\mathrm{supp}(\mu)$. $\square$

We need the following Lemma [H, Lemma 4.2] which is a special case of the mean ergodic theorem for amenable semigroups [B, Theorem 1].

Lemma 4.5. For a compact metrizable space X and a continuous map $T : X \to X$, if ν is an ergodic T-invariant measure on X, then for every Følner sequence $\{F_m\}_{m=1}^{\infty}$ in $\mathbb{N}$, one has

$$\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} f \circ T^j = \int_X f d\nu$$

for every $f \in L^2(X, \nu)$ (note that the identity holds with respect to L^2-norm). Consequently

$$\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \int_X f(T^j x) g(x) d\nu(x) = \int_X f d\nu \int_X g d\nu \tag{4.4}$$

for all f, g in $L^2(X, \nu)$.

Proof of Theorem 4.1.

For any Borel subset E of $\mathbb{T}^n$, write 1_E for the characteristic function of E.

(1) Suppose μ is an ergodic $\times A$-invariant measure on $\mathbb{T}^n$. Applying Lemma 4.5 for $X = \mathbb{T}^n, T = T_A$ and $\nu = \mu$, we have

$$\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \int_{\mathbb{T}^n} f(T_A^j z) g(z) d\mu(z) = \int_{\mathbb{T}^n} f d\mu \int_{\mathbb{T}^n} g d\mu \tag{4.5}$$

for all continuous functions f, g on $\mathbb{T}^n$. Letting $f(z) = z^k$ and $g(z) = z^l$ for z in $\mathbb{T}^n$ and k, l in $\mathbb{Z}^{n\times 1}$, we obtain (4.1), which is the necessity.

Now assume that (4.1) holds for all k, l in $\mathbb{Z}^{n\times 1}$.

Let $k \in \mathbb{Z}^{n\times 1}$. Letting $l = 0$ in (4.1), we get $\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \hat{\mu}(A^j k) = \hat{\mu}(k)$. Replacing k by Ak, we also have

$$\hat{\mu}(Ak) = \lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \hat{\mu}(A^{j+1} k) = \lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m+1} \hat{\mu}(A^j k).$$

Then

$$|\hat{\mu}(Ak)-\hat{\mu}(k)| = \lim_{m\to\infty}\frac{1}{|F_m|}\Big|\sum_{j\in F_m+1}\hat{\mu}(A^jk)-\sum_{j\in F_m}\hat{\mu}(A^jk)\Big|$$
$$\le \lim_{m\to\infty}\frac{|(F_m+1)\Delta F_m|}{|F_m|}=0,$$

whence $\hat{\mu}(Ak)=\hat{\mu}(k)$. By Lemma 4.2, we get that μ is $\times A$-invariant.

From (4.1) we see that (4.5) is true for all $f(z)=z^k$ and $g(z)=z^l$ with k,l in $\mathbb{Z}^{n\times 1}$. By linearity, (4.5) is also true for all f,g in the linear span V of z^k for all $k\in\mathbb{Z}^{n\times 1}$. Since V is dense in $L^2(\mathbb{T}^n,\mu)$, (4.5) is true for all $f,g\in L^2(\mathbb{T}^n,\mu)$. For any Borel subset E of $\mathbb{T}^n$ satisfying $T_A^{-1}E=E$, taking $f=g=1_E$ in (4.5), we get $\mu(E)=\mu(E)^2$. Hence μ is ergodic.

(2) Suppose μ is a weakly mixing $\times A$-invariant measure on $\mathbb{T}^n$, which means $\mu\times\mu$ is an ergodic $T_A\times T_A$-invariant measure on $\mathbb{T}^{2n}$. Let $k,l\in\mathbb{Z}^{n\times 1}$. Taking $f(z',z'')=(z')^k(z'')^{-k}$ and $g(z',z'')=(z')^l(z'')^{-l}$ in (4.4) of Lemma 4.5 with $X=\mathbb{T}^n\times\mathbb{T}^n, T=T_A\times T_A$ and $\nu=\mu\times\mu$, we get

$$\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}|\hat{\mu}(A^jk+l)|^2=|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2. \tag{4.6}$$

Taking $f(z',z'')=(z')^k$ and $g(z',z'')=(z')^l$ in (4.4) of Lemma 4.5 with $X=\mathbb{T}^n\times\mathbb{T}^n, T=T_A\times T_A$ and $\nu=\mu\times\mu$, we also get

$$\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}\hat{\mu}(A^jk+l)=\hat{\mu}(k)\hat{\mu}(l). \tag{4.7}$$

Since

$$\begin{aligned}&|\hat{\mu}(A^jk+l)-\hat{\mu}(k)\hat{\mu}(l)|^2\\ =\ &|\hat{\mu}(A^jk+l)|^2+|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2-\hat{\mu}(A^jk+l)\overline{\hat{\mu}(k)\hat{\mu}(l)}-\overline{\hat{\mu}(A^jk+l)}\hat{\mu}(k)\hat{\mu}(l),\end{aligned}$$

we have

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}|\hat{\mu}(A^jk+l)-\hat{\mu}(k)\hat{\mu}(l)|^2\\
=&\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}[|\hat{\mu}(A^jk+l)|^2+|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2\\
&\qquad -\hat{\mu}(A^jk+l)\overline{\hat{\mu}(k)\hat{\mu}(l)}-\overline{\hat{\mu}(A^jk+l)}\hat{\mu}(k)\hat{\mu}(l)]\\
\overset{(4.6),(4.7)}{=}&|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2+|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2-|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2-|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2=0.\end{aligned}$$

This proves the necessity.

Conversely, suppose that (4.2) holds for all $k,l\in\mathbb{Z}^{n\times 1}$.

Note that $T_A \times T_A = T_{\mathrm{diag}(A,A)}$ on $\mathbb{T}^n \times \mathbb{T}^n = \mathbb{T}^{2n}$. In order to prove that $\mu\times\mu$ is an ergodic $T_A\times T_A$-invariant measure on $\mathbb{T}^n\times\mathbb{T}^n$, by part (1) it suffices to show that

$$\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}\widehat{\mu\times\mu}(\begin{bmatrix}A & \\ & A\end{bmatrix}^j\begin{bmatrix}k'\\k''\end{bmatrix}+\begin{bmatrix}l'\\l''\end{bmatrix})=\widehat{\mu\times\mu}(\begin{bmatrix}k'\\k''\end{bmatrix})\widehat{\mu\times\mu}(\begin{bmatrix}l'\\l''\end{bmatrix})$$

for all $k',k'',l',l''\in\mathbb{Z}^{n\times 1}$. Note that

$$\widehat{\mu\times\mu}(\begin{bmatrix}u\\v\end{bmatrix})=\hat{\mu}(u)\hat{\mu}(v)$$

for all $u,v\in\mathbb{Z}^{n\times 1}$. Thus it suffices to show

$$\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}\hat{\mu}(A^jk'+l')\hat{\mu}(A^jk''+l'')=\hat{\mu}(k')\hat{\mu}(k'')\hat{\mu}(l')\hat{\mu}(l'')$$

for all $k',k'',l',l''\in\mathbb{Z}^{n\times 1}$.

Note that

$$\begin{aligned}
& |\hat{\mu}(A^jk'+l')\hat{\mu}(A^jk''+l'')-\hat{\mu}(k')\hat{\mu}(k'')\hat{\mu}(l')\hat{\mu}(l'')| \\
\leq\ & |\hat{\mu}(A^jk'+l')[\hat{\mu}(A^jk''+l'')-\hat{\mu}(k'')\hat{\mu}(l'')]|+|[\hat{\mu}(A^jk'+l')-\hat{\mu}(k')\hat{\mu}(l')]\hat{\mu}(k'')\hat{\mu}(l'')| \\
\leq\ & |\hat{\mu}(A^jk''+l'')-\hat{\mu}(k'')(l'')|+|\hat{\mu}(A^jk'+l')-\hat{\mu}(k')\hat{\mu}(l')|
\end{aligned}$$

for all $k',k'',l',l''\in\mathbb{Z}^{n\times 1}$, whence

$$\begin{aligned}
& \lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}|\hat{\mu}(A^jk'+l')\hat{\mu}(A^jk''+l'')-\hat{\mu}(k')\hat{\mu}(k'')\hat{\mu}(l')\hat{\mu}(l'')|^2 \\
\leq\ & \lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}[|\hat{\mu}(A^jk''+l'')-\hat{\mu}(k'')\hat{\mu}(l'')|+|\hat{\mu}(A^jk'+l')-\hat{\mu}(k')\hat{\mu}(l')|]^2 \\
\leq\ & 2\lim_{m\to\infty}\frac{1}{|F_m|}\sum_{j\in F_m}[|\hat{\mu}(A^jk''+l'')-\hat{\mu}(k'')\hat{\mu}(l'')|^2+|\hat{\mu}(A^jk'+l')-\hat{\mu}(k')\hat{\mu}(l')|^2] \\
\overset{(4.2)}{=}\ & 0,
\end{aligned}$$

where in the second inequality we use $(a+b)^2\leq 2(a^2+b^2)$ for all real numbers a,b. This proves the sufficiency.

(3) Suppose μ is strongly mixing, which means that $\lim_{j\to\infty}\mu(T_A^{-j}E\cap F)=\mu(E)\mu(F)$ for all Borel subsets E,F of $\mathbb{T}^n$. Then

$$\lim_{j\to\infty}\int_{\mathbb{T}^n}1_E(T_A^jz)1_F(z)d\mu(z)=\int_{\mathbb{T}^n}1_Ed\mu\int_{\mathbb{T}^n}1_Fd\mu$$

for all Borel subsets E, F of $\mathbb{T}^n$. Since the linear combinations of characteristic functions are dense in $L^2(\mathbb{T}^n, \mu)$, we have

$$\lim_{j\to\infty} \int_{\mathbb{T}^n} f(T_A^j z) g(z) d\mu(z) = \int_{\mathbb{T}^n} f d\mu \int_{\mathbb{T}^n} g\, d\mu$$

for all $f, g \in C(\mathbb{T}^n)$. In particular, taking $f(z) = z^k$ and $g(z) = z^l$, we obtain (4.3) for all k, l in $\mathbb{Z}^{n\times 1}$. This proves the necessity.

On the other hand, suppose a measure μ on $\mathbb{T}^n$ satisfies (4.3) for all $k, l \in \mathbb{Z}^{n\times 1}$. Let $l = 0$ and replace k by Ak. Then

$$\hat{\mu}(Ak) = \lim_{j\to\infty} \hat{\mu}(A^{j+1}k) = \lim_{j\to\infty} \hat{\mu}(A^j k) = \hat{\mu}(k)$$

for all $k \in \mathbb{Z}^{n\times 1}$. Hence μ is $\times A$-invariant in view of Lemma 4.2. From (4.3) we have

$$\lim_{j\to\infty} \int_{\mathbb{T}^n} f(T_A^j z) g(z) d\mu(z) = \int_{\mathbb{T}^n} f d\mu \int_{\mathbb{T}^n} g d\mu$$

when $f(z) = z^k$ and $g(z) = z^l$ for k, l in $\mathbb{Z}^{n\times 1}$. Since the linear combinations of z^k for $k \in \mathbb{Z}^{n\times 1}$ are dense in $L^2(\mathbb{T}^n, \mu)$, the above is also true for all $f, g \in L^2(\mathbb{T}^n, \mu)$. In particular it holds for $f = 1_E$ and $g = 1_F$ for any Borel subsets E, F of $\mathbb{T}^n$, that is,

$$\lim_{j\to\infty} \mu(T_A^{-j} E \cap F) = \mu(E)\mu(F).$$

□

5. Measure Rigidity on $\mathbb{T}^n$

In this section we prove Theorem 1.7 and Corollary 1.8. For this we need the following Lemma [H, Lemma 5.1].

Lemma 5.1. Let $T : X \to X$ be a continuous map on a compact metrizable space X. Then a weakly mixing T-invariant measure μ on X with an atom is always a Dirac measure, i.e. $\text{supp}(\mu)$ is a singleton.

Note that a measure μ on $\mathbb{T}^n$ is the Lebesgue measure iff $\hat{\mu}(k) = 0$ for all nonzero $k \in \mathbb{Z}^{n\times 1}$.

Proof of Theorem 1.7.

(1) Suppose μ is an ergodic $\times A$-invariant measure on $\mathbb{T}^n$ and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ which is strongly independent over $\mathbb{Q}$ and a Følner sequence $\Sigma = \{F_m\}_{m=1}^\infty$ in $\mathbb{N}$ such that μ is $\times(A^j + B_i)$-invariant for every $1 \le i \le n$ and j in some $E \subseteq \mathbb{N}$ with $\overline{D}_\Sigma(E) = 1$. Passing to a subsequence of Σ if necessary, we may assume that $\lim_{m\to\infty} \frac{|F_m \cap E|}{|F_m|} = 1$. By Lemma 4.2, we have $\hat{\mu}(A^j k + B_i k) = \hat{\mu}(k)$ for all $j \in E$, $1 \le i \le n$ and $k \in \mathbb{Z}^{n\times 1}$.

Assume that μ is not the Lebesgue measure. Then there exists a nonzero $k \in \mathbb{Z}^{n\times 1}$ such that $\hat{\mu}(k) \neq 0$.

Since μ is an ergodic $\times A$-invariant measure, by Theorem 4.1 (1), we have $\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} \hat{\mu}(A^j k + B_i k) = \hat{\mu}(k)\hat{\mu}(B_i k)$ for every $1 \leq i \leq n$. Note that

$$\begin{aligned}
&\frac{1}{|F_m|} \sum_{j\in F_m} \hat{\mu}(A^j k + B_i k) \\
&= \frac{1}{|F_m|} \sum_{j\in F_m\cap E} \hat{\mu}(A^j k + B_i k) + \frac{1}{|F_m|} \sum_{j\in F_m\setminus E} \hat{\mu}(A^j k + B_i k) \\
&= \frac{|F_m \cap E|}{|F_m|}\hat{\mu}(k) + \frac{1}{|F_m|} \sum_{j\in F_m\setminus E} \hat{\mu}(A^j k + B_i k) \to \hat{\mu}(k)
\end{aligned}$$

as $m \to \infty$. Hence $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(B_i k)$ which implies $\hat{\mu}(B_i k) = 1$ for every $1 \leq i \leq n$. From Lemma 4.4 we get that μ is finitely supported.

(2) Suppose μ is a weakly mixing $\times A$-invariant measure on $\mathbb{T}^n$ and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ which is strongly independent over $\mathbb{Q}$ and a Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ such that μ is $\times(A^j + B_i)$-invariant for every $1 \leq i \leq n$ and j in some $E \subseteq \mathbb{N}$ with $\overline{D}_\Sigma(E) > 0$. By Lemma 4.2, we have $\hat{\mu}(A^j k + B_i k) = \hat{\mu}(k)$ for all $j \in E$, $1 \leq i \leq n$ and $k \in \mathbb{Z}^{n\times 1}$.

Assume that μ is not the Lebesgue measure. Then there exists a nonzero $k \in \mathbb{Z}^{n\times 1}$ such that $\hat{\mu}(k) \neq 0$.

Let $1 \leq i \leq n$. Since μ is a weakly mixing $\times A$-invariant measure, by Theorem 4.1 (2), we have $\lim_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} |\hat{\mu}(A^j k + B_i k) - \hat{\mu}(k)\hat{\mu}(B_i k)|^2 = 0$. Therefore,

$$\begin{aligned}
0 &= \limsup_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m} |\hat{\mu}(A^j k + B_i k) - \hat{\mu}(k)\hat{\mu}(B_i k)|^2 \\
&\geq \limsup_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m\cap E} |\hat{\mu}(A^j k + B_i k) - \hat{\mu}(k)\hat{\mu}(B_i k)|^2 \\
&= \limsup_{m\to\infty} \frac{1}{|F_m|} \sum_{j\in F_m\cap E} |\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(B_i k)|^2 \\
&= |\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(B_i k)|^2 \overline{D}_\Sigma(E).
\end{aligned}$$

Hence $\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(B_i k) = 0$, which implies that $\hat{\mu}(B_i k) = 1$. From Lemma 4.4 we get that μ is finitely supported.

(3) Suppose μ is a strongly mixing $\times A$-invariant measure on $\mathbb{T}^n$ and there exist an n-tuple $(B_1, B_2, \cdots, B_n)$ of matrices in $M_n(\mathbb{Z})$ which is

strongly independent over $\mathbb{Q}$ and an infinite set $E \subseteq \mathbb{N}$ such that μ is $\times(A^j + B_i)$-invariant for every $1 \leq i \leq n$ and j in E.

Assume that μ is not the Lebesgue measure. Then there exists a nonzero $k \in \mathbb{Z}^{n\times 1}$ such that $\hat{\mu}(k) \neq 0$.

Let $1 \leq i \leq n$. Since μ is a strongly mixing $\times A$-invariant measure, by Theorem 4.1 (3), we have

$$\lim_{j\to\infty} \hat{\mu}(A^j k + B_i k) = \hat{\mu}(k)\hat{\mu}(B_i k).$$

Owing to μ being $\times(A^j + B_i)$-invariant for all $j \in E$, by Lemma 4.2 one has $\hat{\mu}(A^j k + B_i k) = \hat{\mu}(k)$ for all $j \in E$. Consequently, $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(B_i k)$, which implies $\hat{\mu}(B_i k) = 1$. From Lemma 4.4 we get that μ is finitely supported.

Suppose μ is a measure on $\mathbb{T}^n$ satisfying (2) or (3) of Theorem 1.7. If μ is not a Lebesgue measure, then μ is finitely supported. According to Lemma 5.1, we conclude that μ is a Dirac measure on $\mathbb{T}^n$. □

Proof of Corollary 1.8.

Take a nonzero B in $M_n(\mathbb{Z})$ with $P_B(t)$ irreducible in $\mathbb{Q}[t]$ (see Remark 3.3). Then B is strongly independent over $\mathbb{Q}$ by Theorem 1.5. The multiplicative semigroup S generated by $\{B, B^j + B^i\}_{0\leq i\leq n-1, j\geq 1}$, where we put $B^0 = I_n$, is what we need. □

Acknowledgement

We thank helpful comments from Huaxin Lin, Kunyu Guo, Wenming Wu, Shengkui Ye and Yi Gu. We also thank the referee for comments and suggestions, which greatly improve the article.

References

[B] T. Bewley. Extension of the Birkhoff and von Neumann ergodic theorems to semigroup actions. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **7** (1971), 283–291.

[EF] M. Einsiedler and A. Fish. Rigidity of measures invariant under the action of a multiplicative semigroup of polynomial growth on $\mathbb{T}$. *Ergodic Theory Dynam. Systems* **30** (2010), no. 1, 151–157.

[F] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory* **1** (1967), 1–49.

[H] H. Huang. Fourier coefficients of $\times p$-invariant measures. *J. Mod. Dyn.* **11** (2017), 551–562.

[J] A. S. A. Johnson. Measures on the circle invariant under multiplication by a nonlacunary subsemigroup of the integers. *Israel J. Math.* **77** (1992), 211–240.

[KK] B. Kalinin and A. Katok. Invariant measures for actions of higher rank abelian groups. *Smooth Ergodic Theory and its Applications (Seattle, WA, 1999)*, 593—637, Proc. Sympos. Pure Math., **69**, Amer. Math. Soc., Providence, RI, 2001.

[KS1] A. Katok and R. J. Spatzier. Invariant measures for higher-rank hyperbolic abelian actions. *Ergodic Theory Dynam. Systems* **16** (1996), 751–778.

[KS2] A. Katok and R. J. Spatzier. Corrections to "Invariant measures for higher-rank hyperbolic abelian actions". *Ergodic Theory Dynam. Systems* **18** (1998), 503–507.

[La] S. Lang. *Algebra.* Revised third edition. Graduate Texts in Mathematics, **211**. Springer-Verlag, New York, 2002.

[L] R. Lyons. On measures simultaneously 2- and 3-invariant. *Israel J. Math.* **61** (1988), 219–224.

[Ro] S. Roman. *Advanved Linear Algebra.* Third edition. Graduate Texts in Mathematics, **135**. Springer, New York, 2008.

[R] D. J. Rudolph. $\times 2$ and $\times 3$ invariant measures and entropy. *Ergodic Theory Dynam. Systems* **10** (1990), 395–406.

[S] E. A. Sataev. On measures invariant with respect to polynomial semigroups of circle transformations. *Uspehi Mat. Nauk.* **30** (1975), no.2, 203–204.

[W] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics, **79**. Springer-Verlag, New York-Berlin, 1982.

Huichi Huang, College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P.R. China
Email address: huanghuichi@cqu.edu.cn

Hanfeng Li, Center of Mathematics, Chongqing University, Chongqing 401331, P.R. China and Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900, USA
Email address: hfli@math.buffalo.edu

Enhui Shi, School of mathematical science, Soochow University, Suzhou, 215006, P.R. China
Email address: ehshi@suda.edu.cn

Hui Xu, School of mathematical science, Soochow University, Suzhou, 215006, P.R. China
Email address: 20184007001@stu.suda.edu.cn