STRICT p-NEGATIVE TYPE OF A METRIC SPACE

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Abstract. Doust and Weston [8] have introduced a new method called *enhanced negative type* for calculating a non-trivial lower bound $\wp_T$ on the supremal strict p-negative type of any given finite metric tree (T, d). In the context of finite metric trees any such lower bound $\wp_T > 1$ is deemed to be non-trivial. In this paper we refine the technique of enhanced negative type and show how it may be applied more generally to any finite metric space (X, d) that is known to have strict p-negative type for some $p \geq 0$. This allows us to significantly improve the lower bounds on the supremal strict p-negative type of finite metric trees that were given in [8, Corollary 5.5] and, moreover, leads in to one of our main results: The supremal p-negative type of a finite metric space cannot be strict. By way of application we are then able to exhibit large classes of finite metric spaces (such as finite isometric subspaces of Hadamard manifolds) that must have strict p-negative type for some $p > 1$. We also show that if a metric space (finite or otherwise) has p-negative type for some $p > 0$, then it must have strict q-negative type for all $q \in [0, p)$. This generalizes Schoenberg [27, Theorem 2] and leads to a complete classification of the intervals on which a metric space may have strict p-negative type.

1. Introduction and Synopsis

The study of positive definite kernels and the related notion of p-negative type metrics dates back to the early 1900s with some antecedents in the 1800s. A major theme that emerged was the search for metric characterizations of subsets of Hilbert space up to isometry. Significant initial results on this classical embedding problem were obtained by Cayley [6], Menger [21, 22, 23] and Schoenberg [26, 27, 28]. We note in particular [28, Theorem 1]: A metric space is isometric to a subset of Hilbert space if and only if it has 2-negative type. This result was spectacularly generalized to the category of normed spaces by Bretagnolle et al. [5, Theorem 2]: A real normed space is linearly isometric to a subspace of some L_p-space $(1 \leq p \leq 2)$ if and only if it has p-negative type. It remains a prominent question to give a complete generalization of this result to the setting of non-commutative L_p-spaces. See, for example, Junge [15].

More recently, difficult questions concerning p-negative type metrics have figured prominently in theoretical computer science. A prime example is the recently refuted *Goemans-Linial conjecture*: Every metric space of 1-negative type bi-Lipschitz embeds into some L_1-space. Although this conjecture clearly holds for normed spaces by [5, Theorem 2] (with $p = 1$), it is not true for arbitrary metric spaces. This was shown by Khot and Vishnoi [18]. Subsequently, Lee and Naor [19]

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have shown that there is no metric version of [5, Theorem 2] (modulo bi-Lipschitz embeddings) for any $p \in [1, 2)$. More precisely, we have [19, Theorem 1.2]: For each $p \in [1, 2)$ there is a metric space (X, d) of p-negative type which does not bi-Lipschitz embed into any L_p-space.

The related notion of strict p-negative type has been studied rather less well than its classical counterpart and most known results deal with the case $p = 1$. The present work is motivated by functional analytic questions that arise naturally from the papers of Hjorth et al. [13, 14]. Both [13] and [14] focus on examples and properties of finite metric spaces of strict 1-negative type. One theme of these papers is to determine global geometric properties of finite metric spaces of strict 1-negative type. As an example we mention [14, Theorem 3.9]: If a finite metric space is of strict 1-negative type, then it has a unique ∞-extender. It is also natural to ask for conditions on a finite metric space which will guarantee that it has strict 1-negative type. One such result is [14, Theorem 5.2]: If a finite metric space is hypermetric and regular, then it is of strict 1-negative type.

The theme of this paper is to determine basic properties of strict p-negative metrics for all $p \geq 0$. In particular, we aim to move beyond the familiar case $p = 1$, thereby setting up the rudiments of a basic theory of strict p-negative type metrics.

Section 2 is dedicated to a review of the salient features of generalized roundness, negative type, and strict negative type. In Definition 2.7 we recall the notion of the (normalized) p-negative type gap Γ_X^p of a metric space (X, d). This parameter was recently introduced by Doust and Weston [8, 9] in order to obtain non-trivial lower bounds on the maximal p-negative of finite metric trees. Basic properties of Γ_X^p will play a vital rôle in our computations in Section 3.

The observation is made in [8, Theorem 5.2] that if the p-negative type gap Γ_X^p of a finite metric space (X, d) is positive for some $p \geq 0$, then (X, d) must have strict q-negative type on some interval of the form $[p, p+\zeta)$ where $\zeta > 0$. However, the authors only provide an explicit value for ζ in the case $p = 1$. Letting $n = |X|$, the value of ζ given in this case is $O(1/n^2)$. (See [8, Theorem 5.1].) The purpose of Section 3 is to give a precise quantitative version of [8, Theorem 5.2] which yields significantly improved values of ζ for all $p \geq 0$. In fact, for each $p \geq 0$, our value of ζ is $O(1)$. The precise statement of this result is given in Theorem 3.3. By way of application, Theorem 3.3 leads to significantly improved lower bounds on the maximal p-negative type of finite metric trees. These are given in Corollary 3.4. Then in Remark 3.5 we point out that the estimates given in Corollary 3.4 are reasonably sharp for finite metric trees that resemble stars. This suggests there is little room for improvement in the statement of Theorem 3.3 (in general).

In Section 4 we use Theorem 3.3 and an elementary compactness argument to derive a key result of this paper: The supremal p-negative type of a finite metric space cannot be strict. This is done in Corollary 4.3 to Theorem 4.1. Using known results we are then able to exhibit large classes of finite metric spaces, all of which must have strict p-negative type for some $p > 1$. For example, any finite isometric subspace of a Hadamard manifold must have strict p-negative type for some $p > 1$. We collate an array of such examples in Corollary 4.5.

The main results of Section 5 are Theorems 5.4, 5.5 and 5.8. These theorems generalize [27, Theorem 2]. For example, in Theorem 5.4 we show that if a metric space (finite or otherwise) has p-negative type for some $p > 0$, then it must have strict q-negative type for all $q \in [0, p)$. This allows us to precisely codify the types of intervals on which a metric space may have strict p-negative type. It is interesting to note that finite metric spaces behave quite differently to infinite metric spaces in this respect. These differences are highlighted in Theorems 5.5 and 5.8. Understanding how strict negative type behaves on intervals leads to further examples of metric spaces that have non-trivial strict p-negative type. We then conclude the paper with the observation in Remark 5.13 that Theorems 3.3, 4.1 and 5.4 (as well as several of our corollaries) actually hold more generally for finite semi-metric spaces. This is because we do not use the triangle inequality at any point in our definitions or proofs.

Throughout this paper the set of natural numbers $\mathbb{N}$ is taken to consist of all positive integers and sums indexed over the empty set are always taken to be zero. Given a real number x, we are using $\lfloor x \rfloor$ to denote the largest integer that does not exceed x, and $\lceil x \rceil$ to denote the smallest integer which is not less than x.

2. A framework for ordinary and strict p-negative type

We begin by recalling some theoretical features of (strict) p-negative type and its relationship to (strict) generalized roundness. More detailed accounts may be found in the work of Benyamini and Lindenstrauss [3], Deza and Laurent [7], Prassidis and Weston [25], and Wells and Williams [29]. These works emphasize the interplay between the classical p-negative type inequalities and isometric, Lipschitz or uniform embeddings. They also indicate applications to more contemporary areas of interest such as theoretical computer science. One of the most important results for our purposes is the equivalence of (strict) p-negative type and (strict) generalized roundness p. These equivalences are described in Theorem 2.5.

Definition 2.1. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has *p-negative type* if and only if for all natural numbers $k \geq 2$, all finite subsets $\{x_1, \ldots, x_k\} \subseteq X$, and all choices of real numbers $\eta_1, \ldots, \eta_k$ with $\eta_1 + \cdots + \eta_k = 0$, we have:

$$\sum_{1 \leq i,j \leq k} d(x_i, x_j)^p \eta_i \eta_j \quad \leq \quad 0. \tag{1}$$

(b) (X, d) has *strict p-negative type* if and only if it has p-negative type and the associated inequalities (1) are all strict except in the trivial case $(\eta_1, \ldots, \eta_k) = (0, \ldots, 0)$.

Remark 2.2. Every metric space obviously has strict 0-negative type. It is also the case that every finite metric space has strict p-negative type for some $p > 0$. This follows from Weston [30, Theorem 4.3], Theorem 2.5 (a) and Theorem 5.4.

It is possible to reformulate both ordinary and strict p-negative type in terms of an invariant known as *generalized roundness* from the uniform theory of Banach spaces. Generalized roundness was introduced by Enflo [11] in order to solve (in the negative) *Smirnov's Problem*: Is every separable metric space uniformly homeomorphic to a subset of Hilbert space? The analog of this problem for coarse embeddings was later raised by Gromov [12] and solved negatively by Dranishnikov et al. [10]. Prior to introducing generalized roundness in Definition 2.4 (a) we shall develop some intermediate technical notions in order to streamline the exposition in the remainder of this paper.

Definition 2.3. Let s, t be arbitrary natural numbers and let X be any set.

(a) An (s,t)-*simplex* in X is an $(s+t)$-vector $(a_1, \ldots, a_s, b_1, \ldots, b_t) \in X^{s+t}$ consisting of $s+t$ pairwise distinct coordinates $a_1, \ldots, a_s, b_1, \ldots, b_t \in X$. Such a simplex will be denoted by $D = [a_j; b_i]_{s,t}$.

(b) A *load vector* for an (s,t)-simplex $D = [a_j; b_i]_{s,t}$ in X is an arbitrary vector $\vec{\omega} = (m_1, \ldots m_s, n_1, \ldots, n_t) \in \mathbb{R}_+^{s+t}$ that assigns a positive weight $m_j > 0$ or $n_i > 0$ to each vertex a_j or b_i of D, respectively.

(c) A *loaded* (s,t)-*simplex* in X consists of an (s,t)-simplex $D = [a_j; b_i]_{s,t}$ in X together with a load vector $\vec{\omega} = (m_1, \ldots, m_s, n_1, \ldots, n_t)$ for D. Such a loaded simplex will be denoted by $D(\vec{\omega})$ or $[a_j(m_j); b_i(n_i)]_{s,t}$ as the need arises.

(d) A *normalized* (s,t)-*simplex* in X is a loaded (s,t)-simplex $D(\vec{\omega})$ in X whose load vector $\vec{\omega} = (m_1, \ldots, m_s, n_1, \ldots, n_t)$ satisfies the two normalizations:

$$m_1 + \cdots + m_s = 1 = n_1 + \cdots n_t.$$

Such a vector $\vec{\omega}$ will be called a *normalized load vector* for D.

Rather than giving the original definition of generalized roundness p from [11], we shall present an equivalent reformulation in Definition 2.4 (a) that is due to Lennard et al. [20] and Weston [30].

Definition 2.4. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has *generalized roundness* p if and only if for all $s, t \in \mathbb{N}$ and all normalized (s,t)-simplices $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{s,t}$ in X we have:

$$\begin{aligned} & \sum_{1 \leq j_1 < j_2 \leq s} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p \\ \leq \; & \sum_{j,i=1}^{s,t} m_j n_i d(a_j, b_i)^p. \end{aligned} \tag{2}$$

(b) (X, d) has *strict generalized roundness* p if and only if it has generalized roundness p and the associated inequalities (2) are all strict.

Two key aspects of generalized roundness for the purposes of this paper are the following equivalences.

Theorem 2.5 ([20], [8])**.** *Let $p \geq 0$ and let (X, d) be a metric space. Then:*

(a) *(X, d) has p-negative type if and only if it has generalized roundness p.*

(b) *(X, d) has strict p-negative type if and only if it has strict generalized roundness p.*

Based on Definition 2.4 (a) and Theorem 2.5 we introduce two numerical parameters $\gamma_D^p(\vec{\omega})$ and Γ_X^p that are designed to quantify the *degree of strictness* of the non-trivial p-negative type inequalities.

Definition 2.6. Let $p \geq 0$ and let (X, d) be a metric space. Let s, t be natural numbers and $D = [a_j; b_i]_{s,t}$ be an (s,t)-simplex in X. Denote by $N_{s,t}$ the set of all normalized load vectors $\vec{\omega} = (m_1, \ldots, m_s, n_1, \ldots, n_t) \subset \mathbb{R}_+^{s+t}$ for D. Then the *(normalized) p-negative type simplex gap* of D is defined to be the function

$\gamma_D^p : N_{s,t} \to \mathbb{R}$ where

$$\begin{aligned}\gamma_D^p(\vec{\omega}) &= \sum_{j,i=1}^{s,t} m_j n_i d(a_j, b_i)^p - \sum_{1 \le j_1 < j_2 \le s} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p \\ &\quad - \sum_{1 \le i_1 < i_2 \le t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p\end{aligned}$$

for each $\vec{\omega} = (m_1, \ldots, m_s, n_1, \ldots, n_t) \in N_{s,t}$.

Notice that $\gamma_D^p(\vec{\omega})$ takes the difference between the right side and the left side of the inequality (2). So, by Theorem 2.5, (X, d) has strict p-negative type if and only if $\gamma_D^p(\vec{\omega}) > 0$ for each normalized (s, t)-simplex $D(\vec{\omega})$ in X.

Definition 2.7. Let $p \ge 0$. Let (X, d) be a metric space with p-negative type. We define the *(normalized) p-negative type gap* of (X, d) to be the non-negative quantity

$$\Gamma_X^p = \inf_{D(\vec{\omega})} \gamma_D^p(\vec{\omega})$$

where the infimum is taken over all normalized (s, t)-simplices $D(\vec{\omega})$ in X.

Recall that a *finite metric tree* is a finite connected graph that has no cycles, endowed with an edge weighted path metric. Hjorth et al. [14] have shown that finite metric trees have strict 1-negative type. Therefore it makes sense to try to compute the 1-negative type gap of any given finite metric tree. Indeed, a very succinct formula was derived in [8, Corollary 4.13]. However, a modicum of additional notation is necessary before stating this result. The set of all edges in a metric tree (T, d), considered as unordered pairs, will be denoted $E(T)$, and the metric length $d(x, y)$ of any given edge $e = (x, y) \in E(T)$ will be denoted $|e|$.

Theorem 2.8 (Doust and Weston [8])**.** *Let (T, d) be a finite metric tree. Then the (normalized) 1-negative type gap $\Gamma = \Gamma_T^1$ of (T, d) is given by the following formula:*

$$\Gamma = \left\{ \sum_{e \in E(T)} |e|^{-1} \right\}^{-1}.$$

In particular, $\Gamma > 0$.

In the remaining sections of this paper we shall show how the notions, equivalences and results of this section may be used to infer some basic properties of metrics of strict p-negative metrics for general values of $p \ge 0$.

3. A quantitative lower bound on supremal strict p-negative type

The observation is made in [8, Theorem 5.2] that if the p-negative type gap Γ_X^p of a finite metric space (X, d) is positive for some $p \ge 0$, then (X, d) must have strict q-negative type on some interval of the form $[p, p + \zeta)$ where $\zeta > 0$. However, the authors only provide an explicit value for ζ in the case $p = 1$. Letting $n = |X|$, the value of ζ given in this case is $O(1/n^2)$. (See [8, Theorem 5.1].) The purpose of the present section is to give a precise quantitative version of [8, Theorem 5.2] which yields significantly improved values of ζ for all $p \ge 0$. In fact, for each $p \ge 0$, our value of ζ is $O(1)$. The precise statement of this result is given in Theorem 3.3. As an application we obtain significantly improved lower bounds on the maximal p-negative type of finite metric trees. These are stated in Corollary 3.4. Then in

Remark 3.5 we point out that the estimates given in Corollary 3.4 are actually close to best possible for finite metric trees that resemble stars. This suggests there is little room for improvement in the statement of Theorem 3.3, the main result of this section.

The proof of Theorem 3.3 is facilitated by the following two technical lemmas which are easily realized using basic calculus or by simple combinatorial arguments. The proofs of these lemmas are therefore omitted.

Lemma 3.1. *Let $s \in \mathbb{N}$. If s real variables $\ell_1, \ldots, \ell_s > 0$ are subject to the constraint $\ell_1 + \cdots + \ell_s = 1$, then the expression*

$$\sum_{k_1 < k_2} \ell_{k_1} \ell_{k_2}$$

has maximum value $\frac{s(s-1)}{2} \cdot \frac{1}{s^2} = \frac{1}{2}(1 - \frac{1}{s})$ which is attained when $\ell_1 = \cdots = \ell_s = \frac{1}{s}$.

Lemma 3.2. *Let $s, t \in \mathbb{N}$ and let $m = s + t$. Then*

$$\frac{1}{2}\left(1 - \frac{1}{s}\right) + \frac{1}{2}\left(1 - \frac{1}{t}\right) \leq 1 - \frac{1}{2}\left(\frac{1}{\lfloor \frac{m}{2} \rfloor} + \frac{1}{\lceil \frac{m}{2} \rceil}\right).$$

Moreover, the function $\gamma(m) = 1 - \frac{1}{2}\left(\frac{1}{\lfloor \frac{m}{2} \rfloor} + \frac{1}{\lceil \frac{m}{2} \rceil}\right)$ increases strictly as m increases.

We will continue to use the notation $\gamma(m) = 1 - \frac{1}{2} \cdot \left(\lfloor \frac{m}{2} \rfloor^{-1} + \lceil \frac{m}{2} \rceil^{-1}\right)$ introduced in the preceding lemma throughout the remainder of this section as it allows the efficient statement and succinct proof of certain key formulas such as Theorem 3.3.

The following basic notions are also relevant to the proof of Theorem 3.3. Let (X, d) be a metric space. If $d(x, y) = 1$ for all $x \neq y$, then d is called the *discrete metric* on X. The *metric diameter* of (X, d) is given by the quantity $\operatorname{diam} X = \sup\{d(x, y) | x, y \in X\}$. Provided $|X| < \infty$, the *scaled metric diameter* of (X, d) is given by the ratio $\mathfrak{D}_X = (\operatorname{diam} X)/ \min\{d(x, y) | x \neq y\}$.

Theorem 3.3. *Let (X, d) be a finite metric space with cardinality $n = |X| \geq 3$ and let $p \geq 0$. If the p-negative type gap Γ_X^p of (X, d) is positive, then (X, d) has q-negative type for all $q \in [p, p + \zeta]$ where*

$$\zeta = \frac{\ln\left(1 + \frac{\Gamma_X^p}{(\operatorname{diam} X)^p \cdot \gamma(n)}\right)}{\ln \mathfrak{D}_X}.$$

Moreover, (X, d) has strict q-negative type for all $q \in [p, p + \zeta)$. In particular, $p + \zeta$ provides a lower bound on the supremal (strict) q-negative type of (X, d).

Proof. For notational ease we set $\Gamma = \Gamma_X^p$ and $\mathfrak{D} = \mathfrak{D}_X$ throughout this proof. We may assume that the metric d is not a positive multiple of the discrete metric on X. Otherwise, (X, d) would have strict q-negative type for all $q \geq 0$. Hence $\mathfrak{D} > 1$.

Since scaling the metric by a positive constant has no effect on whether the space has p-negative type, we may assume that $\min\{d(x, y) | x \neq y\} = 1$. This means that $\mathfrak{D}$ is now the diameter of our rescaled metric space (which we will continue to denote by (X, d)). Moreover, for all $\ell = d(x, y) \neq 0$ and all $\zeta > 0$, we have $\ell^{p+\zeta} - \ell^p \leq \mathfrak{D}^{p+\zeta} - \mathfrak{D}^p$. This is because, for any fixed $\zeta > 0$, the function $f(x) = x^{p+\zeta} - x^p$ is increasing on the interval $[1, \infty)$. This inequality will be used in the derivation of (5) below.

Consider an arbitrary normalized (s,t)-simplex $D = [a_j(m_j); b_i(n_i)]_{s,t}$ in X. Necessarily, $m = s + t \leq n$. For any given $r \geq 0$, let

$$\begin{aligned} L(r) &= \sum_{j_1<j_2} m_{j_1}m_{j_2}d(a_{j_i}, a_{j_2})^r + \sum_{i_1<i_2} n_{i_1}n_{i_2}d(b_{i_1}, b_{i_2})^r, \text{ and} \\ R(r) &= \sum_{j,i} m_j n_i d(a_j, b_i)^r. \end{aligned}$$

By definition of the p-negative type gap Γ we have

$$L(p) + \Gamma \leq R(p). \tag{3}$$

The strategy of the proof is to argue that

$$L(p+\zeta) < L(p) + \Gamma \quad \text{and} \quad R(p) \leq R(p+\zeta) \tag{4}$$

provided $\zeta > 0$ is sufficiently small. If so, then $L(p+\zeta) < R(p+\zeta)$ by (3) and (4). In other words, (X, d) has strict $(p+\zeta)$-negative type under these circumstances. Now, as all non-zero distances in (X, d) are at least one, we automatically obtain the second inequality of (4) for all $\zeta > 0$. Therefore we only need to concentrate on the first inequality of (4). First of all, notice that

$$\begin{aligned} L(p+\zeta) - L(p) &= \sum_{j_1<j_2} m_{j_1}m_{j_2}\big(d(a_{j_1}, a_{j_2})^{p+\zeta} - d(a_{j_1}, a_{j_2})^p\big) \\ &\quad + \sum_{i_1<i_2} n_{i_1}n_{i_2}\big(d(b_{i_1}, b_{i_2})^{p+\zeta} - d(b_{i_1}, b_{i_2})^p\big) \\ &\leq \left(\sum_{j_1<j_2} m_{j_1}m_{j_2} + \sum_{i_1<i_2} n_{i_1}n_{i_2}\right)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big) \\ &\leq \left(1 - \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right)\right)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big) \\ &\leq \left(1 - \frac{1}{2}\left(\frac{1}{\lfloor\frac{m}{2}\rfloor} + \frac{1}{\lceil\frac{m}{2}\rceil}\right)\right)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big) \\ &= \gamma(m)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big) \\ &\leq \gamma(n)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big), \end{aligned} \tag{5}$$

by applying Lemmas 3.1 and 3.2. Now observe that:

$$\gamma(n)\cdot\big(\mathfrak{D}^{p+\zeta} - \mathfrak{D}^p\big) \leq \Gamma \quad \text{iff} \quad \zeta \leq \frac{\ln\left(1 + \frac{\Gamma}{\mathfrak{D}^p\cdot\gamma(n)}\right)}{\ln\mathfrak{D}}. \tag{6}$$

By combining (5) and (6), we obtain the first inequality of (4) for all $\zeta > 0$ such that

$$\zeta < \zeta_0 = \frac{\ln\left(1 + \frac{\Gamma}{\mathfrak{D}^p\cdot\gamma(n)}\right)}{\ln\mathfrak{D}}.$$

Hence $L(p+\zeta) < R(p+\zeta)$ for any such ζ. It is also clear from (4), (5) and (6) that $L(\zeta_0) \leq R(\zeta_0)$. These observations and descaling the metric (if necessary) complete the proof of the theorem. $\square$

Recall that the *ordinary path metric* on a finite tree T assigns length one to each edge in the tree (with all other distances determined geodesically). With this in mind, we see that Theorem 3.3 provides a significant improvement of the estimate given in [8, Corollary 5.5].

Corollary 3.4. *Let T be a finite tree on $n = |T| \geq 3$ vertices that is endowed with the ordinary path metric d. Let $\mathfrak{D}$ denote the metric diameter of the resulting finite metric tree (T, d). Let $\wp_T$ denote the maximal p-negative type of (T, d). Then:*

$$\wp_T \quad \geq \quad 1 + \left\{ \ln\left(1 + \frac{1}{\mathfrak{D} \cdot (n-1) \cdot \gamma(n)}\right) \Big/ \ln \mathfrak{D} \right\}. \tag{7}$$

Proof. By Theorem 2.8, $\Gamma_T^1 = \frac{1}{n-1}$. Now apply Theorem 3.3 with $p = 1$. □

Remark 3.5. The lower bound on $\wp_T$ given in the statement of Corollary 3.4 is basically of the correct order of magnitude when $\mathfrak{D} = 2$. To see this, first of all notice that if $n > 2$ is even and $\mathfrak{D} = 2$, then (7) in Corollary 3.4 simplifies to give:

$$\wp_T \geq 1 + \left\{ \ln\left(1 + \frac{n}{2(n-1)(n-2)}\right) \Big/ \ln 2 \right\}.$$

However, if T denotes a star with $n - 1$ leaves (endowed with the ordinary path metric), then [8, Theorem 5.6] gives the exact value:

$$\wp_T = 1 + \left\{ \ln\left(1 + \frac{1}{n-2}\right) \Big/ \ln 2 \right\}.$$

4. Supremal p-negative type of a finite metric space cannot be strict

If the p-negative type gap Γ_X^p of a metric space (X, d) is positive then (X, d) clearly has strict p-negative type. It is interesting to ask to what extent — if any — the converse of this statement is true. Our next result points out that the converse statement is always true in the case of finite metric spaces. By way of a notable contrast, [8, Theorem 5.7] shows that there exist infinite metric trees (X, d) of strict 1-negative type with 1-negative type gap $\Gamma_X^1 = 0$.

Theorem 4.1. *Let $p \geq 0$ and let (X, d) be a finite metric space. Then (X, d) has strict p-negative type if and only if $\Gamma_X^p > 0$.*

Proof. Let $p \geq 0$ be given. We need only concern ourselves with the forward implication of the theorem since the converse is clear from the definitions.

Assume that (X, d) is a finite metric space with strict p-negative type. By Theorem 2.5, $\gamma_D^p(\vec{\omega}) > 0$ for each normalized (s, t)-simplex $D(\vec{\omega}) \subseteq X$. Referring back to Definitions 2.3 and 2.6 we further note that we may assume that each such p-negative type simplex gap γ_D^p is defined on the compact set $\overline{N_{s,t}} \subset \mathbb{R}^{s+t}$ and is positive at each point of $\overline{N_{s,t}}$. Therefore

$$\min\left\{ \gamma_D^p(\vec{\omega}) \,|\, \vec{\omega} \in \overline{N_{s,t}} \right\} > 0$$

for each (s, t)-simplex D in X. But as $|X| < \infty$ the number of distinct (s, t)-simplexes D that can be formed from X must be finite. Thus the p-negative type gap Γ_X^p is seen to be the minimum of finitely many positive quantities. As such we obtain the desired result: $\Gamma_X^p > 0$. □

Corollary 4.2. *Let $p \geq 0$ and let (X, d) be a finite metric space. If (X, d) has strict p-negative type, then (X, d) must have strict q-negative type for some interval of values $q \in [p, p + \zeta)$, $\zeta > 0$.*

Proof. By Theorem 4.1, $\Gamma = \Gamma_X^p > 0$. Now apply Theorem 3.3. □

As an immediate consequence of Corollary 4.2 we obtain one of the main results of this paper.

Corollary 4.3. *The supremal p-negative type of a finite metric space cannot be strict.*

Moreover, since p-negative type holds on closed intervals, we therefore obtain an interesting case of equality in the classical negative type inequalities as a direct consequence of Corollary 4.3.

Corollary 4.4. *Let (X, d) be a finite metric space. Let $\wp$ denote the supremal p-negative type of (X, d). If $\wp < \infty$ then there exists a normalized (s, t)-simplex $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{s,t}$ in X such that $\gamma_D^{\wp}(\vec{\omega}) = 0$. In other words, we obtain:*

$$\begin{aligned} & \sum_{1 \leq j_1 < j_2 \leq s} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^{\wp} + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^{\wp} \\ = & \sum_{j,i=1}^{s,t} m_j n_i d(a_j, b_i)^{\wp}. \end{aligned}$$

Corollary 4.5. *The following finite metric spaces all have strict q-negative type for some interval of values $q \in [1, 1 + \zeta)$ (where $\zeta > 0$ depends upon the particular space):*

(a) *Any three-point metric space.*
(b) *Any finite metric tree.*
(c) *Any finite isometric subspace of a k-sphere $\mathbb{S}^k$ (endowed with the usual geodesic metric) that contains at most one pair of antipodal points.*
(d) *Any finite isometric subspace of the hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$ (or $\mathbb{H}^k_{\mathbb{C}}$).*
(e) *Any finite isometric subspace of a Hadamard manifold.*

Proof. All of the above finite metric spaces have strict p-negative type for $p = 1$ by results given in [13] and [14]. We may therefore apply Corollary 4.2 *en masse*. □

5. Range of strict p-negative type

It is a classical result of Schoenberg [27, Theorem 2] that p-negative type holds on closed intervals. More precisely, the set of all values of p for which a given metric space (X, d) has p-negative type is always an interval of the form $[0, \wp]$ or $[0, \infty)$. Included here is the possibility that $\wp = 0$, in which case the interval degenerates to $\{0\}$. Examples of Enflo [11] in tandem with Theorem 2.5 (a) imply that all such intervals (degenerate or otherwise) can occur. Moreover, for intervals of the form $[0, \wp]$ with $\wp > 0$, the examples given in [11, Section 1] are finite metric spaces. In the case of the degenerate interval $\{0\}$ the situation is slightly more delicate. It follows from [11, Theorem 2.1] and Theorem 2.5 (a) that the Banach space $C[0, 1]$ does not have p-negative type for any $p > 0$. In Theorems 5.4, 5.5 and 5.8 we provide strict versions of [27, Theorem 2]. These theorems allow us to precisely codify the types of intervals on which a metric space may have strict p-negative type. It is interesting to note that finite metric spaces behave quite differently to infinite metric spaces in this respect. Theorems 5.5 and 5.8 highlight this point.

In order to proceed we must first briefly recall some basic facts about kernels of positive type and kernels conditionally of negative type. (In some important respects we are following Nowak [24, Sections 2–4].)

Definition 5.1. Let X be a topological space.

(a) A kernel of *positive type* on X is a continuous function $\Phi : X \times X \to \mathbb{C}$ such that for any $n \in \mathbb{N}$, any elements $x_1, \ldots, x_n \in X$, and any complex numbers $\eta_1, \ldots, \eta_n$ we have:
$$\sum_{1 \le i,j \le n} \Phi(x_i, x_j)\eta_i \overline{\eta_j} \ge 0.$$

(b) A kernel *conditionally of negative type* on X is a continuous function $\Psi : X \times X \to \mathbb{R}$ with the following three properties:
  (1) $\Psi(x, x) = 0$ for all $x \in X$,
  (2) $\Psi(x, y) = \Psi(y, x)$ for all $x, y \in X$, and
  (3) for any $n \in \mathbb{N}$, any $x_1, \ldots, x_n \in X$, and any real numbers $\eta_1, \ldots, \eta_n$ with $\eta_1 + \cdots + \eta_n = 0$ we have:
$$\sum_{1 \le i,j \le n} \Psi(x_i, x_j)\eta_i \eta_j \le 0.$$

The following fundamental relationship between kernels of positive type and kernels conditionally of negative type was given by Schoenberg [28]. For a short proof of this theorem we refer the reader to Bekka et al. [1, Theorem C.3.2].

Theorem 5.2 (Schoenberg [28])**.** *Let X be a topological space and $\Psi : X \times X \to \mathbb{R}$ be a continuous kernel on X such that $\Psi(x, x) = 0$ and $\Psi(x, y) = \Psi(y, x)$ for all $x, y \in X$. Then Ψ is conditionally of negative type if and only if the kernel $\Phi = e^{-t\Psi}$ is of positive type for every $t \ge 0$.*

Our proof of Theorem 5.4 makes use of the following identity. An explanation of this identity may be found in the proof of Corollary 3.2.10 in Berg et al. [2].

Lemma 5.3. *For each $\alpha \in (0, 1)$ there exists a constant $c_\alpha > 0$ such that*
$$x^\alpha = c_\alpha \int_0^\infty (1 - e^{-tx})t^{-\alpha-1} dt$$
for all $x \ge 0$.

Theorem 5.4. *Let (X, d) be a metric space. If (X, d) has p-negative type for some $p > 0$, then it must have strict q-negative type for all q such that $0 \le q < p$.*

Proof. Every metric space has strict 0-negative type. So we may assume that $q > 0$. Since (X, d) has p-negative type, the function $\Psi : X \times X \to \mathbb{R}$ defined by $\Psi(x, y) = d(x, y)^p$ is conditionally of negative type. Hence, by Theorem 5.2, the function $e^{-t\Psi} : X \times X \to \mathbb{C}$ is of positive type for every $t \ge 0$.

Let $x_1, \ldots, x_n$ $(n \ge 2)$ be distinct points in X and let $\eta_1, \ldots, \eta_n$ be real numbers, not all zero, such that $\sum_j \eta_j = 0$. We need to show that $\sum_{i,j} d(x_i, x_j)^q \eta_i \eta_j < 0$.

For each $t \ge 0$, set
$$f(t) = \sum_{i,j} (1 - e^{-td(x_i,x_j)^p})\eta_i \eta_j.$$

Then

$$\begin{aligned} f(t) &= \sum_{i,j} \eta_i\eta_j - \sum_{i,j} e^{-td(x_i,x_j)^p}\eta_i\eta_j = \Big(\sum_j \eta_j\Big)^2 - \sum_{i,j} e^{-td(x_i,x_j)^p}\eta_i\eta_j \\ &= -\sum_{i,j} e^{-td(x_i,x_j)^p}\eta_i\eta_j \le 0 \end{aligned}$$

for all $t \ge 0$. When $t \to \infty$, one has $f(t) \to -\sum_j \eta_j^2 < 0$. Thus $f(t) < 0$ for all t sufficiently large. Set $\alpha = q/p$. By applying Lemma 5.3 to $x = d(x_i, x_j)^p$, one gets

$$\begin{aligned} \sum_{i,j} d(x_i,x_j)^q \eta_i\eta_j &= \sum_{i,j}\Big(c_\alpha \int_0^\infty (1 - e^{-td(x_i,x_j)^p})t^{-\alpha-1}\, dt\Big)\eta_i\eta_j \\ &= c_\alpha \int_0^\infty f(t)t^{-\alpha-1}dt < 0, \end{aligned}$$

as desired. □

As an immediate consequence of Corollary 4.3, Theorem 5.4, [30, Theorem 4.3] and the examples given in [11, Section 1] we obtain the following theorem.

Theorem 5.5. *Let (X,d) be a finite metric space. The set of all values of p for which (X,d) has strict p-negative type is always an interval of the form $[0,\wp)$, with $\wp > 0$, or $[0,\infty)$. Moreover, all such intervals can occur.*

By way of marked contrast with Theorem 5.5, we note that (a) the set of all values of p for which the Banach space $C[0,1]$ has strict p-negative type is the degenerate interval $\{0\}$ (this follows from [11, Theorem 2.1] and Theorem 2.5), and (b) the supremal p-negative of an infinite metric space may or may not be strict. For example, in [8, Theorem 5.7] the authors construct an infinite metric tree that has strict p-negative type if and only if $p \in [0,1]$. However, the Banach space ℓ_1 has strict p-negative type if and only if $p \in [0,1)$. Our next theorem points out that for each $\wp > 0$ there is an infinite metric space (X,d) that has strict p-negative type if and only if $p \in [0,\wp]$. The following definition introduces the relevant spaces.

Definition 5.6. Let $\wp > 0$. Let $(\wp_k)$ be a strictly decreasing sequence of real numbers that converges to $\wp$. Let n be a natural number that satisfies the condition:

$$\left(1 - \frac{1}{n}\right)^{1/\wp} \ge 1/2.$$

Let Y be the union of a sequence of pairwise disjoint sets $(Y_1, Y_2, Y_3, \ldots)$ such that $|Y_k| = n$ for all $k \in \mathbb{N}$. Let Z be the union of a sequence of pairwise disjoint sets $(Z_1, Z_2, Z_3, \ldots)$ such that $|Z_k| = n$ for all $k \in \mathbb{N}$ and $Y \cap Z = \emptyset$. Set $X = Y \cup Z$. We metrize X in the following way:

$$d(y,z) = \left(1 - \frac{1}{n}\right)^{1/\wp_k}$$

if $y \in Y_k$ and $z \in Z_k$ for some $k \in \mathbb{N}$. All other non-zero distances in the space are taken to be one. We call (X,d) an *Enflo $\wp$-space.*

Notice that in Definition 5.6 the condition placed on n ensures that (X,d) really is a metric space. Moreover, from the examples noted in [11, Section 1], together with Theorem 2.5 (a), it follows that each subspace $Y_k \cup Z_k$ of an Enflo $\wp$-space

(X,d) has maximal p-negative type exactly equal to $\wp_k$. In order to proceed we need to develop a slightly stronger statement about certain subspaces of (X,d).

Lemma 5.7. *Let $\wp > 0$. Let (X,d) be an Enflo $\wp$-space as in Definition 5.6. For each $m \in \mathbb{N}$, the subspace $X_m = \bigcup\{Y_k \cup Z_k | 1 \leq k \leq m\}$ of (X,d) has $\wp_m$-negative type. In fact, $\wp_m$ is the maximal p-negative type of the subspace X_m.*

Proof. Let D be a given normalized simplex in X_m. Without loss of generality, we may assume that $D = [y_{k,j}(\alpha_{k,j}), z_{k,j}(\beta_{k,j}); \bar{y}_{k,j}(\gamma_{k,j}), \bar{z}_{k,j}(\eta_{k,j})]$, where $y_{k,j}, \bar{y}_{k,j} \in Y_k$ and $z_{k,j}, \bar{z}_{k,j} \in Z_k$. In other words, our simplex has the points $y_{k,j}, z_{k,j}$ on one side and the remaining points $\bar{y}_{k,j}, \bar{z}_{k,j}$ on the other side (with weights as indicated).

Let $L(\wp_m)$ and $R(\wp_m)$ denote the left side and right side of (2) computed relative to the simplex D with exponent $\wp_m$ (respectively). By setting $\alpha_k = \sum_j \alpha_{k,j}, \beta_k = \sum_j \beta_{k,j}, \gamma_k = \sum_j \gamma_{k,j}$ and $\eta_k = \sum_j \eta_{k,j}$ we may then compute $L(\wp_m)$ and $R(\wp_m)$:

$$\begin{aligned} L(\wp_m) &= \sum_k (\alpha_k\beta_k + \gamma_k\eta_k)\left\{\left(1-\frac{1}{n}\right)^{\wp_m/\wp_k} - 1\right\} + 1 \\ &\quad -\frac{1}{2}\sum_k\left(\sum_j \alpha_{k,j}^2 + \sum_j \beta_{k,j}^2 + \sum_j \gamma_{k,j}^2 + \sum_j \eta_{k,j}^2\right), \end{aligned} \tag{8}$$

while

$$R(\wp_m) = \sum_k (\alpha_k\eta_k + \gamma_k\beta_k)\left\{\left(1-\frac{1}{n}\right)^{\wp_m/\wp_k} - 1\right\} + 1. \tag{9}$$

If we let s_k denote the number of points $y_{k,j}$ in Y_k and let t_k denote the number of points $\bar{y}_{k,j}$ in Y_k, then it follows that we have:

$$\begin{aligned} \sum_j \alpha_{k,j}^2 + \sum_j \gamma_{k,j}^2 &\geq \frac{\alpha_k^2}{s_k} + \frac{\gamma_k^2}{t_k} \\ &\geq \frac{\alpha_k^2 + \gamma_k^2}{n}. \end{aligned}$$

Similarly,

$$\sum_j \beta_{k,j}^2 + \sum_j \eta_{k,j}^2 \geq \frac{\beta_k^2 + \eta_k^2}{n}.$$

As a result, comparing the expressions (8) and (9), we see that it will follow that $L(\wp_m) \leq R(\wp_m)$ provided that we can establish the following non-linear inequality:

$$\begin{aligned} &\sum_k \Big\{(\alpha_k\beta_k + \gamma_k\eta_k) - (\alpha_k\eta_k + \gamma_k\beta_k)\Big\}\left\{\left(1-\frac{1}{n}\right)^{\wp_m/\wp_k} - 1\right\} \\ \leq\ & \frac{1}{2}\sum_k \frac{\alpha_k^2 + \gamma_k^2 + \beta_k^2 + \eta_k^2}{n}. \end{aligned} \tag{10}$$

We claim that (10) holds term by term. That is to say,

$$\begin{aligned} &\Big\{(\alpha_k\beta_k + \gamma_k\eta_k) - (\alpha_k\eta_k + \gamma_k\beta_k)\Big\}\Big\{\Big(1-\frac{1}{n}\Big)^{\wp_m/\wp_k} -1\Big\} \\ \le\ & \frac{1}{2}\cdot\frac{\alpha_k^2+\gamma_k^2+\beta_k^2+\eta_k^2}{n}, \end{aligned} \tag{11}$$

for each k. In fact, for each k,

$$\begin{aligned} &\left|\Big\{(\alpha_k\beta_k + \gamma_k\eta_k) - (\alpha_k\eta_k + \gamma_k\beta_k)\Big\}\Big\{\Big(1-\frac{1}{n}\Big)^{\wp_m/\wp_k} -1\Big\}\right| \\ \le\ & \max\big(\alpha_k\beta_k+\gamma_k\eta_k, \alpha_k\eta_k+\gamma_k\beta_k\big)\cdot\frac{1}{n} \\ \le\ & \frac{1}{2}\cdot\frac{\alpha_k^2+\beta_k^2+\gamma_k^2+\eta_k^2}{n}. \end{aligned}$$

As (11) implies (10) we conclude that $L(\wp_m) \le R(\wp_m)$. Thus X_m has maximal p-negative type at least $\wp_m$. However the subspace $Y_m \cup Z_m$ of X_m has maximal p-negative type $\wp_m$ by [11, Section 1] and Theorem 2.5 (a). We conclude that the maximal p-negative type of X_m is $\wp_m$. □

Theorem 5.8. *Let $\wp > 0$. Let (X,d) be an Enflo $\wp$-space as in Definition 5.6. Then (X,d) has strict p-negative type if and only if $p \in [0,\wp]$.*

Proof. For each k the subspace $Y_k \cup Z_k$ of (X,d) has maximal p-negative type $\wp_k$. And since $\wp_k \searrow \wp$ as $k \to \infty$ it follows that (X,d) does not have p-negative type for any $p > \wp$.

However, each subspace $X_m = Y_m \cup Z_m$ of (X,d) has $\wp_m$-negative type by Lemma 5.7. By Theorem 5.4, X_m has strict p-negative type for all $p \in [0,\wp_m)$. Thus (X,d) has strict p-negative type for all $p \in [0,\wp]$ as asserted. □

We conclude this paper with some final applications of Theorem 5.4. Recall that the maximal q-negative type of certain classical (quasi-) Banach spaces has been computed explicitly. For example, suppose $0 < p \le 2$ and that μ is a non-trivial positive measure, then the maximal q-negative type of $L_p(\mu)$ is simply p. A short proof of this result, which is due to Schoenberg [28] for $1 \le p \le 2$, may be found in [20, Corollary 2.6 (a)]. Theorem 5.4 therefore applies as follows.

Corollary 5.9. *Let $0 < p \le 2$ and let μ be a positive measure. Then any metric space (X,d) which is isometric to a subset of $L_p(\mu)$ must have strict q-negative type for all $q \in [0,p)$.*

Corollary 4.4 and Theorem 5.4 combine to provide the following characterization of the supremal p-negative type of a finite metric space in terms of zeros of the simplex gap functions γ_D^q.

Corollary 5.10. *If the supremal p-negative type $\wp$ of a finite metric space (X,d) is finite, then:*

$$\wp = \min\{q \,|\, q > 0 \textit{ and } \gamma_D^q(\vec{\omega}) = 0 \textit{ for some normalized (s,t)-simplex } D(\vec{\omega}) \subseteq X\}.$$

In certain instances Theorem 5.4 provides a second description of the maximal p-negative type of a metric space.

Corollary 5.11. *Let $\wp > 0$. If a metric space (X,d) has $\wp$-negative type but not strict $\wp$-negative type, then $\wp$ is the maximal p-negative type of (X,d).*

It follows from Kelly [16, 17] that any k-sphere $\mathbb{S}^k$ endowed with the usual geodesic metric is ℓ_1-embeddable and therefore of 1-negative type. On the other hand, Hjorth et al. [14, Theorem 9.1] have shown that a finite isometric subspace (X, d) of a k-sphere $\mathbb{S}^k$ is of strict 1-negative type if and only if X contains at most one pair of antipodal points. These comments and Corollary 5.11 imply the next corollary.

Corollary 5.12. *A finite isometric subspace (X, d) of a k-sphere $\mathbb{S}^k$ has maximal p-negative type $= 1$ if and only if X contains at least two pairs of antipodal points.*

Recall, following Blumenthal [4], that a *semi-metric space* is required to satisfy all of the axioms of a metric space except (possibly) the triangle inequality.

Remark 5.13. In closing we note that Theorems 3.3, 4.1 and 5.4 hold (more generally) for all finite semi-metric spaces (X, d). The same goes for Corollaries 4.2, 4.3, 4.4, 5.10 and 5.11. This is because the triangle inequality has played no rôle in any of the definitions or computations of this paper.

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