Apportionment and the 2000 Election

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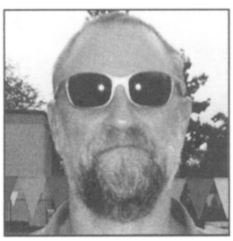

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Introduction

A paradox produced by the 2000 Presidential election is that although Al Gore won the popular vote, he lost the election to George W. Bush. This is, of course, because the President is not elected by a plurality of votes cast but rather by a plurality of votes of the 538 member Electoral College (EC). Each state, except for Maine and Nebraska, elects its members of the EC by the winner-take-all method, i.e., the winner of the plurality vote in a state is entitled to all the electors from that state. (Maine and Nebraska give an elector to the winner of the plurality of votes in each congressional district and in addition give two electors to the winner of the plurality of the statewide vote. Therefore, the electors in Maine and Nebraska may be split between several candidates.) The size of each state's delegation to the EC equals the size of the state's delegation in the House of Representatives (HR) plus two for each of its two Senators. The reasons for this method of apportionment of the EC members are rooted in the Connecticut compromise of 1787 [**11**], [**5**], [**10**]. Furthermore, the 23rd amendment to the U.S. Constitution gives the District of Columbia the same number of members in the EC as the smallest state has, currently three.

In 1941 the size of the HR was fixed at 435 and has not been changed since. (For a short time, 1961–63, after Hawaii and Alaska joined the union the House size was 437 to give the two new states representation in the HR. However, the apportionment following the 1960 census was again based on a House size of 435.) Though the size of the HR has not increased during the last sixty years the population increased from 130 million in 1940 to 250 million in 1990, so the number of people per representative increased from 300,100 to 572,000. After the 2000 census it is 647,000. If we wanted the same ratio of representatives to people today as existed in 1940 then the HR based on the 1990 census should have had about 830 members throughout the 1990s. To maintain the 1940 ratio after the 2000 census, the House size should be about 940 now.

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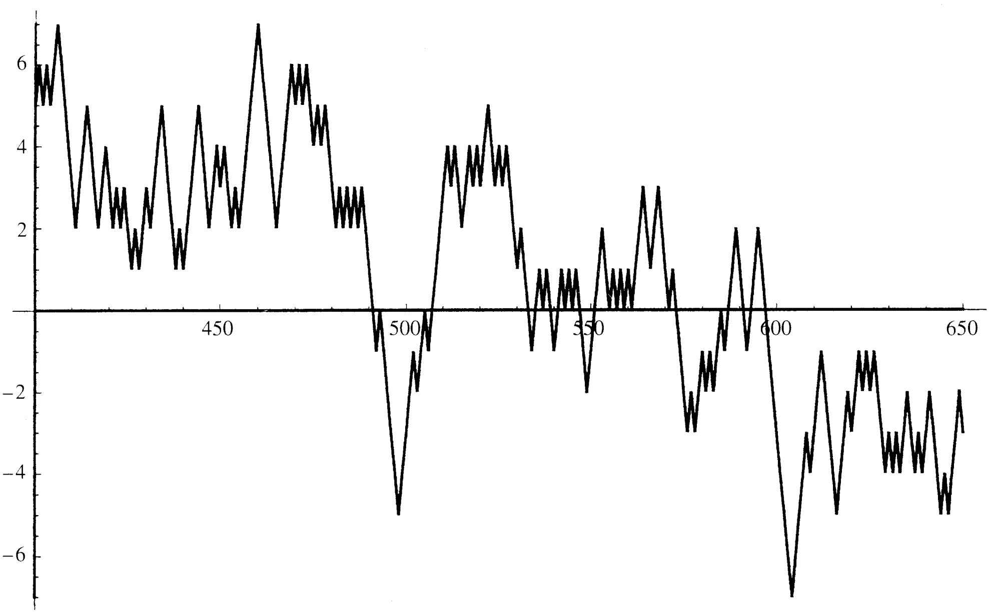

Who Wins?

IN THIS ISSUE:

- House size matters
- Parrondo's paradox—two losers make a winner
- A formula for the square root of (some) matrices
- The Gudermannian, a neglected function

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Since our interest is in the 2000 presidential election we use the 1990 census figures since the number of electoral votes for each state was based on the 1990 census. First we carried out an apportionment of a hypothetical HR with 830 members based on the 1990 census using the current apportionment method. Then, based on the official results of the 2000 presidential election from all fifty states and the District of Columbia we added up the electoral votes for Bush and Gore. The result of this thought experiment is that Gore has 471 of the 934 EC votes versus 463 votes for Bush. Hence Gore would have been the winner of the election if the HR had had 830 members in 2000. It is this surprising result that inspired this article.

The main result of this investigation is: The winner of a presidential election depends inherently on the size of the House of Representatives.

Background

Before we explain this we need to discuss the current apportionment laws that determine the composition of the HR and hence the EC. The size of a state's delegation in the HR is apportioned based on the size of its population. The U.S. Constitution prescribes the apportionment of the EC based on the apportionment of the HR but it does not specify a method of apportionment for the HR. During the last 225 years of U.S. history several methods of apportionment have been used; see [**17**] for a nice summary. The apportionment method currently in use is the Method of Equal Proportions, also known as the Huntington-Hill method. How it got to be named Method of Equal Proportion is a story that is told very well in [**3**]. Let it be said here only that naming it the Method of Equal Proportions was for the purpose of making it appear to be the only unbiased among several apportionment methods. That it is, in fact, not unbiased and favors smaller states has been argued repeatedly (see, e.g., [**3**], [**13**]). How the Huntington-Hill [**9**] method of apportionment works will be described in the next section. While the apportionment of the members of the HR is an interesting mathematical and political problem, the apportionment of the Senate is prescribed by the U.S. Constitution and each state is entitled to two senators.

We first discuss some general features of the composition of the EC and their implications for the determination of the winner of a presidential election.

Because the size of each state's EC delegation is two more than the size of its delegation in the HR, smaller states have a larger representation in the EC than they would be entitled to if EC members were apportioned based on population size alone. In the 2000 election the 22 smallest states had a total of 98 votes in the EC while their combined population [**4**] was roughly equal to that of the state of California, which had only 54 votes in the EC. Californians could certainly claim that their votes did not count as much as the votes of citizens in the smaller states. Of those 98 EC votes, 37 went for Gore while 61 went for Bush. In the final tally Bush won 30 states while Gore won 20 states plus the District of Columbia.

Though these particular numbers result from the use of the Huntington-Hill method, the results are very similar if we replace the Huntington-Hill method with any other reasonable apportionment method and the general argument that the collection of smaller states have a greater influence in determining the winner of presidential elections still holds.

The size of a state's delegation in the HR in 2000 was based on the size of its population as determined by the 1990 census. The same was true for the 1992 and 1996 elections. The requirement for the federal government to hold a census every ten years is for the purpose of apportioning the seats in the HR. The census data for all censuses are available from the Census Bureau web site [**4**] and the size of the delegation in the HR for each state after the 1990 census is available (see, e.g., [**17**]). It can also be easily computed.

If we assume for the moment that the size of a state's delegation was the same as the size of its delegation in the HR, then Gore would have been elected by the EC with a vote of 224 to 211 for Bush. The 224 votes do not even count the votes cast in the District of Columbia, which Gore won, since there is no congressional representation for the District of Columbia. The 224 to 211 result closely resembles the result of the popular vote. Such a scheme has little to no hope of ever becoming law. Changing the rules under which the members of the EC are apportioned would require a constitutional change which is very unlikely to succeed since 37 states would have to ratify such a change. Most states are small and it would seem unlikely for such states to vote to decrease their influence in presidential elections.

On the other hand, if we assume that each state gets the same number of members in the EC then Bush would win such an election regardless of how we deal with the District of Columbia. Such a situation could arise if we fixed the House size at 50, each state getting exactly one representative. Given the vast differences in the sizes of the populations in the different states, e.g., Wyoming's population of about 460,000 versus California's population of some 30 million, it is unlikely that such an apportionment of representatives and EC members would have broad support either.

The Huntington-Hill method of apportionment

The Huntington-Hill method apportions to each state a number of seats such that no transfer of one seat from one state to another state lowers the relative difference in the size of the average constituency of the two states. That is, if state X has population P_X and M_X seats while state Y has population P_Y and M_Y seats then

$$\left|\left(\frac{P_X}{M_X}-\frac{P_Y}{M_Y}\right)\Big/\min\left\{\frac{P_X}{M_X},\frac{P_Y}{M_Y}\right\}\right| < \left|\left(\frac{P_X}{M_X+1}-\frac{P_Y}{M_Y-1}\right)\Big/\min\left\{\frac{P_X}{M_X+1},\frac{P_Y}{M_Y-1}\right\}\right| \tag{1}$$

for all states X and Y.

The Huntington-Hill method can also be described [**8**] as follows: Given a divisor H we divide each state's population by H to find the *modified quota* for each state. (The *natural quota*, also known as *standard quota*, is defined with the *natural divisor* D, the total population of the country divided by the number of seats in the House.)

The modified quotas will have fractional parts. The Huntington-Hill method deals with them in the following way. If the modified quota of a state is between n and $n+1$ we round the modified quota up if it is larger than $\sqrt{n(n+1)}$, the geometric mean of n and $n+1$. In contrast, using conventional rounding we round up if the modified quota is larger than $(n+(n+1))/2 = n+1/2$, the arithmetic mean of n and $n+1$. To apply the Huntington-Hill method we need to determine the above mentioned divisor H so that when we round using the geometric mean as our cutoff we apportion the correct number of seats.

The Huntington-Hill method automatically takes care of the constitutional requirement that every state receive at least one representative. If the modified quota of a state is less than 1, i.e., $n = 0$, then the Huntington-Hill method prescribes that it is rounded up to 1 since $\sqrt{1 \cdot 0} = 0$. It is well-known that the geometric mean of a and b is not bigger than the arithmetic mean of a and b with equality if and only if $a = b$. It is also easy to show that $\lim_{n\to\infty} n + \frac{1}{2} - \sqrt{n(n+1)} = 0$. To illustrate, the geometric mean of 1 and 2 is 1.414, the geometric mean of 2 and 3 is 2.449, while the geometric mean of 52 and 53 is 52.498. Thus a state with a modified quota between 2 and 3 needs only to raise its quota to 2.45 to receive a 3rd seat, while California with a modified quota between 52 and 53 needs to raise its modified quota to almost 52.5 to receive a 53rd seat. It is this property that leads the Huntington-Hill method to favor small states.

Finding a correct divisor D_M for a given House size M may seem like a daunting task. Three factors make it a much more manageable problem. First, a divisor H is usually fairly close to the natural divisor, which is easy to compute. For the 1990 census the natural divisor is 572466.17 while 574850 works as a Huntington-Hill divisor. Second, in almost all cases there is a range of divisors that result in the same apportionment for a given House size. Third, with a spreadsheet many different divisors can be checked quickly and a divisor H can be found. E.g., we found that $D = 247900$ results in a House with 1000 members. California as the largest state would have 120 representatives in such a HR while Wyoming, the smallest, would have 2 representatives.

There is, of course, a more formal way to calculate the divisor that is also more efficient for a range of House sizes. For a state X with population p and each $n \geq 2$ we can calculate the quotient $d_{X,n} = p/\sqrt{n(n-1)}$. Any divisor larger than $d_{X,n}$ gives state X fewer than n seats in the HR while any divisor smaller than $d_{X,n}$ gives the state X more than n seats in the HR.

For example, with state X being California we have

$$d_{X,2} = P_X/\sqrt{2} = 21043512.6573 \quad \text{and} \quad d_{X,3} = P_X/\sqrt{6} = 12149776.9742.$$

If we choose a divisor larger than $d_{X,2}$ then California receives only one seat while any divisor between $d_{X,3}$ and $d_{X,2}$ will result in an apportionment with California getting two seats.

Now form the collection of pairs $(d_{X,n}, X)$ for all states X and all n below a certain value N. (We worked with $N = 120$ as no state would have more than 120 representatives in the HR as long as the size of the HR is not above 1000.) Order the collection from largest to smallest and call the ordered collection S. Denote the first coordinate of the nth pair by S_n. The first ten entries of S are (21043512, CA), (12721172, NY), (12149477, CA), (12011276, TX), (9148495, FL), (8590978, CA), (8401590, PA), (8082656, IL), (7670068, OH), (7344572, NY). (The complete list can be found at [**12**].) Finding the sorted list involves calculating some 6000 numbers and ordering the resulting list, a trivial amount of calculation by today's standards.

If we choose $D = 7600000$ as our divisor, i.e., a divisor between the 10th and 9th largest divisors, then we see that California will be apportioned four seats, New York, Texas, Florida, Pennsylvania, Illinois and Ohio will be apportioned two seats each; all other states will be apportioned one seat each. The House size for $D = 7600000$ is 59. In general, any divisor D between s_{n+1} and s_n will result in a House size of $50 + n$. The apportionment of a state X is $1 + |\{(d_k, X) \mid d_k > D\}|$.

Conversely, given a House size M we can easily find a corresponding divisor D_M. In fact for D_M we can use any number strictly between s_{M-50} and s_{M-49}. For $M = 435$, the current House size, we find that any D_M between 573546.6159 and 573643.1525 yields the correct apportionment.

At this point the reader might be worried about what happens if $s_n = s_{n+1}$. Then we would not be able to find an apportionment for a House of size $n + 50$. While this is certainly a theoretical concern, it seems rather unlikely to occur, and in fact has not occurred in the history of apportionment in the US.

The results

We have carried out the apportionment of the HR for all House sizes between 50 and 1000 which for the Huntington-Hill method correspond to divisors between 247,900 and 22,000,000. The divisor 247,900 yields a House size of 1000 while the divisor 22,000,000 yields a House size of 50. We expect Gore to win for large House sizes as he did for the House size 830 and Bush to win for small House sizes as he did with the House size at 435. The actual results are more surprising than that.

As the House size ranges from 50 to 1000 the 2000 election would have produced ties for the following 25 sizes: 491, 493, 505, 507, 533, 535, 537, 539, 541, 543, 545, 547, 551, 555, 557, 559, 561, 571, 573, 585, 587, 591, 593, 597 and 655. For all sizes larger than 597, except for 655, which results in a tie, Gore would have won. For all sizes smaller than 491 Bush would have won the election, as he did, in fact, with a House size of 435. We list the would-be winner for each House size in Table 1:

Table 1.

House size	winner	House size	winner	House size	winner
< 491	Bush	541	tie	562–570	Bush
491	tie	542	Bush	571	tie
492	Gore	543	tie	572	Bush
493	tie	544	Bush	573	tie
494–504	Gore	545	tie	574–584	Gore
505	tie	546	Bush	585	tie
506	Gore	547	tie	586	Gore
507	tie	548–550	Gore	587	tie
508–532	Bush	551	tie	588–590	Bush
533	tie	552–554	Bush	591	tie
534	Gore	555	tie	592	Gore
535	tie	556	Bush	593	tie
536	Bush	557	tie	594–596	Bush
537	tie	558	Bush	597	tie
538	Bush	559	tie	598–654	Gore
539	tie	560	Bush	655	tie
540	Gore	561	tie	> 655	Gore

Perhaps the most troubling aspect of the information in Table 1 is the fact that for House sizes between 492 and 596 the winner goes back and forth many times without much rhyme or reason. For those 105 different House sizes the election ends in a tie 23 times, Gore wins 29 times, and Bush wins 53 times. The winner of the 2000 presidential election was determined in 1941 when the House size was fixed at 435.

To put it as a punch line: Had the House size been set at 500 in 1941 (and not been changed since) then Gore would have won the 2000 election!

Next we describe the reasons for the different outcomes for the House sizes 490, 491, and 492. With a House size of 490, Bush wins the election by one vote, a House size of 491 results in a tie, while a House size of 492 results in Gore winning the election by one vote. The apportionment of the House with 490 seats based on the 1990 census figures is given in Table 2. The "B" or "G" indicates which candidate won the respective state in 2000.

Table 2.

Wyoming	3	B	South Carolina	9	B
Alaska	3	B	Arizona	9	B
Vermont	3	G	Kentucky	9	B
North Dakota	3	B	Alabama	10	B
Delaware	3	G	Louisiana	10	B
South Dakota	3	B	Minnesota	11	G
Montana	4	B	Maryland	11	G
Rhode Island	4	G	Washington	12	G
Idaho	4	B	Tennessee	12	B
New Hampshire	4	B	Wisconsin	12	G
Hawaii	4	G	Missouri	12	B
Nevada	4	B	Indiana	13	B
Maine	4	G	Massachusetts	14	G
New Mexico	5	G	Virginia	14	B
Nebraska	5	B	Georgia	15	B
Utah	5	B	North Carolina	15	B
West Virginia	6	B	New Jersey	17	G
Arkansas	7	B	Michigan	20	G
Kansas	7	B	Ohio	23	B
Mississippi	7	B	Illinois	25	G
Iowa	8	G	Pennslyvania	25	G
Oregon	8	G	Florida	28	B
Oklahoma	8	B	Texas	36	B
Connecticut	9	G	New York	37	G
Colorado	9	B	California	61	G

Increasing the House size from 490 to 491 causes the state of New York to gain an extra seat while all other states have the same. Since Gore won New York the election now results in a tie. When we increase the House size from 491 to 492 the extra seat is gained by Pennsylvania. Since Gore won that state he now wins the election by one vote.

At the web site [**12**] readers will find a table that for all House sizes between 50 and 1000 lists the difference in the electoral votes using the 1990 election results. Figure 1 shows the difference as a function of the House size.

Though the global picture clearly shows the general tendency for the difference to decrease this is not necessarily obvious when we look at smaller intervals. Figure 2 shows the graph for the House sizes between 480 and 530. Notice the steep increase in the difference from a House size of 498 and a House size of 511. Over this interval of House sizes the difference increases from -5 to $+4$. This is of course reflected in the fact that over this interval the states gaining seats are all states Bush won in 2000 with the exception of New York and California (see [**12**]).

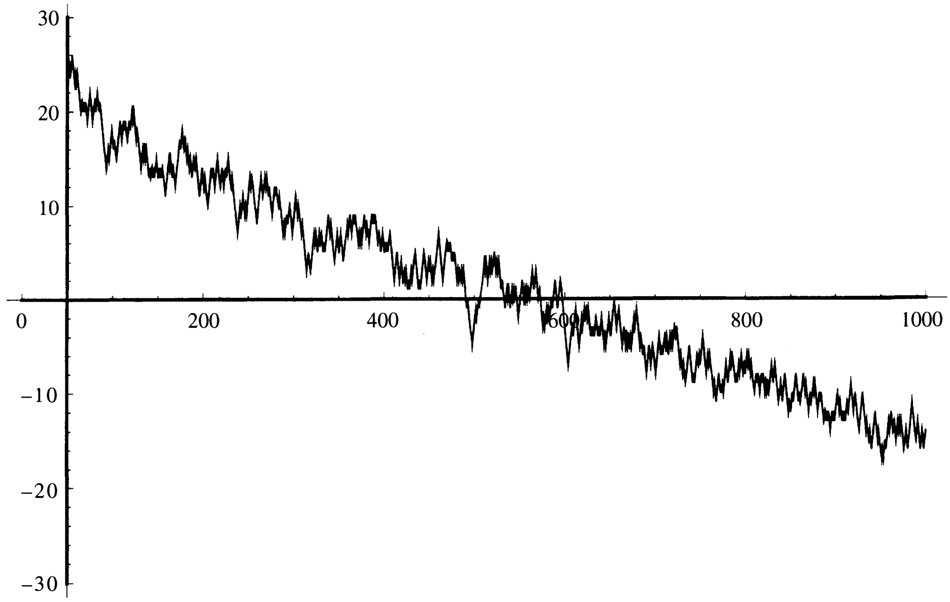

Figure 1.

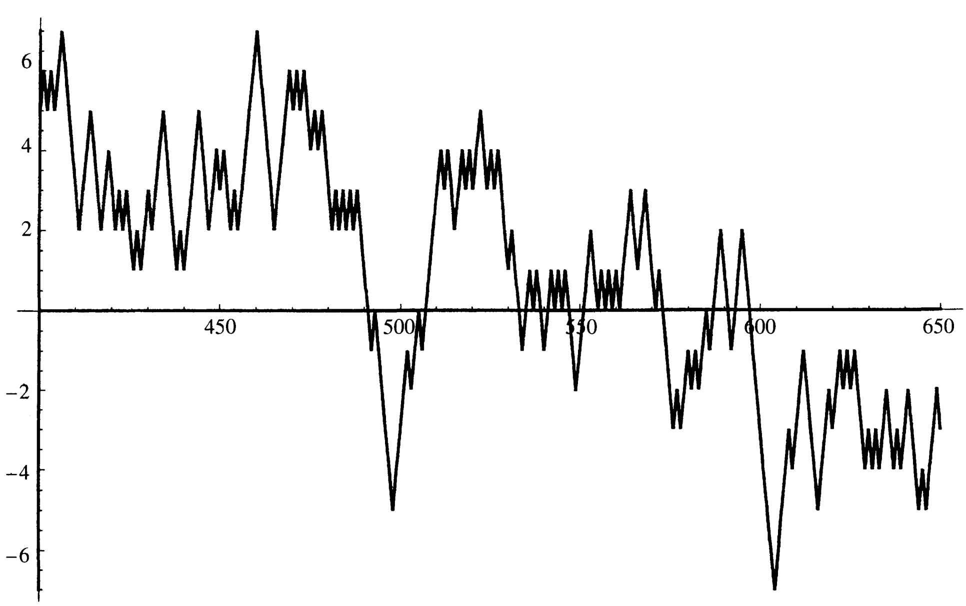

Figure 2.

Summary

The number of electoral votes for Bush minus the electoral votes for Gore changes by one each time the House size increases by one seat. The direction of the change depends on which state gains the extra seat. As the House size increases from 491 to 597, the winner changes repeatedly depending on which order the states gain the additional seats. This order depends very intricately on the population sizes and cannot be discerned a priori. We can make the following observations that explain the behavior for small House sizes and large House sizes. For large House sizes the relative representation of the states in the EC becomes closer to their relative representation in the HR, which is why Gore would have won the election for large House sizes. Our earlier example shows that for small House sizes smaller states have a relatively larger part of the members in the EC. This is the reason Bush would have won the election for small House sizes. The interval between 491 and 597 falls between the two extremes, which is reflected in the repeated change of the winner.

The non-monotonicity of the difference in the electoral votes for the two candidates with respect to the House size is not unlike the Alabama paradox which lead to Hamilton's method being abandoned in 1901 as the method of apportionment in the U.S. **[17]**. Without going into detail, with Hamilton's method of apportionment it may happen, as it did with Alabama after the 1880 census, that the House size increases yet one or more states lose a seat [**17**].

Though the absolute numbers of EC votes the candidates receive do not decrease when the number of House seats increases, the difference in the number of votes for two candidates shows the same non-monotonic behavior with respect to the House size that we see with the Alabama paradox in Hamilton's method.

Bush's electoral strategy of winning many small states, without winning a plurality of the votes in the whole nation and without winning the big states California and New York, worked! However, it worked only because the House size was small enough. We suggest the above as an argument for reconsideration of the EC, its composition, and its use in electing the President.

An obvious way to avoid pathologies is a direct election of the president by a plurality of the votes, eliminating the EC altogether. Since this would require a change in the U.S. Constitution the adoption of such a method is far from being politically realistic. Furthermore, a direct vote by a plurality of votes would be subject to the "Nader factor", that is a third party candidate that draws votes disproportionately away from one candidate over the other, thereby influencing the election. The perils of voting theory are probably better known through Arrow's Theorem [**1**] and the more recent work of Saari [**14**], [**15**], [**16**], and shall not be discussed further here. In the last decade the topics of apportionment theory and voting theory have made their way into some textbooks used for liberal arts mathematics courses. Very accessible overviews of the issues involved both in apportionment perils and in Arrow's theorem are given in [**17**], [**6**], [**7**]. A nice introduction to apportionment theory and the Huntington-Hill method can found in [**8**].

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Mathematics Without Words

There is no end to mathematics, or to mathematical ingenuity. In the January 2002 issue (p. 13) was a wordless demonstration that if $0 < m < n$, then

$$\arctan\left(\frac{m}{n}\right) + \arctan\left(\frac{n-m}{n+m}\right) = \frac{\pi}{4}.$$

Roger Nelsen (Lewis & Clark College, nelsen@lclark.edu) saw it, thought (I think), "There must be a simpler way," and the following is the result.