

## Lecture 9

### Review

1. Suppose  $U, V$  are  $kG$ -modules. Then so is  $U \otimes V = U \otimes_k V$  with action  $g(u \otimes v) = gu \otimes gv$ , extended linearly to  $kG$ .
2.  $\text{Hom}_k(U, V) \cong U^* \otimes_k V$   
 $\text{Hom}_{kG}(U \otimes V, W) \cong \text{Hom}_{kG}(U, V^* \otimes W)$
3.  $P$  free (projective)  $\Rightarrow U \otimes_k P$  is free (projective)
4.  $\otimes_k$  is exact, given  $0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$  and  $N \in kG\text{-mod}$  we get  
 $0 \rightarrow U \otimes N \rightarrow M \otimes N \rightarrow V \otimes N \rightarrow 0$ .

### Induction & Restriction

Def  $H \leq G$ ,  $U \in kG\text{-mod}$  then  $U_H = U|_H = \text{Res}_H^G U$  is restriction of  $U$  to  $H$ .

Problem Suppose  $H \leq G$ ,  $V$  an  $H$ -module. Produce a corresponding  $G$ -module. Note: Very rarely is  $V$  the restriction of some  $kG$ -module.

Setup: Let  $B \subset A$  be a subalgebra, and  $V \in B\text{-mod}$ .

Def The induced module  $\text{Ind}_B^A V = V^A \cong A \otimes_B V$ .

Rmk For all we know this could be zero!

Def The coinduced module  $\text{CoInd}_B^A V \cong \text{Hom}_B({}_B A, V)$  with action:

$$(a\psi)(a') = \psi(a'a)$$

### General Properties

1  $\text{Ind}_B^A - = A_B \otimes_B -$  is right exact

2 coind is left exact

3  $\text{Ind}_B^A -$  is left adjoint to restriction, i.e.  $M \in A\text{-mod}, N \in B\text{-mod}$

$$\text{Hom}_A(\text{Ind}_B^A N, M) \cong \text{Hom}_B(N, \text{Res}_B M)$$

4 Coind is right adjoint to restriction, i.e.

$$\text{Hom}_A(M, \text{CoInd}_B^A N) \cong \text{Hom}_B(\text{Res}_B M, N)$$

Special case:  $H \leq G, B \cong kH \subseteq A \cong kG$

Recall 1  $B_B \otimes_B U \cong U$

2  $kG \downarrow_H \cong kH \oplus \dots \oplus k1$

COR Let  $H \leq G$  be finite. Then  $\text{Ind}_H^G$  is an exact functor and is  $\cong$  to  $\text{CoInd}$ .

Suppose  $G/H = \{e, g_2, g_3, \dots, g_n\}$  coset reps

$RG \otimes_{RH} U$  spanned by  $g \otimes_H U$   
 spanned by  $g_i h \otimes_H U \cong g_i \otimes_H U$

Thm  $\text{Ind}_H^G U \cong \bigoplus_{s \in G/H} s \otimes U$ ,  $\dim U^G = [G:H] \dim U$

Prop 1 Action is:  $gs = zh su$   $g(s \otimes u) = gs \otimes u$   
 $= zh \otimes u$   
 $= zh \otimes u \in z \otimes U$

2. One copy of  $U$  for each coset, action of  $g$  permutes around copies

3. This  ~~$\dim U^G =$~~   $(U^G)_H \cong U \oplus \dots$

Example  $\text{Ind}_H^G R$  is just the permutation representation on the cosets of  $H$ .

Example

Let  $U$  be the  $R\Sigma_3$ -module  $(12) \rightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$   $(23) \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   
 $(123) \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\Sigma_4 \cong \Sigma_3 \ltimes V(14) \cong \Sigma_3 \ltimes V(24) \cong \Sigma_3 \ltimes V(34) \cong \Sigma_3$

$(12)e = e(12)$   
 $(12)(14) = (142) = (24)(12)$   
 $(12)(24) = (124) = (14)(12)$   
 $(12)(34) = (34)(12)$

$(12) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

$$(1234)e = (14)(123)$$

$$(1234)(14) = (234) = (24)(23)$$

$$(1234)(24) = (12)(34) = (34)(12)$$

$$(1234)(34) = (123) = e \cdot (123)$$

$$(1234) \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Omnibus Lemma  $V_1, V_2, V$  RH modules,  $U$  a RG-module.

1) If  $V$  is free (projective) then  $V^{\vee}$  is free (projective).

$$2) (V_1 \oplus V_2)^{\vee} \cong V_1^{\vee} \oplus V_2^{\vee}$$

$$3) (V^{\vee})^{\vee} \cong V^{\vee}$$

4) If  $L \leq H \leq G$ ,  $W$  an  $L$ -module, then  $(W^{H/L})^{\vee} \cong W^{\vee}$

$$5) U \otimes V^{\vee} \cong (U \otimes V)^{\vee}$$

Proof

$$2) RG \otimes_H (V_1 \oplus V_2) \cong (RG \otimes_H V_1) \oplus (RG \otimes_H V_2)$$

$$1) RG \otimes_H RH \cong \bigoplus RG, \text{ now use 2}$$

$$3) RG \otimes_{RH} V^{\vee} \cong (RG^{\vee} \otimes_H V)^{\vee} \cong (RG \otimes_H V)^{\vee}$$

$$4) \text{ ~~RG~~ } RG \otimes_H RH \otimes_L \text{ ~~RG~~ } W \cong RG \otimes_L W$$

$$5) U \otimes RG \otimes_H V \cong (U \otimes V) \otimes RG \otimes_H (U \otimes V)$$

A universal property of induced modules;

$$\text{Ind}_H^G V \cong e \otimes V \oplus g \otimes V \oplus \dots$$

Suppose  $\Psi: e \otimes V \rightarrow U$  is a RH-homomorphism,  $U \in \text{mod } RG$ .

Define  $\tilde{\Psi}(g \otimes V) = g \Psi(e \otimes V)$ , check  $\tilde{\Psi}$  is a  $RG$ -module map  $V^G \rightarrow U$ .

Def (Alperin)  $H \leq G$ . A  $RG$ -module  $U$  is relatively  $H$ -free if  $\exists$  a RH submodule  $X$  of  $U$  such that any RH homomorphism  $\Psi: X \rightarrow M$  extends uniquely to a  $RG$ -hom  $U \rightarrow M$ .

COR  $V^G$  is relatively  $H$ -free

Rmk Free  $\iff$  Relatively  $\{e\}$  free

Thm Suppose  $U$  is a  $RG$ -module, generated by a RH submodule  $X$  and  $\dim U = |G:H| \cdot \dim X$ . Then  $U \cong X^G$ .

Rmk Alperin uses this "test" in many proofs

Mackey Thm Let  $L, H \leq G$ ,  $V \in \text{RH mod}$ .

$$(V^G)_L \cong \bigoplus_{s \in L \backslash G/H} \left( (s \otimes V)_{L \cap s H s^{-1}} \right)_L$$