

## Lecture 8

### Review

•  $U$  a  $KG$ -module,  $U^*$  is also via  $(g\psi)(u) = \psi(g^{-1}u)$

•  $0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$  gives  $0 \rightarrow V^* \rightarrow M^* \rightarrow U^* \rightarrow 0$  so

$$\begin{array}{c} \begin{array}{c} \left. \begin{array}{c} M \\ \hline V \cong M/U \\ \hline U \end{array} \right\} \\ \Rightarrow \\ \left. \begin{array}{c} M^* \\ \hline U^* \\ \hline V^* \end{array} \right\} \end{array}$$

• We proved  $KG \cong KG^*$ , and as a corollary that any projective  $KG$ -module is injective, and vice-versa. In particular  $PIM$ 's have simple socle.

### Symmetric Algebras

Def: Let  $A$  be a f.d. algebra/ $K$ . Say  $A$  is Frobenius if  $\exists$  a linear map  $\lambda: A \rightarrow K$  such that  $\text{Ker } \lambda$  contains no nonzero left or right ideals.

Def: If further  $\lambda(ab) = \lambda(ba) \forall a, b \in A$ , say  $A$  is a symmetric algebra.

Prop: Define a bilinear form by  $(a, b) = \lambda(ab)$ . Notice  $(ab, c) = (a, bc)$ . Having a nondegenerate bilinear form of this form is equiv. to Frobenius. If form is symmetric then so is  $A$ .

Def.  $A$  is self-injective if  $AA$  is an injective module (equivalently if projective  $\leftrightarrow$  injective in  $A\text{-mod}$ ).

Then  $A$  Frobenius  $\Rightarrow A$  self-injective.

Proof Mimic our proof that  $RG \cong (eG)^A$

Example  $KG$  is a symmetric algebra, define  $\lambda(\sum_{g \in G} a_g g) = a_e e$

Lemma Let  $e = e^2 \in A$  and  $M$  any  $A$ -module. (Rmk:  $Ae$  is an  $A$ -submodule of  $A$ )  
Then:

$$eM \cong \text{Hom}_A(Ae, M) \text{ as vector spaces.}$$

Proof LHS  $\rightarrow$  RHS  $em \rightarrow (ae \rightarrow aem)$

RHS  $\rightarrow$  LHS  $\alpha \rightarrow \alpha(e) = e\alpha(e) \in eM$

Check //

Rmk Note  $A \cong \text{Hom}_A(A, A) = \text{End}_A(A)$  as vector spaces, but opposite algebras.

Thm Suppose  $P$  is a PIM for a symmetric algebra  $A$ . Then  $P/\text{rad}P \cong \text{soc}P$ .

Proof Let  $A \cong P \oplus R$  and  $1 = e + e'$ . Check  $e^2 = e$  and  $P = Ae$ .

Thus  $\text{soc}(P) = \text{soc}(P)e$  is a left ideal in  $A$ . Let  $\lambda$  be as in the def. of symmetric, so  $\text{soc}(P) \not\subseteq \ker \lambda$ . Thus  $\exists x \in \text{soc}(P)$  with  $\lambda(xe) \neq 0$  so  $\lambda(x) \neq 0$  so  $e \text{soc}(P) \neq 0$  so

$\text{Hom}_A(P, \text{soc}(P)) \neq 0$ . Thus  $P/\text{rad}P \cong \text{soc}(P)$  but

we know  $\text{soc}(P)$  is simple.

# Tensor Products

Easiest Case:  $U, V$  vector spaces, bases  $\{u_1, \dots, u_n\}, \{v_1, \dots, v_m\}$ ,  
 $U \otimes V$  is vector space w/ basis  $\{u_i \otimes v_j\}$

Def  $U \otimes V$  is a  $kG$ -module via  $g(u \otimes v) = gu \otimes gv$ .

WARNING: Not true that  $a(u \otimes v) = au \otimes av \quad \forall a \in kG$ .  
In general cannot take  $\otimes$  of two  $R$ -modules  
and get an  $R$ -module.

Prop 1. Easy to see  $\otimes_R$  is commutative, associative.

2. For arbitrary  $A$ -modules, no  $\otimes$ , need a Hopf algebra structure.

Thm Let  $V$  be faithful  $kG$ -module, and  $P$  an indecomposable projective.  
Then  $\exists$  some  $n$  so  $V \otimes V \otimes \dots \otimes V \cong P \oplus Q$ .

Proof Some nontrivial Galois Theory.

Recall  $\text{Hom}_k(U, V)$  is a  $kG$ -module via  $(g\psi)(u) = g\psi(g^{-1}u)$ .

Observation:  $\psi \in \text{Hom}_{kG}(U, V) \iff \begin{aligned} \psi(gu) &= g\psi(u) \quad \forall g, u \\ &\iff g^{-1}\psi(gu) = \psi(u) \quad \forall g, u \\ &\iff g\psi = \psi \end{aligned}$

Thus  $\text{Hom}_{kG}(U, V)$  is the set of fixed points of  $G$  in  $\text{Hom}_k(U, V)$ .



Key Thm Let  $V$  be an arbitrary  $RG$ -module and  $P$  projective. Then  $V \otimes P$  is projective.

Proof Since  $V \otimes P \mid V \otimes F$ , ETS  $V \otimes RG$  is free.

idea Let  $V = \langle v_1, \dots, v_n \rangle$ . Let  $F_i$  be submodule gen by  $v_i \otimes 1$  so  $F_i \cong RG$ .

Check  $V \otimes RG \cong F_1 \oplus \dots \oplus F_n$

In Algebras

1.  $V_p$  is  $p$ -dimensional  $SL(2, p)$  module.  
- Sylow is  $\langle (1 \ 0) \rangle$ ,  $V_p$  uniserial so projective for Sylow so  $p \mid \dim V_p$ .

2.  $V_6 \otimes V_p$  is projective  $\cong P_{p+1}$

Keep going to get all projective covers!