Lecture 7

Review

Thm Bijection btw simple A-modules and $\cong$ classes of indec projectives $S \longleftrightarrow P(S)$ given by
$S \cong P(S)/\mathrm{rad}\, P(S)$

The modules $\{P(S)\}$ are called PIMS (projective, indec. modules)

COR $_AA \cong \bigoplus_{i=1}^{n} P(S_i)^{\oplus \dim S_i}$ times

Example $G = \Sigma_3$ $p=3$ simples are sgn and k

$$_{\Sigma_3}k\Sigma_3 \cong \begin{matrix} k \\ \mathrm{sgn} \\ k \end{matrix} \oplus \begin{matrix} \mathrm{sgn} \\ k \\ \mathrm{sgn} \end{matrix} \qquad k\Sigma_3/\mathrm{rad} \cong k \oplus k$$

$G = \Sigma_3$ $p=2$, simples are k and a 2-dimensional simple projective $\underline{S}$

$$_{\Sigma_3}k\Sigma_3 \cong \begin{matrix} k \\ k \end{matrix} \oplus S \qquad k\Sigma_3/\mathrm{rad} \cong k \oplus M_2(k)$$

General Problem When $A \cong kG$, what additional properties do the PIMS have?

Prop ("Lagranges Thm)

1. Let $H \leq G$. Then kG is free as a left H-module

2. Let P be projective kG-module. Then P_H is projective kH-mod.

3. (Cor) Let $|G| = p^a r$, $p \nmid r$. Then every projective kG module has dimension divisible by p^a.

Proof 1. Let $H = He, Hg_2, \ldots, Hg_n$ be distinct right cosets of $H \leq G$. Check that $\{hg_i \mid h \in H\}$ is a kH submodule $\cong kH$. Thus

$$\operatorname{Res}^G_H kG \cong kH^{\oplus [G:H]} \qquad //$$

2. P proj $\Rightarrow \oplus kG \cong P \oplus P'$, restrict to H and use #1

3. kP has one simple module, so $kP \cong P(k)$ and any projective kP module has dimension div. by p^a //

Example Let $G = \langle g \rangle$ be cyclic of order $p^a r$, $p \nmid r$.

Simple modules S_i are 1-dim, $1 \leq i \leq r$, g acts by λ_i. Indec were uniserial of length $1 \to p^a$. Thus

$$P(S_i) \cong \left.\begin{matrix} S_i \\ S_i \\ \vdots \\ S_i \end{matrix}\right\} p^a$$

Alperin p.35-37 + G has a cyclic normal Sylow p-subgroup, Classify all indec kG-modules

Duality

Review V a vectorspace $/k$, then $V^* \cong \mathrm{Hom}_k(V,k)$ is space of linear functionals. Further, given a linear map $T: V \to W$, $\exists\ T^*: W^* \to V^*$ given by $T^*(\psi)(v) = \psi(T(v))\ \ \forall\ \psi \in W^*$

Exercise. Let V, W be fin dim. Then T^* has matrix transpose to that of T, if we use dual bases

Prop Let V be a kG-module, and let $\varphi \in V^*$. Define $g\varphi$ by

$$g\varphi(v) = \varphi(g^{-1}v). \quad \text{Then this}$$

makes V^* a kG-module.

Proof Note $((g_1g_2)\varphi)(v) = \varphi(g_2^{-1}g_1^{-1}v) = (g_2\varphi)(g_1^{-1}v)$
$= g_1(g_2\varphi)(v)$ //

Rmk Let V, U be kG-modules then $\mathrm{Hom}_k(V,U)$ is a G-module via

$$(gf)(v) = gf(g^{-1}v)$$

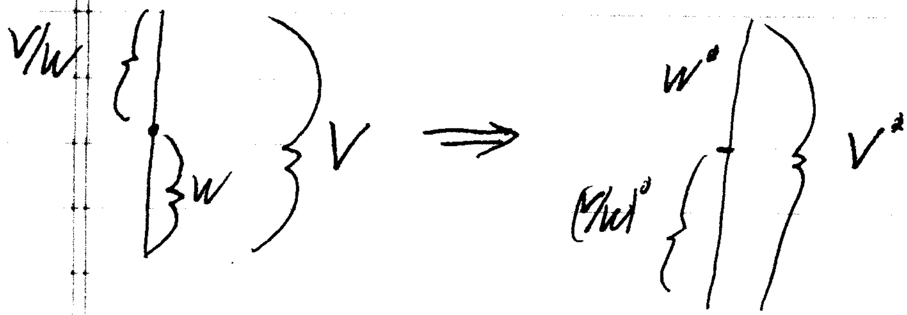
This is special case $U \cong k$.

Basic properties

- Subspaces of $V \longleftrightarrow$ quotients of V^* and same for modules

Ex Suppose $U \subset V$ submodules. Consider

$$\{\varphi \in V^* \mid \varphi|_U = 0\} \text{ naturally } \cong (V/U)^*$$



COR U is simple $\Leftrightarrow$ U^* is simple

- $(V_1 \oplus V_2)^* \cong V_1^* \oplus V_2^*$
- $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$
- $(V^*)^* \cong V$ (this $\cong$ is natural)

Key Lemma $(kG)^* \cong kG$ as left G-modules. Thus the dual of a free module is free.

Proof Consider map $kG \to (kG)^*$ taking $g \to \varphi_g$ so $\varphi_g(h) = \delta_g^h$. This is bijection. Check

$$\varphi(g_1g_2)(h) = \delta_h^{g_1g_2}$$

$$(g_1\varphi(g_2))(h) = \varphi(g_2)(g_1^{-1}h) = \delta_{g_1^{-1}h}^{g_2}$$

$=$

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We now know, since $(kG)^* \cong kG$, that duals of projectives are projective.

COR Let I be a kG-module, TFAE

1. I is a summand of a free module.

2. Given $0 \to I \to V \to Q \to 0$ then $V \cong I \oplus Q$

3. Given
$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, "\psi"] \arrow[d, "\varphi"] & V \arrow[dl, dashed, "\rho"] \\ & I & \end{tikzcd}$$
then $\exists ! \rho$

Def Modules satisfying 2 or 3 are called <u>injective</u>

Restate A kG-module is projective iff it is injective.

COR Suppose P is a PIM. Then $soc(P)$ is simple.

Proof P^* is a PIM so $P^*/rad(P^*)$ simple.

However $P^*/rad(P^*) \cong soc(P)^*$.

Problem $P(S)^* \cong P(S')$. What is S'?

Thm P a PIM then $P/rad(P) \cong soc(P)$.

In particular $P \cong P^*$.