Lecture 6

• Warn about T_n in notes vs book

Recall $G = C_p \times C_p$ has infinite repr type (actually wild if $p > 2$)

Let $G = \langle x, y \mid x^p = y^p = e,\ xy = yx \rangle$

Consider $2n$-dimensional vector space V_n with matrices:

$$x \to \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \quad y \mapsto \begin{pmatrix} I_n & 0 \\ \begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ 0 & 1 & 0 \end{smallmatrix} & I_n \end{pmatrix}$$

Check $x^p = I_{2n} = y^p$, $xy = yx$ (note $y^2 = \begin{pmatrix} I & 0 \\ \begin{smallmatrix} 0 & & \\ 2 & \ddots & \\ 0 & 2 & 0 \end{smallmatrix} & I \end{pmatrix}$ etc.)

Q: What is $\mathrm{End}_{kG}(V_n)$? Need a linear map $\psi : V_n \to V_n$ such that

$$\psi(gv) = g\psi(v) \quad \forall g \in G, \text{ so ETS } \begin{array}{l} \psi(xv) = x\psi(v) \\ \psi(yv) = y\psi(v) \end{array}$$

Thus $\mathrm{End}_{kG}(V_n) \cong \{2n \times 2n \text{ matrices } A \mid A \text{ commutes w/ } X \text{ \& } Y\}$

Exercise

$$\mathrm{End}_{kG}(V_n) = \left\{ \begin{pmatrix} B & 0 \\ C & B \end{pmatrix} \;\middle|\; B = \begin{pmatrix} \lambda_1 & & & 0 \\ \lambda_2 & \ddots & & \\ \lambda_3 & \ddots & \ddots & \\ \lambda_n & \lambda_3 & \lambda_2 & \lambda_1 \end{pmatrix} \right\}$$

Check radical is codimension 1 so

$\mathrm{End}_{kG}(V_n)$ is local $\Rightarrow V_n$ is indecomposable.

* Review exact Sequences, splitting.

Projective Modules

Review *Free* A-modules are of the form $A \oplus A \oplus \cdots \oplus A$, have basis $\{e_i\}$ that can be sent anywhere and extend to an A-module homom.

Prop Let U be an A-module. Then U is free iff U has a subspace X such that for any A-module V and any linear map $\psi: X \to V$, ψ extends *uniquely* to an A-module homomorphism $\tilde{\psi}: U \to V$.

Rmk Stating as prop. of subspace X no diff than prop of basis of X.

Thm Let P be an A-module. TFAE, and we say P is *projective*.

1. P is a direct summand of a free module.
2. Any surjective module homo $\psi: U \to P$ *splits*, i.e. $U \cong P \oplus \operatorname{Ker}\psi$.
3. Given any surjective module homom. $\varphi: V \to W$ and any homomorphism $\psi: P \to W$, $\exists!\ \rho: U \to V$ such that $\varphi \circ \rho = \psi$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\rho"'] \arrow[d, "\psi"] & \\ V \arrow[r, "\varphi"] & W \arrow[r] & 0 \end{tikzcd}$$

4. *Sketch of Proof*

Rmk Krull-Schmidt implies an <u>indecomposable</u> projective module is a direct summand of A, don't need $A \oplus A \oplus \ldots \oplus A$

<u>Key Theorem</u> There is a 1-1 correspondence between $\cong$ classes of indecomposable projective A-modules and $\cong$ classes of simple A-modules given by $P \longrightarrow P/\mathrm{rad}\,P$.

- Says $P/\mathrm{rad}\,P$ is simple. Also says $P/\mathrm{rad}\,P \cong Q/\mathrm{rad}\,Q \Rightarrow P \cong Q$.
- Rmk: $U/\mathrm{rad}\,U$ simple $\longleftrightarrow$ U has a unique max'l submodule.

<u>Proof.</u> <u>Step 1</u> Show $P/\mathrm{rad}\,P$ is simple

<u>Proof</u> Let $\varphi \in \mathrm{End}_A(P)$. Then $\varphi(\mathrm{rad}\,P) = \varphi(\mathrm{rad}\,A\,P) \subseteq \mathrm{rad}\,(\varphi(P)) \subseteq \mathrm{rad}\,P$

Thus φ induces an endomorphism $\bar{\varphi} \in \mathrm{End}_A(P/\mathrm{rad}\,P)$.

Moreover given $\tilde{\varphi} : P/\mathrm{rad}\,P \to P/\mathrm{rad}\,P$

$$\begin{tikzcd} P \arrow[r, dashed, "\exists!\,\hat{\varphi}"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ P/\mathrm{rad}\,P \arrow[r, "\tilde{\varphi}"] & P/\mathrm{rad}\,P \end{tikzcd}$$

$\leftarrow$ surjection

Thus Alg hom $\mathrm{End}(P) \twoheadrightarrow \mathrm{End}(P/\mathrm{rad}\,P)$ so $P/\mathrm{rad}\,P$ is indec.

But it's semisimple! Thus simple //

<u>Step 2</u> Suppose P, Q are indec proj and $P/\mathrm{rad}\,P \cong Q/\mathrm{rad}\,Q$.

$$\begin{tikzcd} P \arrow[r, dashed, "\exists!"] \arrow[d, "\pi"'] & Q \arrow[d, "\pi"] \\ P/\mathrm{rad}\,P \arrow[r, "\cong"] & Q/\mathrm{rad}\,Q \end{tikzcd}$$

check $\mathrm{Im}\,P = Q$, not $\subset \mathrm{rad}\,Q$

so $P \twoheadrightarrow Q$

so $P \cong Q$

<u>Step 3</u> Show every simple occurs.

Consequences

1. Let $S_1, S_2, \ldots, S_n$ be simple A-modules. Let $P(S_i)$ be indec projective cover $S_i \cong P(S_i)/\mathrm{rad}(S_i)$.

   Then ${}_AA \cong \bigoplus_{i=1}^{n} P(S_i)^{\oplus \dim S_i}$

2. Suppose $U/\mathrm{rad}\,U \cong S$ is simple. Then $\exists$ a surjection $P(S) \twoheadrightarrow U$.

Goal When $A \cong kG$, what additional things can we say about PIM's?

Prop (Lagrange's Thm)

1. Let $H \leq G$. Then kG is ~~projecti~~ free as a left kH-module.
2. Let P be projective kG-module. Then $\mathrm{Res}_H^G P = P_H$ is projective kH-module.
3. Cor Let $|G| = p^a r$, $p \nmid r$. Every projective kG-module ~~is divisible~~ has dimension divisible by p^a.

EX G cyclic. Cor 3 above gives us the PIM's