Lecture 4

p.7 #2-4
12 #5
20 #4,4

Recall

Thm A semisimple k-algebra, $k=\bar{k}$. Then

1. $A \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_s}(k)$

2. $M_{n_i}(k)$ has one simple module of dimension n_i, $\cong$ to column vectors of length n_i. Moreover $M_{n_i}(k) \cong \underbrace{S \oplus S \oplus \cdots \oplus S}_{n_i}$ as a left $M_{n_i}(k)$ module.

3. Thus A has s-simple modules, $S_1, S_2, \ldots, S_s$ and
$$ {}_A A \cong \bigoplus_{i=1}^{s} (S_i)^{\oplus n_i} $$

Thm The group algebra kG is semisimple if and only if $\operatorname{char} k = p \nmid |G|$.

Proof Last time: $p \mid |G| \Rightarrow$ not semisimple.

So assume $p=0$ or $p \nmid |G|$. Let $U \subset V$ be kG-modules. ETS $V \cong_{kG} U \oplus W$. Define linear map $\pi: V \to U$ such that $\operatorname{Im}(\pi) = U$ and $\pi^2 = \pi$.

Define $\tilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v)$. Check

1. $\tilde{\pi}$ is a kG-module map
2. $\tilde{\pi}^2 = \tilde{\pi}$
3. $\operatorname{Im} \tilde{\pi} = U$ (more, $\tilde{\pi}|_U = id$)

Thus $V \cong U \oplus \ker \tilde{\pi}$

Thus: When $p \mid |G|$, $\operatorname{rad} kG \neq 0$ and $kG/\operatorname{rad}(kG)$ is a direct sum of matrix algebras.

Recall simple kG modules $\longleftrightarrow$ simple $(kG/\operatorname{rad}(kG))$ modules

Thm The # of simple kG-modules is the # of conjugacy classes of G with elements of p'-order.

Proof p17-20 of Alperin, generalizes char 0 proof #conjclasses $= \dim(Z(\mathbb{C}G))$

Ex 1 $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_s}(\mathbb{C})$, $s =$ # of conjugacy classes.
$n_i = \dim(S_i)$

Note $|G| = \sum_{i=1}^{s} n_i^2$

Rmk $n_i \mid |G|$, hard proof, character theory.

Eg $G = S_3$, $k = \mathbb{C}$ $\quad \psi_1 =$ trivial module $\quad \psi_2 : \begin{matrix}(12) \to (-1) \\ (123) \to (1)\end{matrix}$ sign

ψ_3 from day 1 $\quad (12) \to \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad (23) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So $\mathbb{C}S_3 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ as algebras

change of basis?

Ex 2 Suppose $G = P$ is a p-group, then trivial module is the only simple module.

$\operatorname{rad}(kP) = \Delta(P)$ and $kP/\operatorname{rad}(kP) \cong k$

- Direct Proof

- kP modules can still be very complicated.

Ex 3 G cyclic, $|G| = n = p^a r$, $p \nmid r$. Check that unique subgroup of order r contains all elts of p' order, so $\exists$ r simple modules

Now $X^r - 1$ is separable, so $\exists$ r distinct r^{th} roots of unity $\lambda_1, \dots, \lambda_r$. Moreover $(\lambda_i)^n = (\lambda_i)^{rp^a} = 1$ so if $G = \langle g \rangle$ then $\psi_i(g) = (\lambda_i)$ is simple,

so all simples are 1-dimensional

Rmk Can prove directly that simples are 1-dim for (abelian) cyclic group.

Rmk $G \cong Z_{p^a} \times Z_r$ $\quad kG \cong k(Z_{p^a}) \otimes k(Z_r)$

Example 4

$G = SL(2,p)$ Fact p conjugacy classes of p' order.

Natural module $V = \{ \binom{\alpha}{\beta} \}$ Let $X = \binom{1}{0}$ $Y = \binom{0}{1}$

Recall! $Sym^a(V) \cong V^{\otimes a}/(e)$ think of as homog. poly in X, Y of degree a.

Note $\begin{pmatrix} a & b \\ c & d \end{pmatrix} X = aX + cY \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y = bX + dY$

V_1 = trivial module $V_2 = V = \langle X, Y \rangle$ $V_3 = Sym^2(V) = \langle X^2, XY, Y^2 \rangle$

$V_4 = Sym^3(V) = \langle X^3, X^2Y, XY^2, Y^3 \rangle$ .. so $\dim V_n = n$.

<u>Thm</u> $V_1, V_2, \ldots, V_p$ are simple, and thus give all simple $SL_2(p)$-modules.

"<u>Proof</u>" Let $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Let $V_{n+1} = \langle X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n \rangle \quad 1 \le n < p$

<u>Step 1</u> Analyze V_{n+1} as a $k\langle g \rangle$ module.

Note $g: X \to X+Y$
$\qquad Y \to Y$

Check $0 \subset \langle Y^n \rangle \subset \langle XY^{n-1}, Y^n \rangle \subset \ldots \subset V_{n+1}$ is unique comp series.

Similarly for $k\langle h \rangle$

<u>COR</u> Any $SL_2(p)$ submodule contains X^n and Y^n so is all of V_{n+1}.

Hidden in linear algebra: Need $\binom{n}{i} \neq 0$ for $1 \le i \le n$. //

<u>Rmk</u> $V_{p+1} = \langle X^p, X^{p-1}Y, \ldots, Y^p \rangle$ has submodule $\langle X^p, Y^p \rangle$

Since $(aX+bY)^p = a^pX^p + b^pY^p$ so V_{p+1} not simple!

* We will return to this example

* These are Lie theoretic techniques

Clifford Thm

Suppose S is a simple kG-module and $H \trianglelefteq G$.
Then $\mathrm{Res}_H S = S_H$ is semisimple.

Proof

Lemma Let V be a kH submodule of S_H. Then so is gV.

Proof $hgV = g\underbrace{g^{-1}hg}_{\in H} V \subseteq gV$. //

Choose T a simple kH submodule of S_H.
Then gT also simple.

Now $\sum_{g\in G} gT$ clearly a kG-submodule

Thus $V = \sum_{g \in G} gT$ is semisimple kH module.