Lecture 24

Review B a block of Σ_n, p-core $b = (b_1, b_2, \ldots, b_r)$ and weight w, so $b \vdash n - pw$

Put b on abacus with $pw+r$ beads, let Γ_i = # beads on runner i?
Suppose $\Gamma_i = \Gamma_{i-1} + k$, $k \geq w$.

$\bar{B}$ block of Σ_{n-k} w/ core obtained by swapping runners $i-1$ and i.

Thm (Scopes) B and $\bar{B}$ are Morita equivalent.

$$\begin{array}{ccccccccc} \operatorname{mod} B & \xrightarrow{G_1} & \operatorname{mod} \Sigma_n & \xrightarrow{G_2} & \operatorname{mod} \Sigma_{n-k} \times \Sigma_k & \xrightarrow{G_3} & \operatorname{mod} \Sigma_{n-k} & \xrightarrow{G_4} & \bar{B} \\ & \xleftarrow[G_1']{} & & \xleftarrow[G_2']{} & & \xleftarrow[G_3']{} & & \xleftarrow[G_4']{} & \end{array}$$

$G_2 = \operatorname{Res}$ $\qquad G_3 = - \otimes_{\Sigma_k} R$ $\qquad \left(\begin{array}{c}\text{quotients at}\\ \text{action of } \Sigma_k\end{array}\right)$
G_2' = Coinduction
G_3' = extend action by letting Σ_k act trivially

Recall Indecomposable kG-modules U have a vertex Q, which is minimal subgroup such that $U \mid V_Q^G$ some V,
WLOG $V = U_Q$.

Problem Compute vertices and sources if possible, for various modules.

Rmk Morita Equiv preserves Endomorphism algebras
hence indecomposability.

Thm (Wildon '03)

1. Let S^λ be an indecomposable Specht module. Then the vertex of S^λ is nontrivial cyclic iff λ has p-weight 1

2. Let $\lambda = (n-r, 1^r)$. If S^λ is indecomposable and $p \nmid n$ then S^λ has as vertex a Sylow p-subgroup of $\Sigma_{n-r-1} \times \Sigma_r$.

Rmks 1. S^λ is always indec. when $p > 2$ and also when $p = 2$ and λ is 2-reg. For $p=2$ and λ 2-singular, it is not known when S^λ is indec.

2. Let $n = p^a r$, $p \nmid r$. Suppose λ has p-weight w. Then p^{a-w} is the largest possible power which can divide $\dim S^\lambda$.

Pf $\dim S^\lambda = \dfrac{n!}{\prod h.l.}$

Thm 5 Let V be an indecomposable B module. Then V and $G(V)$ have the same vertex Q.

Lemma 4 Let $J_0 \leq J$ and K be subgroups of G. Let U be a $k(J_0 \times K)$ module. Then

$$U\Big\uparrow_{J_0 \times K}^{J \times K} \otimes_K R \cong (U \otimes_K R)\Big\uparrow_{J_0}^{J} \quad \text{as } kJ \text{ modules,}$$

Pf $\quad k(J \times K) \otimes_{J_0 \times K} U \otimes_K R \cong kJ \otimes_{J_0} U \otimes_K R$

Proof of Thm Part 1

Step 1 Use dimensions

Recall Suppose $P \in Syl_p(G)$, $Q \le P$ a vertex of U. Then

$$\frac{|P|}{|Q|} \mid \dim U. \quad \text{(Follows from Green's Index Thm)}$$

Let λ have p-weight w. Then

$$\frac{p^a}{|Q|} \mid \dim S^\lambda \Rightarrow \frac{p^a}{|Q|} \mid p^{a-w} \Rightarrow p^w \mid Q.$$

Suppose now Q is cyclic of order p^c, $c \ge 1$, so $w \le c$.

We know $Q \le \underset{\text{group}}{\text{Defect}} = Syl_p(\Sigma_{pw})$

Now Σ_{pw} has an element of order p^c iff $wp \ge p^c$, i.e. $w \ge p^{c-1}$

Thus $p^{c-1} \le w \le c$ (*) Impossible unless $c = 1$ or 2.

$c = 1 \Rightarrow w = 1$ is known case, Specht modules do have cyclic vertices

Proof of Thm 5

Suppose V has vertex Q and source U

$$G_3 \circ G_2 \left(U\uparrow_Q^{\Sigma_n} \right) = \left(\left(U\uparrow_Q^{\Sigma_n} \right)\downarrow_{\Sigma_{n-k} \times \Sigma_k} \right) \otimes_{\Sigma_k} R$$

by Mackey $=$

$$\left(\bigoplus (g(U)) \uparrow_{\Sigma_{n-k} \times \Sigma_k \cap Q^g}^{\Sigma_{n-k} \times \Sigma_k} \right) \otimes_{\Sigma_k} R$$

* Set $J_0 \leq \Sigma_{n-k}$ as projection of $(\Sigma_{n-k} \times \Sigma_k) \cap gQg^{-1}$, so $|J_0| \leq |Q|$

Apply Lemma 4

$$\left(g(U)\uparrow_{\Sigma_{n-k} \times \Sigma_k \cap Q^g}^{\Sigma_{n-k} \times \Sigma_k} \right) \otimes_{\Sigma_k} R \cong \left(g(U)\uparrow_{\Sigma_{n-k} \times \Sigma_k \cap Q^g}^{J_0 \times \Sigma_k} \otimes_{\Sigma_k} R \right)\uparrow_{J_0}^{\Sigma_{n-k}}$$

Thus $G(V)$ is a summand of a module, each summand of which is induced from subgroup of J_0, and hence

$$|\text{vertex } G(V)| \leq |Q|.$$

Conversely suppose X is an indec $\overline{B}$ module w/ vertex R, source Y

Then

$$G'(X) \mid G_2' G_3' \left(Y\uparrow_R^{\Sigma_{n-k}} \right) = (Y \otimes R_{\Sigma_k})\uparrow_R^{\Sigma_n} \quad \text{so}$$

$G'(X)$ has vertex in R. Thus $|\text{vertex } G'(X)| \leq |\text{vertex } X|$.

But $G'G(V) \cong V$ so $|\text{vertex } G(V)| = |\text{vertex } V|$

Thus $(*) \Rightarrow gQg^{-1} \leq \Sigma_{n-k}$ is vertex of $G(V)$ //

$\underline{c=2}$ $\quad 2 \geq w \geq p \quad$ so $\quad p = 2 = w$

By Scopes, all weight 2 blocks arise for Σ_n, some $n \leq 5$.

Thus WLOG S^λ is indec Specht module for Σ_4 or Σ_5 in principal block.

$\lambda = 4, (3,1), (2,1^2), (1^4), (5), (3,2), (2,1^3), (1^5)$ have odd dimension
so $|\text{vertex}| = 8$ noncyclic

Left

1. $S^{(2,2)}$ is indec w/ vertex $V = \{e, (12)(34), (13)(24), (14)(23)\}$

2. $S^{(3,1,1)}$ is indec w/ vertex $\neq Q = \{e, (12), (34), (12)(34)\}$

Proof of Part 2

- Define p-permutation modules (aka trivial source)
- Choose U indec module w/ p-perm basis B invariant under Sylow P

Thm U, B as above, $Q \leq P$. Then U has a vertex containing Q iff $B^Q \neq \emptyset$

- Construct p-perm basis of S^λ