Review

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) \( \vdash \), \( \lambda \neq 0 \). For any \( r \geq 1 \) we can represent \( \lambda \) on an abacus with \( r \) beads.

Then let \( \lambda \vdash d \). Then \( \lambda \) has a well-defined \( p \)-core \( \tilde{\lambda} \vdash d \text{-pw} \), where \( w \) is the \( p \)-weight of \( \lambda \). For \( \lambda, \lambda' \vdash d \), \( \tilde{\lambda} \) and \( \tilde{\lambda}' \) are in the same block iff \( \tilde{\lambda} = \tilde{\lambda}' \).

Recall

1. \( \lambda \) is uniquely determined by its \( p \)-core and \( p \)-quotient, a sequence of partitions \( \lambda_{(0)}, \lambda_{(1)}, \lambda_{(p)} \) of total weight \( w \).

2. The number of \( p \)-irred modules in a block is a function only of \( w \).
   Similarly for \( k \mathbb{Z}_d \).

3. Often use \( WP + E \) beads to represent \( \lambda \).

4. The Sylow \( p \)-subgroup of a block of weight \( w \) is \( \cong \) to a Sylow \( p \)-subgroup of \( \mathbb{Z}_{pw} \).

5. Adding/removing nodes at rim \( p \)-hooks easy to represent on abacus.
Def: Two categories $C$ and $D$ are equivalent if there exist functors $F: C \rightarrow D$ and $L: D \rightarrow C$ such that $FL$ is naturally isomorphic to the identity functor $1_C$ and similarly $LF\cong 1_D$.

Equivalently: $F: C \rightarrow D$ is an equivalence if for any objects $c, d \in C$, the map $\text{Hom}_C(c, d) \rightarrow \text{Hom}_D(F(c), F(d))$ is a bijection and every object in $D$ is isomorphic to an object of form $F(c)$.

Def: Algebras $A_1, A_2$ are Morita equivalent if $\text{mod} A_1$ and $\text{mod} A_2$ are equivalent.

Ex: $\mathbb{R}$ and $M_n(\mathbb{R})$, a ring with 1.

Donovan Conjecture: Fix a $p$-group $D$. There are only finitely many block algebras, up to Morita equivalence, with defect group $\cong D$.

Thm (Scopes '91): Fix $w \geq 0$. There are only finitely many $kG$ blocks of weight $w$ up to Morita equivalence.

Moreover:
- The number of non equivalent blocks is $\leq \prod_{i=1}^p \left( (i-1)(w-1) + 1 \right)$.
- Every Morita type occurs for some $\Sigma_d$,
  \[
  d \leq \frac{p^2(p-1)^2(w-1)^2}{4+wp}
  \]
Cartan Matrix

Recall: Decomposition matrix \( D \) has irreducible \( \mathfrak{sl}_2 \) module:

\[
\begin{pmatrix}

\end{pmatrix}
\]

Def: Cartan matrix \( C \) has rows and columns indexed by irreducible \( \mathfrak{sl}_2 \) modules (equiv. indec. proj. modules) and \( \mathfrak{sl}_2 \):

\[
C_{ij} = \text{mult} (P_i, D_j) = \text{Hom} (P_i, P_j) = \text{Hom} (P_i, P_j)^* = C_{ji}
\]

Ex: \( \mathfrak{sl}_3 \), \( p = 3 \)

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

\[
P \text{IMs } R \text{ spin } R \text{ spin } C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

Then \( C = D^\text{tr} D \).

Rank: Just as decomposition matrices of a block make sense,
so do Cartan matrices of a block.

Rank: Cartan matrix is invariant under Morita Equivalence (up to...
**Scores' Idea**

Fix a block $B$ with $p$-core $(b_1, b_2, \ldots, b_r)$ and weight $w$.
Take an $r+p$-vector $B$-set and represent on a abacus.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be # of beads on each runner.

**Ex:** $p=3$, core = $(9, 7, 5, 3, 1^2)$

Weight = 2

$r+p=12$

$$
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

**h.l.** 1, 3, 5, 8, 11, 14

$p_1 = 2$ $p_2 = 3$ $p_3 = 7$

**Thm (Scores)**

Suppose for some $i \geq 2$, $\Gamma_i = \Gamma_{i-1} + \kappa$ with $\kappa \geq w$.

Let $\overline{B}$ be block of $\Sigma_{n-k}$ with core $\overline{B}$, where $\overline{B}$ has abacus obtained by swapping runners $i-1$ and $i$.

Then $B$ and $\overline{B}$ are Morita equivalent.
Ex. B block of $\Omega_{22}$ w/ core (9,7,5,3,1$^2$)

\[ \begin{array}{cccc}
0 & 0 & 0 & \\
0 & 0 & \\
\vdots & 0 & \\
0 & 0 & \\
\end{array} \]

block of $\Omega_{22}$ w/ core (8,6,4,3$^2$)

Repeat!

\[ \begin{array}{cccc}
0 & 0 & 0 & \\
0 & 0 & \\
\vdots & 0 & \\
0 & 0 & \\
\end{array} \]

block of $\Omega_{23}$ core (7,5,3$^2$)

Proof of Theorem. Assume Matita equivalences as above.

Suppose $B_i B_i'$ blocks at weight $w$ at $\Omega$, $\Omega$ w/ $N > M$.

Suppose $B_0 = B, B_1, ..., B_w = B$ so each pair Matita equiv as above. Each family has a unique block ancestor of all in block.

Take such a block. Write out its 1st column back lengths.

Since it's a p-core, 1st run col is empty.

Ex. h.w. 1,2,3,5,8,11

\[ \begin{array}{cccc}
0 & 0 & 0 & \\
0 & 0 & \\
\vdots & 0 & \\
\end{array} \]

Keep going.

Let $\Theta_i$ be # of beads on each runner.

$\Theta = 0$, $\Theta_i \leq \Theta_i + w - 1$.

Moreover, largest 1st col back length is at most $p(w-1)(w-1)$.