

Lecture 1

Review $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ $\lambda_t \neq 0$.
First column hook lengths $h_{i1}^\lambda = \lambda_i - i + t$

Def Let $r \geq t$. Then $(\lambda_1 - 1 + r, \lambda_2 - 2 + r, \dots, \lambda_r - r + r)$ is called a sequence of B-numbers for λ .

Props Easy to recover λ from a sequence of B #'s.

Ex $\lambda = (6, 2, 2, 1)$ hook lengths ($r=4$) $(9, 4, 3, 1)$
 $r=5$ $(10, 5, 4, 2, 0)$
 $r=6$ $(11, 6, 5, 3, 1, 0)$
 $r=7$ $(12, 7, 6, 4, 2, 1, 0)$

Abacus Given λ as above, and $r \geq t$, represent λ on an abacus with r beads by putting a bead at each spot in seq of B #'s

Ex $\lambda = (6, 2, 2, 1)$ $p=3$

• 0 •	0 • 0	0 0 •	0 0 0
0 0 •	• 0 0	0 • 0	• 0 •
• • •	• • •	0 • •	0 0 •
0 • •	• 0 •	• • 0	• • •
0	0	0	0
$r=4$	$r=5$	$r=6$	$r=7$

Props Recover 1st col hook lengths by counting 1st gap as "0" then count from there

Prop Removing a rim p -hook from λ corr to replacing a B -number with $B_i - p$, i.e. sliding a bead up one spot.

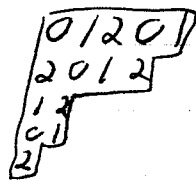
Adding (Removing) an addable (removable) node corresponds to sliding a bead one spot right (left) on the abacus.

COR The p -core $\tilde{\lambda}$ is well defined!

p -content

Def Fix p . The p -content residue of a node (i, j) is $j - i \pmod p$.
The p -content of λ is $(c_0, c_1, \dots, c_{p-1})$ where $c_i = \#$ nodes in λ with residue i .

Ex $\lambda = (54221)$ $p = 3$



p -content $(4, 5, 5)$

Thm Let $\lambda, \mu \vdash d$. Then $\tilde{\lambda} = \tilde{\mu}$ iff λ, μ have same p -content.

Proof A rim p -hook clearly contains one node of each p -residue.
Thus $\tilde{\lambda} = \tilde{\mu} \Rightarrow \lambda, \mu$ same p -content.

Conversely, suppose we have abacus w/ same # of beads on each runner, corr to empty diagram. Build diagram of λ by adding one node at a time. Check node of res i corr to $\overset{i-1}{\cdot} \rightarrow \overset{i}{\cdot}$.
Thus λ, μ have same # of beads on each runner,
so $\tilde{\lambda} = \tilde{\mu}$.

Def Given $\lambda \vdash d$, the weight^w is the # of rim p-hooks removed to get $\tilde{\lambda}$. Thus $\tilde{\lambda} \vdash d - pw$.

(Murphy 183)

Thm Let R be an integral domain. Then $Z(R\Sigma_d)$ is the ring of completely symmetric polynomials in the operators L_2, L_3, \dots, L_n .

Prob Exercise to show these commute w/ all $(a, a+1)$, harder is to show we have right dimension.

COR (After some work)

Nakayama's Conjecture S^λ and S^μ are in same block of $R\Sigma_d$ if & only if $\tilde{\lambda} = \tilde{\mu}$.

"Proof" Recall S^λ has G-Z basis, we know how the L_i act.

Murphy constructs explicitly the block idempotents from the JM elements π

• First proof 1947 Brauer / Robinson

• We already proved λ is a p-core \leftrightarrow no rim p-hooks \leftrightarrow max power $p \mid \dim S^\lambda$
 $\leftrightarrow S^\lambda$ in own block

Thm Let $\lambda \vdash d$ have weight w and p -core $\tilde{\lambda} \vdash d-pw$.
 The # of partitions of d with p -core $\tilde{\lambda}$ depends only on w and is:

$$b(w) = \sum_{(w_0, \dots, w_{p-1})} p(w_0) \cdots p(w_{p-1})$$

sum over all tuples of nonnegative integers so $\sum w_i = w$,
 where $p(u) = \#$ partitions of u .

Proof Given M with $\tilde{M} = \tilde{\lambda}$, the $p(w_i)$ tells us how beads on runner i move.

Ex $\begin{matrix} 0 & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & & \end{matrix}$

$w = 7$
 $(2,1,1), (2,1), \emptyset$

called the p -quotient. //

Rmk λ is uniquely determined by its p -core & p -quotient.

Ex $p=3$ $d=10$ p -cores $x, \begin{matrix} xxx \\ x \end{matrix}, \begin{matrix} xxx \\ x \\ x \end{matrix}$

(Rmk no 3-cores of 7)

Thm (Granville, Ono '96) For every integer d and $t \geq 4$, \exists a partition of d which is a t -core.

Core \times $w=3$, core $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$

(3), \emptyset , \emptyset $\begin{matrix} 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$ \rightarrow ~~$\begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{matrix}$~~ $\begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & \times & & & \\ \times & & & & \end{matrix}$ (4, 2, 1, 1)

\emptyset , (3), \emptyset $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$ \rightarrow $\begin{matrix} \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \end{matrix}$ (10)

\emptyset , \emptyset , (3) $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$ $\begin{matrix} \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & & & & & & \end{matrix}$ (8, 2)

(2), \emptyset , \emptyset $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$ $\begin{matrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & & & \\ \times & & & \end{matrix}$ (4, 4, 1, 2)
 (2), (1), \emptyset $\begin{matrix} 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & & \end{matrix}$ etc...

Thm The # of p -regular $\lambda \vdash d$ with core $\nu \vdash d - pw$ also depends only on w , and is

$$b^p(w) = \sum p(w_1) \dots p(w_{p-1})$$

(w_1, \dots, w_{p-1}) as above.

Thm Let B be a block of $R \Sigma_n$ with weight w . A defect group of B is \cong to a ~~sub~~ Sylow p subgroup of Σ_{pw} .

Ex 1. $p=2$ no blocks of defect 2.

2. $w < p$ then defect group is elementary abelian.