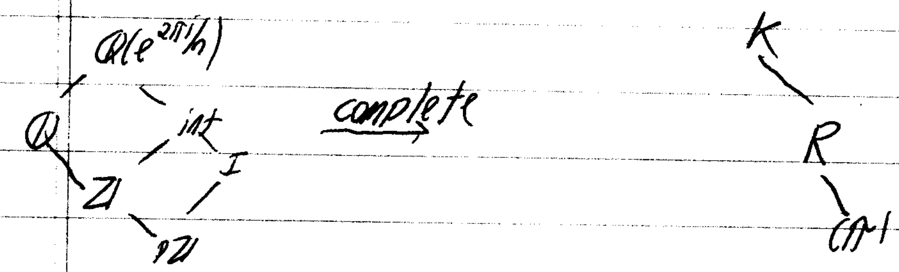
p-modular Reduction

Fact Let S be an irred $\mathbb{Q}G$-module, $|G|=n$. Then S can be written over $\mathbb{Q}(e^{2\pi i/n})$.



Setup R is a complete d.v.r. (D.V.R = PID + local) with max ideal (π)
K is field of quotients
$k = R/(\pi)$ field of char p.

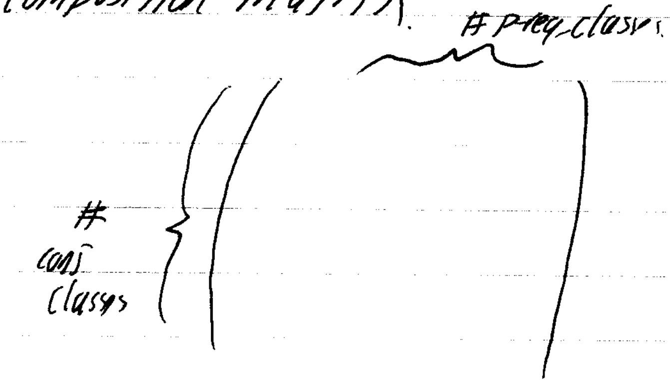
p-modular reduction

$$S \xrightarrow{\text{choose basis}} \rho : G \to GL_n(R) \xrightarrow{\text{reduce}} \bar{\rho} : G \to GL_n(k)$$

to get kG-module $\bar{S}$.

Thm The composition factors of $\bar{S}$ are independent of the choice of basis above.

Define Decomposition matrix.



Thm Let S be an irreducible $\mathbb{C}G$-module. Then all comp factors of $\overline{S}$ lie in the same p-block.

Proof Sketch

$$\begin{array}{c} RG \\ \downarrow \\ kG = B_1 \oplus \cdots \oplus B_t \\ e = e_{B_1} + \cdots + e_{B_t} \end{array}$$

Thm central idempotents lift to RG, i.e.
$$e = \tilde{e}_{B_1} + \cdots + \tilde{e}_{B_t}.$$

If S_R is the RG-module then $S_R = \tilde{e}_{B_1} S_R \oplus \cdots \oplus \tilde{e}_{B_t} S_R$

Now $\otimes_R k$ $\quad \overline{S} = \cdots$

So only one $\tilde{e}_{B_i}$ is $\neq 0$ on S.

COR If you partition irreducibles into blocks, you get

$$\text{Decomp matrix} = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right)$$

Can ask about dec. #'s in a block.

Lecture 20

Review

$\lambda \vdash d$ means λ is a partition of d.

$M^\lambda \cong \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} R$ is permutation module, basis of λ-tabloids

S^λ = Specht module, basis $\{e_t = \kappa_t\{t\} \mid t \text{ is a SYT of shape } \lambda\}$

$\dim S^\lambda = \frac{n!}{\prod h_i}$

Jucys-Murphy Elements:

$L_1 = 0 \quad L_2 = (12) \quad L_3 = (13) + (23) \ldots \quad L_d = (1d) + (2d) + \cdots + (d-1, d)$

Facts

1. $S^\lambda / S^\lambda \cap S^{\lambda^\perp}$ is nonzero, denoted D^λ, $\leftrightarrow$ λ is p-regular.

2. $\{D^\lambda \mid \lambda \text{ p-reg}\}$ gives all $\not\cong$ simple modules in char p.

3. Over $\mathbb{C}$ the JM elements act diagonally on a GZ basis of S^λ.
   Weights $\leftrightarrow$ residue sequences.

   Ex S^{31}

Problem 1. When are D^λ & D^μ in same block?

2. Except $p=2$, λ p-sing, S^μ is indec. So
   When are S^λ, S^μ in same block?

Thm Let S be a simple $\mathbb{Q}G$ module.
$\bar{S}$ is simple and forms its own block iff $p^a \mid \dim S$
where $|G| = p^a r$, $p \nmid r$.

Proof Can use defect gps...

Ex S^λ is in its own p-block $\Leftrightarrow$ no hook lengths are div. by p.

p-cores and abaci

• Define rim p-hook, corr between hooks & rim hooks.
• Define p-core

COR S^λ is its own block $\Leftrightarrow$ λ is a p-core

Def $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ $\lambda_t \neq 0$

$$h_i^\lambda = h_{i1}^\lambda = \lambda_i - i + t \qquad 1^{st} \text{ col hook lengths}$$

Fact $\{h_i^\lambda\}$ determine λ

$$\lambda_t = h_t^\lambda, \quad \lambda_{t-1} = h_{t-1} - 1 \quad \lambda_{t-2} = h_{t-2} - 2$$

More generally given $\beta_1 > \beta_2 > \ldots > \beta_r = 0$
get $\lambda_i = \beta_i + i - r$ partition w/ some 0's

call $\beta_1, \ldots, \beta_r$ a seq. of β #s.

- Abacus
- Renumbering/adding nodes
- Rim p-hooks - replace B_H by $B_i - P$

CCB p-core is unique

Define weight