Fact. Let $S$ be an irreducible $kG$-module, $|G| = n$. Then $S$ can be written over $\mathbb{Q}(e^{2\pi i/n})$.

Setup. $R$ is a complete d.v.r. (D.V.R. = PID + local) with maximal ideal $\mathfrak{m}$. $K$ is field of quotients of $R$. $p = R((t))$ field of char $p$.

$p$-modular reduction

Choose basis $\rho: G \to GL_n(R) \rightarrow \rho: G \to GL_n(k)$ to get $kG$-module $S$.

Then the composition factors of $S$ are independent of the choice of basis above.

Define Decomposition matrix. # prep classes

$$\begin{pmatrix} \# \text{conjugacy classes} \end{pmatrix}$$
Theorem: Let \( S \) be an indecomposable \( RG \)-module. Then all composition factors of \( S \) lie in the same \( p \)-block.

\( \text{Part Sketch} \)

\[
\begin{array}{c}
R G \\
\downarrow \\
R G = R_1 \oplus \cdots \oplus R_6 \\
e = e_{R_1} + \cdots + e_{R_6}
\end{array}
\]

The central idempotents lift to \( RG \), i.e.,

\[
e = \tilde{e}_{R_1} + \cdots + \tilde{e}_{R_6}.
\]

If \( S_R \) is the \( RG \)-module then

\[
S_R = \tilde{e}_{R_1} S_R \oplus \cdots \oplus \tilde{e}_{R_6} S_R.
\]

Now \( e_S \neq 0 \) in \( S \).

So only one \( \tilde{e}_{R_1} \) is \( \neq 0 \) on \( S \).

Corollary: If you partition irreducibles into blocks, you are

\[
\text{Decomp matrix} = \begin{pmatrix}
& 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix}
\]

Can ask about deg. #s in a block.
Review

\( \lambda \vdash d \) means \( \lambda \) is a partition of \( d \).

\( M^{\lambda} = \text{Ind}_{\mathbb{Z}^{\lambda}}^{\mathbb{Z}} \) is permutation module, basis of \( \lambda \)-tableaux.

\( S^\lambda = \text{Specht module, basis } \{ e_\alpha \} \) is a SYT of shape \( \lambda \).

\[ \dim S^\lambda = \frac{d!}{n! n_1! n_2! \cdots} \]

Jucys-Murphy Elements:

\[ L_1 = 0, \quad L_2 = (2), \quad L_3 = (13) + (23), \ldots, \quad L_d = (1d) + (2d) + \cdots + (d-1)d \]

Facts

1. \( S^\lambda / S^\mu \) is nonzero, denote \( D^\lambda \leftrightarrow \lambda \) is p-regular.

2. \( \{ D^\lambda \} \) gives all \( \neq \) simple modules in char \( p \).

3. Over \( \mathbb{C} \) the JM elements act diagonally on a GZ basis of \( S^\lambda \).

Weights \( \leftrightarrow \) residue sequences.

EX: \( S^3 \)

Problem: When are \( D^\lambda \oplus D^\mu \) in same block?

2. Except \( p = 2, \) \( p \)-socle \( S^\lambda \) is indecomposable. So?

When are \( S^\lambda / S^\mu \) in same block?
Thm. Let $\mathcal{S}$ be a simple $G$-module. $\mathcal{S}$ is simple and forms its own block iff $p^\alpha \mid \dim \mathcal{S}$ where $16 = p^\alpha \mid p^r$. 

Proof. Can we detect $\mathcal{S}$?

Ex. $\mathcal{S}^2$ is in its own block $\iff$ no hook lengths are div. by $p$.

p-core and abaci

- Define rim p-hook, core between hooks $\leftrightarrow$ rim hooks.
- Define p-core

Cor. $\mathcal{S}^2$ is in its own block $\iff \lambda$ is a p-core

Def. $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_9 = 0$

\[ h_i = \frac{\lambda_i}{i+1} \]

Fact. \{hi\} determine $\lambda$.

\[ \lambda_e = h_e, \quad \lambda_{e-1} = h_{e-1} - 1, \quad \lambda_{e-2} = h_{e-2} - 2 \]

More generally given $\lambda_9 > \lambda_8 > \cdots > \lambda_1 = 0$

Get $\lambda_i = \lambda_i + i - r$ partition w/ some 0's

Call $B_{1,1}$ by a seq. of $B$'s.
Abacus
Renaissance numb.
Rim p-holes - replace by $b_i - P$

case p-one is unique

defn. weight