Lecture 19

Review

- B a <u>block</u> of kG means $kG \cong B \oplus \_$ as algebras and B is indecomposable.
- Any indec. kG-module is in a unique block
- $S \sim T$ if $\exists \begin{smallmatrix} S \\ T \end{smallmatrix}$ or $\begin{smallmatrix} T \\ S \end{smallmatrix}$ generates equiv relation of being in same block among simple modules.

- Think of kG as a left $G \times G$-module via $(g_1, g_2) \circ g = g_1 g g_2^{-1}$, the blocks are just indecomposable summands.

Thm As a $k[G \times G]$ module B has a vertex of the form $\delta(D)$ for some p-subgroup D.

Rmks D is unique up to conjugacy and is called the <u>defect group</u> of the block. If $|D| = p^d$ then d is the defect of the block.

Def The block containing the trivial module k is called the <u>principal block</u>.

Thm Let B be a block of kG with defect group D. Then any indecomposable kG module in B has vertex contained in D

Rmk Also true that D actually occurs as the vertex of some module in B. Thus B is semisimple iff it has defect zero.

Proof

Recall from Omnibus Lemma that $U \otimes V^G \cong (U_H \otimes V)^G$

Cor U is rel Q-projective $\Rightarrow$ so is any $U \otimes M$.

Now suppose U is in a block B, we must show U is rel D-projective for D the defect group

Consider B as a kG-module via conjugation $g \circ B = gBg^{-1}$.

Note ${}_G B \cong \mathrm{Res}^{G \times G}_{\delta(G)} B$ using obvious $\cong$ $G \cong \delta(G)$

Claim B is relatively D-projective.

Pf $B \cong \mathrm{Res}^{G \times G}_{\delta(G)} B$ but by def ${}_{G \times G}B$ is rel $\delta(D)$ proj so

$$B \mid \mathrm{Res}^{G \times G}_{\delta(G)} \mathrm{Ind}^{G \times G}_{\delta(D)} *$$

which, by Mackey is a $\oplus$ of modules induced from

$$\delta(G) \cap (g_1, g_2)\, \delta D\, (g_1, g_2)^{-1} \qquad (*)$$

But if $(g_1, g_2)(d, d)(g_1^{-1}, g_2^{-1}) \in \delta G$ then $g_1 d g_1^{-1} = g_2 d g_2^{-1}$

Thus $(g_1, g_2)(d, d)(g_1^{-1}, g_2^{-1}) = (g_1, g_1)(d, d)(g_1^{-1}, g_1^{-1}) \in \delta(g_1)\, \delta(D)\, \delta(g_1)^{-1}$

Thus all subgroups in $(*)$ are conjugate in δG to subgroups of $\delta(D)$ //

Since B is relatively D-projective then so is $B \otimes U$. Thus the following is sufficient

<u>Claim</u> $U \mid B \otimes U$

Pf Define $\pi : B \otimes U \to U$ by $\pi(B \otimes u) = Bu$

- Check balanced map
- $\pi(g \circ (B \otimes u)) = \pi(gBg^{-1} \otimes gu) = gBg^{-1}gu = gBu = g\,\pi(B \otimes u)$

Since U is in the block, $\pi(e_B \otimes u) = e_B u = u$ so π is onto

<u>Define</u> $i : U \to B \otimes U$ by $i(u) = e_B \otimes u$, clearly $\pi \circ i = id_U$

$i(gu) = e_B \otimes gu$

$g(i(u)) = g(e_B \otimes u) = g e_B g^{-1} \otimes gu = e_B \otimes gu$ since e_B is central.

Thus $B \otimes U \cong U \oplus \_\_$ //

<u>Thm</u> Suppose B is a block w/ defect group D. Then

1. Let $D \leq P \in Syl_p(G)$. Then $\exists c \in C_G(D)$ with $D = P \cap cPc^{-1}$
2. D contains every normal p-subgroup of G
3. D is largest normal p-subgroup of $N_G(D)$

Example

Claim: The principal block B_0 has Sylow subgroups of G as vertex

Proof: ETS the vertex of the trivial module is $P \in Syl_p(G)$, we know the source is trivial. Suppose vertex is $Q < P$. Then

$$k_P \mid \mathrm{Ind}_Q^P k. \quad \text{But}$$

$$\mathrm{Hom}_Q(k_Q, \mathrm{Ind}_Q^P k) \cong \mathrm{Hom}_Q(k_Q, k_Q) \cong k$$ Thus $\mathrm{Ind}_Q^P k$ is indec. //

The principal block is in many ways the most complicated and most important.

Brauer Correspondence

Suppose $H \le G$ and B is a block of G and b a block of H.

Def If $\mathrm{Res}^{G \times G}_{H \times H} B \cong b \oplus \_\_$ for a unique B, write $B = b^G$.

Say B corresponds to b. Given b, say b^G is defined if such a B exists

Lemma Let b as above w/ defect D.

1. If b^G is defined, then $D \le$ some def gp of b^G
2. If $H \le K \le G$ and $b^K, (b^K)^G$ and b^G all def then $b^G = (b^K)^G$.
3. If $C_G(D) \le H$ then b^G is defined (For example $H = C_G(D)$ or $N_G(D)$)

Brauer's 1st Main Thm

Suppose $D \le N_G(D) \le H \le G$, D a p-subgroup. Then

$b \to b^G$ gives a 1-1 correspondence between blocks of kG with

defect D and blocks of kH with defect G.

Say B & b are Brauer correspondents.

Rmk 1. Respects Green corr