

Lecture 18

Recall A an algebra, write $A = A_1 \oplus \dots \oplus A_s$ a direct sum of 2-sided ideals, each of which is indecomposable as an algebra. The A_i are called block algebras of A .

Properties

- $1 = e_1 + \dots + e_s$ where $\{e_i\}$ are primitive central idempotents.
 - e_i is an identity in A_i
 - $A_i = Ae_i = e_i A$, $A_i A_j = 0$ $i \neq j$.
- Exercise: If $A = B_1 \oplus C$ check $B = B_1 A_1 \oplus \dots \oplus B_1 A_s$ so the decomposition into blocks is unique

Prop Let M be an indecomposable A -module. Then

$$M \cong e_1 M \oplus e_2 M \oplus \dots \oplus e_s M.$$

In particular if M is indecomposable then $e_i M \neq 0$ for a unique i . Say M is in the block A_i .

Prop Any A -module M has a unique decomposition $M = M_1 \oplus \dots \oplus M_s$ where $M_i, i \neq 0$, lies in the block A_i .

- Suppose M lies in the block A_i , this is iff e_i acts as identity on M . Thus every submodule and quotient module of M lies in same block, moreover $\text{Hom}_A(M, N) = 0$ if M, N in diff. blocks.
- Think of a block as a basket that indec modules are put in.

Remarks

1. If S, T are simple and \exists nonsplit $\frac{S}{I}$ or $\frac{T}{J}$ then S, T are in the same block. Moreover if S, T are comp factors of any indec, then they are in same block
2. If $A = P_1 \oplus P_2 \oplus \dots \oplus P_t$ then a block A_i is just the direct sum of all the P_i 's contained in that block.

Thm Let S, T be simple A -modules. TFAE:

1. S, T are in same block
2. \exists a chain of simple modules $S = S_1, S_2, \dots, S_n = T$ such that S_i, S_{i+1} are comp factors in the same PIM.
3. \exists a chain $S = T_1, T_2, \dots, T_n = T$ of simple modules so for each T_i, T_{i+1} there is a nonsplit extension of one by the other.

Proof Recall that given $\frac{T_i}{T_{i+1}}$ that $\exists P(T_i) \rightarrow \frac{T_i}{T_{i+1}} \rightarrow 0$. Thus $3 \rightarrow 2$.
 Also $2 \rightarrow 1$ is clear from remark 1.

1 \rightarrow 2 Suppose S, T are in same block B . Write

$$B = P_1 \oplus \dots \oplus P_s \oplus Q_1 \oplus \dots \oplus Q_t$$

where all the comp factors in P_i 's are related to S by 2 and all in the Q are not. Thus $\{P_i\}$ have no composition factors in common with $\{Q_i\}$ so all maps in $\text{End}(A)$ preserve this decomp, i.e. $\bigoplus P_i$ is an ideal \neq

Thus all the simples in the block are related as in 2.

2-73 Let P be a PIM, it is enough to show any 2 comp factors in P are related as in 3, so enough to show all are related to top.

$P = \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix}$ radical layers.

ETS given a module w/ 2 radical layers, every module in bottom is related to some in top

WLOG $S_1 S_2 \dots S_m = U \quad \text{rad } U = T, \text{ is there an } \frac{S_i}{T} ?$

By correspondence Thm, for each $i \exists$ a submodule $W_i \subseteq U$ with

$U/W_i \cong \bigoplus_{K \neq i} S_K \quad \text{and} \quad U = \sum W_i.$

Thus $\text{rad } U = \text{rad } A U = \sum \text{rad } A W_i$, so some W_i has nontrivial radical, i.e. some $W_i = \frac{S_i}{T} \quad //$

Defect Groups

For each block B we will associate a p -subgroup $D \leq G$ controlling in many respects the rep theory of the block.

Examples

- 1. CG each block has one simple module
- 2. $K\Sigma_3 \quad p=3 \quad K\Sigma_3 \cong \begin{matrix} R & \\ \text{sm} & \\ R & \end{matrix} \oplus \begin{matrix} \text{sm} \\ R \\ \text{sm} \end{matrix}$ only one block
- 3. Simple module is projective \leftrightarrow ~~unique simple~~ Block is a matrix algebra
- 4. $SL_2(p)$ 3 blocks (p 48-50)

Notation $\delta: G \rightarrow G \times G \quad g \mapsto (g, g)$ diagonal homom.

Remark KG is a ^{left} $G \times G$ module via $(g_1, g_2) \circ g = g_1 g g_2^{-1}$.

* Thought of as a left $G \times G$ module, the indecomposable summands of KG are the blocks!

Thm Let B be a block. As a $k[G \times G]$ module B has a vertex of the form $\delta(D)$ for some p -subgroup D .

Def D is called a defect group of the block if $|D| = p^d$, d is the defect of the block.

Remark δH and δK are conjugate in $G \times G \iff H, K$ are conjugate.
Thus D is well-defined up to conjugacy.

Proof

Key idea is $KG \cong \text{Ind}_{\delta G}^{G \times G} K$, cosets of δG are of form $(g, e) \cdot \delta G$.

Thus KG is relatively δG -projective $G \times G$ -module.
Thus any summand is as well. //

Thm Let B be a block of KG with defect group D . Then any indec. KG -module in B has vertex contained in D .

Remark Also true that D actually occurs as vertex of some module in B . Thus B is semisimple iff it has defect zero.

Proof

Recall from the omnibus lemma that $U \otimes V^G \cong (U_H \otimes V)^G$

Cor U is rel \mathcal{O} -proj \Rightarrow so is any $U \otimes M$.

Suppose U is in block B . We consider B as a KG -module via conjugation:

$$g \cdot B = gBg^{-1} \quad \forall B \in \mathcal{B}$$

Note $B \cong \text{Res}_{SG}^{G \times G} B$ if we use obvious $\cong G \cong SG$.

Define $\pi: B \otimes U \rightarrow U$ by $\pi(B \otimes u) = Bu$.

• balanced

$$\bullet \quad g(B \otimes u) = gBg^{-1} \otimes gu \xrightarrow{\pi} gBg^{-1}gu = gBu = g\pi(B \otimes u)$$

Thus π is a G -module map

Define $i: U \rightarrow B \otimes U \quad u \mapsto e_B \otimes u$ e_B is the block idempotent acting as identity on modules in B

$$g(iu) = g(e_B \otimes u) = g e_B g^{-1} \otimes gu = e_B \otimes gu \quad \text{since } e_B \text{ central}$$

$$i(gu) = e_B \otimes gu$$

Finally check that $\pi(iu) = \pi(e_B \otimes u) = e_B u = u$.

Thus $B \otimes U \cong U \oplus \dots$ so ETS $B \otimes U$ is rel \mathcal{D} -projective
so ETS B is rel \mathcal{D} -projective.

Claim B is rel D projective

pf Equivalently $\text{Res}_{SG}^{G \times G} B$ is rel SD proj.

Now by def the $G \times G$ module B is rel SD proj. Thus

B is induced from some $k[SD]$ module.

$\text{Res}_{SG}^{G \times G} \text{Ind}_{SD}^{G \times G}$ — by Mackey is a \oplus of modules induced from

$$\rightarrow S(G) \cap (g_1, g_2) SD (g_1, g_2)^{-1}$$

Suppose $(g_1, g_2)(d, d)(g_1, g_2)^{-1} \in SG$. Then
 $g_1 d g_1^{-1} = g_2 d g_2^{-1}$

$$\text{so } (g_1, g_2)(d, d)(g_1, g_2)^{-1} = (g_1, g_1)(d, d)(g_1, g_1)^{-1} \\ \in S(g_1) S(D) S(g_1)^{-1}$$

So all these are conjugate in $S(G)$ to subgroups of $S(D)$ //

Thm Suppose B is a block of defect group D

1. Let $D \leq P \in \text{Syl}_p(G)$. Then $\exists c \in C_G(D)$ with $D = \bigcap_{P^c} P^c$
2. D contains every normal p -subgroup of G .
3. $D =$ largest normal p -subgroup of $N_G(D)$.

Rmk $1 \rightarrow 2, 3$