Recall $P \in \text{Sub}(G)$. Suppose $gPNP = gP \cap NP = \{ e \} \quad \forall g \in G$.

Let $L = N_G(P)$. Then

Then Induction + Restriction induce a 1-1 corresp. map, namely,

RG and RL modules

$V_L = V \oplus \text{Pro} \quad V^c = U \oplus \text{Pro}$

Moreover, induction + restriction induce a stable equivalence.

Green Correspondence - Generalizes above

Notation Fix a $p$-subgroup $Q$ and choose $N_G(Q) \leq H \leq G$.

\[ \chi = \{ x \leq G \mid x \leq Q \land gQg^{-1} \text{ for some } g \in G, g \notin H \} \]

\[ \gamma = \{ y \leq G \mid y \leq H \land gQg^{-1} \text{ for some } g \in G, g \notin H \} \]

Note that $\chi \leq \gamma$ and $Q \not\leq \chi$.

Note also each subgroup in $\chi$ is proper in $Q$. 

Lemma
Assume: $M$ is an indec $RH$-module that is rel $Q$-projective.

1. $M_H^c = M \oplus M'$ where each summand in $M'$ is pro relative to a subgroup in $X$.

2. Write $M = V \oplus V'$ with $V$ index and $M/VH$. Then every summand of $V'$ is projective relative to a subgroup in $X$.

Proof. Choose $U$ indec $Q$-module with $U^c = M \oplus M_0$.

By Mackey $(U^c)_H = U^c_H \oplus U'$ where $U'$ is rel $Y$-projective.

Combine \[ (M^c)_H \oplus (M_0^c)_H = M \oplus M_0 \oplus U' \]

Cancel $M, M_0$ to get\[ (M^c)_H = \mathbb{M} \oplus M' \] with $M'/U'$ as desired.

2. Let $M^c = V \oplus V'$ with $V$ index, $M/VH$.

Choose $V_i$ a summand of $V'$ it is rel $Q$-projective since $V$ is.

Choose $Q_i \leq Q$ a vertex and a source $S_i$. So

2. Let $M_H^c = V \oplus V'$ with $V$ index, $M/VH$.

Choose $V_i$ a summand of $V'$ it is rel $Q$-projective since $V$ is.

Choose $Q_i \leq Q$ a vertex and a source $S_i$. So

$S_i \mid V_i \downarrow_{Q_i}$. Choose $M, (V_i)_H$ so $S_i \mid M_{(V_i)_H}$

Now $M$ is rel $Q$-projective

Now $M_1 \downarrow (V_i)_H \downarrow (M^c)_H$ so $M_1$ is rel $Y$-proj by 2.

i.e. has vertex $H/\gamma \gamma_Q^{-1}$

Now $S_i \mid M_{(V_i)_H}$ so $S_i \mid (\_\_\_\_H^c)^H$ $\Rightarrow S_i$ is rel $X$ proj.

This $V_i$ is rel $X$ proj.
Then

(Green Correspondence)

Suppose \( Q \leq G \) is a p-subgroup and \( Q \trianglelefteq N_G(Q) \leq H \leq G \). Then \( \exists \) a 1-1 correspondence between indecomposable \( kH \)-modules with vertex \( Q \) and indecomposable \( kH \)-modules with vertex \( Q \), given as follows:

1. \( V \) indec \( kQ \)-module w/vertex \( Q \) then \( V^H \) has unique summand \( f(V) \) w/vertex \( Q \). Remaining summands have vertex in \( H \).

2. \( M \) indec \( kH \)-module w/vertex \( Q \) then \( M^Q \) has a unique summand \( g(Q) \) with vertex \( Q \), and other summands have vertices in \( H \).

3. \( f(g(Q)) = M \), \( f(f(V)) = V \)

4. Correspondence preserves being trivial source.

Remark In the TI case \( X = \{ e \} \) by definition, so \( M^Q = g(Q) \oplus \mathbb{Q} e \).

Suppose \( V \) is not just \( \{ e \} \). So we have a TI Sylow \( P \) and

\[ e \neq gP^{-1}NH, \ g \in H. \]

Then \( gP^{-1}NH \) is a \( P \)-subgroup of \( H \), hence conjugate into \( P \).

Thus \( \exists x \in H \) with

\[ x(gP^{-1}NH)x^{-1} \leq P \]

\[ xgP \neq g^{-1}NH \leq P \Rightarrow xgP \neq (xgP) \leq P \]

\[ \Rightarrow xg \notin N_G(P) \leq H \]

Thus \( V = \{ e \} \) and \( V^H = \{ \mathbb{Q} e \} \Rightarrow g \notin H \ast f(V) \oplus \mathbb{Q} e \).
1. Given \( V \) in \( kG \)-mod w/ vertex \( Q \), source \( S \) so \( V \leq S \). Let \( S' = M \cap M' \) where \( M \) is indec and \( V \leq M \).

   By Lemma part 1, \( (M \leq M')_H = M \oplus \text{rel}_Y \). Want \( V_H = M \cap Y \).

Now \( V \leq (V_H)_H \) so \( V_H \) has a summand with vertex \( Q \). But \( V \leq M \)

and \( Q \neq Y \) so \( M \mid V_H \) and \( M \) has vertex \( Q \), other summands rel \( Y \).

Let \( f(V) = M \).

2. Suppose \( M \) is indec kG-module w/ vertex \( Q \). Always \( M \mid (M')_H \) so choose \( V \) indec, \( V \leq M \) so \( M \mid V_H \).

   By Lemma part 2, \( M' = V \oplus \text{rel}_Y \), so choose \( g(M) = V \).

3. Start with \( V \in kG \)-mod, with source \( S \).

   \( G \rightarrow V \) \quad \text{\( M \) is rel \( Y \) pro} \)

   \( H \rightarrow S \leq M \) \quad \text{This \( V \leq M \), \( V \leq M' \).

   If start with \( M \), \( V \leq M \).

   Choose some \( V \).

   Source is constant.
Green Correspondence Remarks

1. It is ubiquitous in representation theory.

2. There are results comparing $\text{Hom}_k \left( V, V_1 \right)$ and $\text{Hom}_k \left( \left( V \right), \left( V_1 \right) \right)$
   but not directly, i.e. not cut by maps factoring
   though relatively IT-projective modules.

3. Then Suppose $U$ indec $kG$ module w/ vertex $Q$ and $M$ the
   corresponding $kH$-module.

   1. For a $kG$-module $W$, $U \cap W \iff M \cap W$.

   2. $BV(G) \iff U$ indec and $M \cap W \iff W \equiv U$.

   i.e. 2 is a sort of converse, if you "care" the Green car
   then you are the Green car.