Lecture 16

Review: Def $H \leq G$. A kG-module U is relatively H projective if $U | V^G$ for some H-module V. In this case we always have $U | (U_H)^G$.

Thm Given U indec $\exists$ a p-subgroup Q and an indec. Q-module S such that

1. $U | S^G$; $S | U_Q$
2. If U is rel H-projective then $gQg^{-1} \leq H$ for some $g \in G$
3. If S' is indec Q-module and $U | S'^G$ then $S' \cong g(S)$ some $g \in N_G(Q)$

Rmk For $g \in N_G(Q)$ then $q \to gqg^{-1} \in \mathrm{Aut}\,Q$, this is an example of twisting a module by a group automorphism

Def Q is a vertex of U, S is a source

Properties of Vertices & Sources

Lemma Let U be an indec kG-module w/ vertex Q, $Q \leq H$. Then $\exists$ an indecomposable kH module V satisfying any 2 of:

1. $V | U_H$ 2. $U | V^G$ 3. V has vertex Q

Rmk Eventually can get all 3 at once

Proof $1 \Rightarrow 2$

Since U is rel H proj, $U | (U_H)^G$ so choose summand $V | U_H$ with $U | V^G$ //

$2 \Rightarrow 3$ Let S be a source, so $U | S^G = (S^H)^G$. Chose $V | S^H$ so $U | V^G$.
Need V to have vertex Q.

Since $V | S^H$, V is rel Q projective. Suppose a vertex $R \lneq Q$. Then
$\exists$ W an R-module so $V | W^H \Rightarrow U | W^G \Rightarrow$ vertex of $U \leq W$ $\ast$.
Thus $R = Q$

$$\begin{array}{c} G \\ | \\ H \\ | \\ Q \end{array}$$

$1 \Rightarrow 3$
As above, our source S can be chosen so $S | U_Q$. Write

$$U_H = \oplus\_\_\_ \quad \text{and chose a summand } V \text{ with } \boxed{S | V_Q}$$
$$\text{so } \boxed{V | U_H}$$

Claim V has vertex Q

Pf $V | U_H$ so $V | (S^G)_H$ so $V | \left(s(S)_{H \cap sQs^{-1}}\right)^H$ some s

Thus V has vertex $R \leq H \cap sQs^{-1}$. ETS R & Q conjugate in H.

Now $V | W_R^{\uparrow H}$ and $S | V_Q$ so $S | (W^H)_Q$. Thus

S is relatively $Q \cap hRh^{-1}$ for some $h \in H$. But S has vertex Q
so $Q \cap hRh^{-1} = Q \Longrightarrow Q \leq hRh^{-1}$.

But $R \leq sQs^{-1}$. Thus $|R| = |Q|$ and $Q = hRh^{-1}$. //

Trivial Intersections and the Stable Category

Assume Sylow subgroup P is trivial intersection, i.e. $P \cap gPg^{-1}$ is always P or 1. (ex: $|P| = p$)

Let $L = N_G(P)$

Thm $\exists$ a 1-1 correspondence between $\cong$ classes of nonprojective indecomposable kG & kL modules, such that if $U \in \operatorname{mod} kG$ corresponds to $V \in \operatorname{mod} kL$ then

$$U_L \cong V \oplus \text{PROJ}$$
$$V^G \cong U \oplus \text{PROJ}.$$

Proof Apply Mackey to an indec V.

$$(V^G)_L \cong \bigoplus_{s \in L\backslash G/L} \left(s(V)_{L \cap sLs^{-1}}\right)^L. \quad \text{Note } P \trianglelefteq L$$

If $s \notin L$ then P & sPs^{-1} are unique Sylow's of L and sLs^{-1} so $P \cap sPs^{-1}$ is Sylow of $L \cap sLs^{-1}$ so $|L \cap sLs^{-1}|$ is coprime to p

Thus $\quad (V^G)_L \cong V \oplus \text{Projective}$

Write $\quad V^G \cong U_1 \oplus U_2 \oplus \dots \oplus U_n$. Since Sylow $\leq L$, then WLOG U_1 is not projective, $U_2 \to U_n$ are projective, So

$$\boxed{V^G \cong U \oplus \text{Proj} \quad \text{and} \quad U|_L \cong V \oplus \text{Proj}}$$

So we have a bijection, does every U arise

For $U \in kG$-mod, U is rel L-projective. (not proj) So $U \mid V^G$ for some nonprojective kL module V. //

COR Let U_1, U_2 nonproj indec kG-modules; V_1, V_2 corr kL modules.

There exists non-split $0 \to U_1 \to U \to U_2 \to 0$ if & only if
There exists nonsplit $0 \to V_1 \to V \to V_2 \to 0$.

Pf Tedious and not enlightening.

<u>Stable Maps</u>

Def Let U_1, U_2 be kG-modules and $f: U_1 \to U_2$ a module homomorphism.
Say f <u>factors through a projective</u> if $\exists$ a projective module P such that
+ maps ψ, ρ

$$\begin{tikzcd} & P \arrow[dr, "\rho"] & \\ U_1 \arrow[ur, "\psi"] \arrow[rr, "f"] & & U_2 \end{tikzcd}$$

<u>Check</u> The set of such f is a subspace of $\mathrm{Hom}_{kG}(U_1, U_2)$.

Def $$\overline{\mathrm{Hom}}_{kG}(U_1, U_2) = \mathrm{Hom}_{kG}(U_1, U_2) \big/ \text{subspace factoring through a proj.}$$

Thm Suppose U_1, U_2 are nonproj indec kG modules corr to V_1, V_2 kL modules.
Then
$$\overline{\mathrm{Hom}}_{kG}(U_1, U_2) \cong \overline{\mathrm{Hom}}_{kL}(V_1, V_2)$$

<u>Proof</u> ETS $\overline{Hom}_{kG}(V_1^G, U_2) \cong \overline{Hom}_{kL}(V_1, U_{2L})$.

But $Hom_{kG}(V^G, U) \cong Hom_{kL}(V, U_L)$ so ETS this $\cong$

preserves property of factoring through a projective.

So suppose $v \in Hom_{kL}(V, U_L)$ and $\hat{v} \in Hom_{kG}(V^G, U)$

$V^G \cong V \oplus gV \oplus \ldots$

Then $\hat{v}$ extends v. So if $\hat{v}$ factors through proj P
then v factors through P_L, also proj.

Suppose
$$\begin{tikzcd} & Q \arrow[dr, "\beta"] & \\ V \arrow[ur, "\alpha"] \arrow[rr, "v"] & & U_L \end{tikzcd}$$
Q proj kL module

Check
$$\begin{tikzcd} & Q^G \arrow[dr, "\beta^G"] & \\ V^G \arrow[ur, "\alpha^G"] & & U \end{tikzcd}$$
//

<u>Example</u> $G = SL(2,p)$ $P = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\}$ $L = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid ac = 1\}$

- <u>Define</u> stable module category stmod kG.
  - Schanuel's Lemma
  - stable equivalence kG-mod & kL-mod in TI case