Review: Def. $H \leq G$. A $kG$-module $M$ is relatively $H$-projective if $M/V$ is $H$-injective for some $H$-module $V$. In this case we always have $M/\text{Im}(f)$. 

Given $M \cong kQ$ and an indec. $G$-module $S$, such that:
1. $M/S \cong S/\ker f$
2. If $M$ is rel. $H$-projective then $Q^*H \leq H$ for some $Q^*H$
3. If $S$ is indec. $G$-module and $M/S$ thin then $S \cong Q(S)$ for some $Q \in \text{Pic}(Q)$.

Remark. For $g \in \text{Pic}(Q)$ then $Q \cong Q^g \cong \text{Aut}(Q)$, thus this is an example of twisting a module by a group automorphism.

Def. $Q$ is a vertex of $M$, $S$ is a source.

Properties of Vertices & Sources:

Lemma. Let $M$ be an indec. $kG$-module with vertex $Q$, $Q \leq H$. Then there exists an indecomposable $kH$-module $V$ satisfying any 2 of:
1. $V/\text{Im}(f)$
2. $V/V^a$
3. $V$ has vertex $Q$

Remark. Eventually can get all 3 at once.
Let $S$ be a source, so $U(S) = (S^u)^e$. Choose $V(S^u)$ so $U(V) = U(S)$.

Need $V$ to have vertex $Q$.

Since $V(S^u)$, $V$ is rel $Q$-projective. Suppose a vertex $R \neq Q$. Then

$\\exists W$ an $R$-module so $V(W) \Rightarrow U(W) \Rightarrow \text{vertex at } Y \in W$ $\neq$.

Thus $R = Q$.

Claim: $V$ has vertex $Q$

Proof: $V(U)$ so $V(S(S^u))$ so $V(S(S^u(U)))$ some $S$

This $V$ has vertex $R = H(SQ^{-1})$, $ETS R \neq Q$ is not in $U$.

Now $V(W)$, and $S(Va)$ so $S(W)$, $Q$.

This $S$ is relatively $Q \cap H^{-1}$ for some $H \not= H$. But $S$ has vertex $Q$ so $Q \cap H^{-1} = Q \Rightarrow Q \subseteq H^{-1}$.

But $R \leq SQ^{-1}$. Thus $|R| = |Q|$ and $Q = H^{-1}$. //
Trivial Intersections and the Stable Category

Assume $Syl_p$ subgroup $P$ is trivial, i.e. $PN_P^{-1}$ is always $P$ or $1$. (ex: $1P = p$)

Let $L = N_G(P)$.

Then a 1-1 correspondence between $\equiv$ classes of nonprojective indecomposable $RG \otimes K_L$ modules, such that if $U \equiv mod_R K_L$ corresponds to $V \equiv mod_K L$ then

$$U_L \equiv V \oplus \text{proj}$$

$$V^c \equiv U \oplus \text{proj}$$

Proof: Apply Mackey to an indec $V$.

$$(V^c)_L \equiv \bigoplus_{s \in \Delta L} (s(V)_L)_{sLs^{-1}}$$

Note PSL

It $s \notin L$ then $P = sP^{-1}$ are unique Sylow's of $L$ and $sLs^{-1}$ so $PN_P$ is Sylow of $N_L(sLs^{-1})$ so $|LsL^{-1}|$ is coprime to $p$.

Thus $$(V^c)_L \equiv V \oplus \text{Projective}$$

Write $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$. Since Sylow $S_L$, then $U_1 \cdots U_n$.

$U_1$ is not projective, $U_2 \cdots U_n$ are projective. So

$$V^c \equiv U \oplus \text{proj} \quad \text{and} \quad U_L \equiv V \oplus \text{proj}$$

So we have a bijection, does every $U$ arise
For $U \in kG$-mod, $U$ is rel $L$-projective so $U/V$ for some nonprojective $kL$-module $V$. \\

**Cor** Let $U_1, U_2$ nonproj indec $kG$-modules, $V_1, V_2$ car $kL$-modules.

There exists non-split $0 \to U_1 \to U \to U_2 \to 0$ if and only if

There exists non-split $0 \to V_1 \to V \to V_2 \to 0$.

It tedious and not enlightening.

**Stable Maps**

**Def** Let $U_1, U_2$ be $kG$-modules and $f: U_1 \to U_2$ a module homomorphism. Say $f$ factors through a projective $\tilde{f}$ if $\tilde{f}$ is a projective module $P$ such that $f \to f \to \pi P \quad P \quad \pi P \quad U_1 \quad f \to U_2$.

Check: The set of such $f$ is a subspace of $\text{Hom}_{kG}(U_1, U_2)$.

**Def** $\frac{\text{Hom}_{kG}(U_1, U_2)}{\text{subspace factors through a proj.}}$

**Thm** Suppose $U_1, U_2$ are nonproj indec $kG$-modules corr to $V_1, V_2$ $kL$-modules. Then $\frac{\text{Hom}_{kG}(U_1, U_2)}{kG} \cong \frac{\text{Hom}_{kL}(V_1, V_2)}{kL}$.
Proof:

$ETS \overline{\text{Hom}}_{\text{rel}}(V_i, V_j) = \overline{\text{Hom}}_{\text{rel}}(V_i, V_j)$.

But $\text{Hom}_{\text{rel}}(V_i, V_j) = \text{Hom}_{\text{rel}}(V_i, V_j)$, so $ETS$ this preserves property of factoring through a projective.

So suppose $v \in \text{Hom}_{\text{rel}}(V_i, V_j)$ and $a \in \text{Hom}_{\text{rel}}(V_i, V_j)$

$v^a = v \oplus g_0 v \oplus \ldots$

Then $v$ extends $V$. So if $v$ factors through $\text{proj } P$

then $v$ factors through $P$, also proj.

Suppose

$\begin{array}{c}
\downarrow \\
U_L
\end{array}$

$\begin{array}{c}
V \\
\downarrow v \\
\oplus a
\end{array}$

$\begin{array}{c}
Q
\end{array}$

A proj kL module.

Check $v^a = Q^a \oplus a u$.

Example: $G = SL(2, p)$, $P = \{ (1, 1) \}$, $L = \{ (a, b) | ab = 1 \}$

Define stable module category $\text{stmod } RG$.

- Schur's Lemma

- Stable equivalence $RG$-mod $\cong kL$-mod in TI case.