Review

1. $H \leq G$, $U$ an $H$-module. Then $\text{Ind}_H^G U = U^G \cong \bigoplus_{g \in G/H} U_g$

2. $(U^G)^\circ = U^G$

3. $(U \oplus V)^\circ = U^G \oplus V^G$

Mackey Theorem Suppose $M$ is a $K$-module. Then

$$(M^G)^\circ_H = \bigoplus_{H \leq K} (g(M) \downarrow_{H \leq K})^\circ$$

Remark $g(M)$ is $g \circ M$ is naturally a $gK$-module, a.k.a. $M_g$.

Aperin calls this "Transport of structure."

Prop Let $H \leq G$, $U$ a $K$-module. Then

1. $U$ is a direct summand of a module induced from $H$ (called relatively $H$-free).
2. Given $V \rightarrow^\psi U \rightarrow^\iota 0$, if $\psi$ splits over $H$ then it splits over $G$.
3. Given $V \rightarrow^\psi W \rightarrow^\iota 0$ suppose $\exists \psi'$ a $K$-map from $V \rightarrow W$ such that $\psi = \psi' \circ \iota$. Then $\exists \psi'$ a $K$-map.

4. $(U^G)^\circ = U \oplus -$ (henceforth write $U / (U^G)^\circ$)

Ref Such modules are called relatively $H$-projective.

Remark

1. Projective = "relatively free projective."
2. Equivalent notion of relatively $H$-injective.
Proof 1 $\leftrightarrow$ 2 $\leftrightarrow$ 3 copy similar result for projective modules.

4 $\rightarrow$ 1 trivial.

Claim 2 $\rightarrow$ 4

There is a natural $kG$-module map $U_{i}^{G} \to U$ given by

$g \cdot u \to gu$

By 2 this splits over $H$, namely $u \mapsto 1 \otimes u \in U$

By $\bullet 2$ it splits over $G$ so $(U_{i}^{G}) = U \otimes -$.

Then, let $P \in \text{Syl}_{p}(G)$ and $P \leq H \leq G$. Then every $kG$-module is relatively $H$-projective.

Proof: Arranging again, we'll prove $\bullet 2$, so suppose

$\xrightarrow{u} U \to 0$

Define

$\tilde{s}(u) = \frac{1}{[G:H]} \sum_{g \in G/H} s(g^{-1}u)$

and check.

Corollary P $\leq$ H $\leq$ G as above, $U \in kG$-mod. Then $U$ is projective iff $U_{i}$ is projective. In particular we can test projectivity on a Sylow.

Proof: $U$ proj $\to$ $U_{i}$ proj already done. Suppose $U$ is proj. By then $U / (U_{i})^{G}$ and $(U_{i})^{G}$ is projective by Corollary. Lemma. \hfill
Let \( U \) be an indecomposable \( kG \)-module.

1. If a \( p \)-subgroup \( Q \leq G \), unique up to conjugacy, such that \( U \) is relatively \( H \)-projective, then \( gQg^{-1} \leq H \) for some \( g \in G \).

2. If an indecomposable \( kG \)-module \( S \) unique up to conjugacy in \( N_G(Q) \), such that \( U/S \).

**Def.** \( Q \) is called a vertex of \( U \), \( S \) is a source of \( U \).

**Rmk.**
1. Smaller vertex \( \approx \) closer to projective
2. Suppose \( U/S = \bigoplus g \otimes S \). Check that \( U/\text{Ind}_{gQg^{-1}}^G g(s) \), hence the conjugacy in \( Q \vdash S \).

**Proof.**
We know \( U \) is rel. Pro. Choose \( Q \) of minimal order so \( U \) is rel. \( Q \)-projective, thus \( U/(U_Q)^c \). The \( S \) some indecomposable summand \( S/U_Q \) such that \( U/S \). Need "uniqueness" of \( Q, S \).

- If \( Q \leq H \) then \( U/(S^+)^c \) so \( U \) is rel. \( H \)-proj.
- As above, \( U \) is also then rel. \( gHg^{-1} \)-projective.

Suppose \( H \leq G \) and \( U \) is rel. \( H \)-proj, so \( U/V \), \( V \) an indec. \( H \)-module.
$S|u_q$ and $u|v^c$ so $S(v^c) < Q$. By Mackey:

\[(v^c)_Q \cong \bigoplus_{s \in Q/\text{Qsh}_c} (s(v)_{Q/\text{Qsh}_c})^Q\]

so $S(s(v)_{Q/\text{Qsh}_c})^Q$, for some $s$.

However if $Q/\text{Qsh}_c < Q$ this contradicts minimality of $|Q|$.

This $Q \leq \text{Qsh}_c$ as desired.

In the case where $H = Q$, then $Q/\text{Qsh}_c = Q \implies s \in N_G(Q)$. //

Remark

1. Finding vertices and sources of modules is difficult and an active area of research. For instance ∼10 pages in last 5 years, for $D_5$, $S_5^m$, only very special cases.

2. A module is trivial source if $u|v^c$, i.e. $u$ is a direct summand of a permutation module. These are very interesting modules!

3. Sample Thm: (Pur) $G$-p-solvable, then the source is an endopermutation module.
**Properties of Vertices & Sources**

**Lemma**

Let $U$ be an indec $K$-module with vertex $Q$ and $Q \leq H$. Then there exists an indec $K$-module $V$ satisfying any two of:

1. $V/Utt$,
2. $u/V^c$,
3. $V$ has vertex $Q$.

Eventually we find a $V$ for all $3$.

**Proof**

1.2 Easy! $U/(u^c)$, so choose $V/Utt$ so $u/V^c$.

2.3 Let $S$ be a source of $U$ so $U/S^c = (S^c)^c$. Choose a summand $V/S^c$ so $u/V^c$. Need $V$ to also have vertex $Q$.

Since $V/S^c$, $V$ is relatively $Q$ projective. Choose a vertex $R \neq Q$.

Then $V/\text{Ind}_R^W \Rightarrow U/W^c \Rightarrow U$ is rel $R$ proj. Thus $R \neq Q$.

1.3 We have source $S$ with $S/Utt$ and $U/S^c$.

Write $Utt = \Theta$ and choose a summand $V/Utt$ with $S/V^c$.

**Claim** $V$ has vertex $Q$.

By $1/4$,

$V/Utt$ so $V/(S^c)^c$ so $V/\text{some } (S(S)_{H\leq Q})^c$, by Mackey.

Thus $V$ has a vertex $R \leq H/A$ so $ETS$ $R$ and $Q$ are conjugate in $H$. 
Now \( V \) is a module induced \( r^H \) and \( S/V_q \).

So again by Mackey \( S \) is relatively \( \mathfrak{A} \mathfrak{n}rH^{-1} \)
projective for some \( h \in H \).

But \( S \) has vertex \( \mathfrak{A} \) so \( \mathfrak{A} H r H^{-1} = \mathfrak{A} \Rightarrow \mathfrak{A} s h r H^{-1} \)
so

But \( R = s q s^{-1} \) so \( |R| = |q| \) and \( \mathfrak{A} = h r H^{-1} \).