Lecture 15

Review 1. $H \leq G$, U an H-module. Then $\mathrm{Ind}_H^G U = U^G \cong kG \otimes_{kH} U = \bigoplus_{g \in G/H} g \otimes U$

2. $(U^H)^G \cong U^G$

3. $(U \oplus V)^G \cong U^G \oplus V^G$

Mackey's Thm Suppose M is a K-module. Then

$$(M^G)_H \cong \bigoplus_{HgK} \left(g(M)\downarrow_{H \cap gKg^{-1}}\right)^H$$

Rmk $g(M)$ is $g \otimes M$ is naturally a gKg^{-1} module, a.k.a. M^g.
Alperin calls this "Transport of structure".

Prop Let $H \leq G$, U a kH-module. TFAE

1. U is a direct summand of a module induced from H (called relatively H free)
2. Given $V \xrightarrow{\varphi} U \to 0$, if φ splits over H, then it splits over G.
3. Given
$$\begin{array}{ccc} & U & \\ & \downarrow \psi & \\ V & \xrightarrow{\varphi} W & \to 0 \end{array}$$
Suppose $\exists\ \rho$ a kH-map from $U \to V$ such that $\psi = \varphi \circ \rho$. Then $\exists\ \tilde{\rho}$ a kG-map.
4. $(U_H)^G \cong U \oplus$ ___ (henceforth write $U \mid (U_H)^G$)

Def Such modules are called relatively H-projective.

Rmk 1. Projective = "relatively $\{e\}$ projective"
2. Equivalent notion of relatively H-injective.

Proof $1 \leftrightarrow 2 \leftrightarrow 3$ copy similar result for projective modules.

$4 \to 1$ trivial

Claim $2 \to 4$

There is a natural kG-module map $U_H^G \twoheadrightarrow U$ given by

$$g \otimes u \to gu$$

~~By 2 this~~ This splits over H, namely ~~gu~~ $U \to 1 \otimes u$

By ~~#~~2 it splits over G so $(U_H^G) \cong U \oplus$ — //

Thm Let $P \in Syl_p(G)$ and $P \leq H \leq G$. Then every kG-module is relatively H-projective.

Proof Averaging again. We'll prove "2", so suppose

$$\begin{tikzcd} V \arrow[r, "\varphi"] & U \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

and s is only a kH-module map with ~~so~~ $\varphi \circ s = id$

Define

$$\tilde{s}(u) = \frac{1}{|G:H|} \sum_{g \in G/H} g s(g^{-1} u)$$ and check.

COR $P \leq H \leq G$ as above, $U \in kG$-mod. Then U is projective iff U_H is projective. In particular we can test projectivity on a Sylow.

Proof U proj $\to U_H$ proj already done. Suppose U_H is proj. By Thm

$U \mid (U_H)^G$ and $(U_H)^G$ is projective by Omnibus Lemma. //

Thm Let U be an indecomposable kG-module.

1. $\exists$ a p-subgroup $Q \le G$, unique up to conjugacy, such that U is relatively H-projective $\leftrightarrow gQg^{-1} \le H$ for some $g \in G$.

2. $\exists$ an indecomposable kQ-module S, unique up to conjugacy in $N_G(Q)$, such that $U | S^G$.

Def. Q is called a __vertex__ of U, S is a __source__ of U.

Rmks
1. Smaller vertex $\approx$ closer to projective

2. Suppose $U | S^G = \bigoplus g \otimes S$. Check that $U | \mathrm{Ind}_{gQg^{-1}}^G g(S)$, moreover $g(S)^G \cong S^G$, hence the ambiguity in Q & S

Proof

We know U is rel. P-projective. Choose Q of minimal order so U is rel Q-projective, thus $U | (U_Q)^G$. Thus $\exists$ some indecomposable summand $S | U_Q$ such that $U | S^G$. Need "__uniqueness__" of Q, S.

- If $Q \le H$ then $U | (S^H)^G$ so U is rel. H-proj
- As above, U is also then rel gHg^{-1} projective

Suppose $H \le G$ and U is rel H-proj, so $U | V^G$, V an indec H-module

$S \mid U_Q$ and $U \mid V^G$ so $S \mid (V^G)_Q$. By Mackey:

$$(V^G)_Q \cong \bigoplus_{s \in Q\backslash G/H} \left(s(V)_{Q \cap sHs^{-1}} \right)^Q$$

so $S \mid \left(s(V)_{Q \cap sHs^{-1}} \right)^Q$ for some s

However if $Q \cap sHs^{-1} < Q$ this contradicts minimality of $|Q|$.
Thus $Q \leq sHs^{-1}$ as desired.

In the case where $H = Q$, then $Q \cap sQs^{-1} = Q \Rightarrow s \in N_G(Q)$. //

<u>Remark</u>

1. Finding vertices and sources of modules is difficult and an active are of research. For instance ≈ 10 papers in last 5 years for D^λ, S^μ, only very special cases.

~~Sampl~~

2. A module is <u>trivial source</u> if $U \mid k^G$, i.e. U is a direct summand of a permutation module. These are <u>very interesting modules</u>!

3. Sample Thm: (Puig) G p-solvable, then the source is an <u>endopermutation module</u>.

Properties of Vertices & Sources

Lemma Let U be indec kG-module w/ vertex Q and $Q \le H$.
Then $\exists$ an indec kH module V satisfying any 2 of:

1. $V \mid U_H$ 2. $U \mid V^G$ 3. V has vertex Q

$$\begin{array}{cc} G & U \\ | & \\ H & \\ | & \\ Q & S \end{array}$$

• Eventually we find a V for all 3

Proof 1 & 2
Easy! $U \mid (U_H)^G$ so choose $V \mid U_H$ so $U \mid V^G$.

2 & 3 Let S be a source of U so $U \mid S^G = (S^H)^G$.
Choose a summand $V \mid S^H$ so $U \mid V^G$. Need V to also have vertex Q.

Since $V \mid S^H$, V is relatively Q projective. Choose a vertex $R \lneq Q$.

Then $V \mid \mathrm{Ind}_R^H W \Rightarrow U \mid W^G \Rightarrow U$ is rel R proj $\divideontimes$. Thus $R = Q$ //

1 & 3 We have source S with $S \mid U_Q$ and $U \mid S^G$.

Write $U_H = \oplus$ __ and choose a summand $\boxed{V \mid U_H \text{ with } S \mid V_Q}$

Claim V has vertex Q.
Pf.
$V \mid U_H$ so $V \mid (S^G)_H$ so $V \mid$ some $\left(s(S)_{H \cap sQs^{-1}}\right)^H$ by Mackey.

Thus V has a vertex $R \le H \cap sQs^{-1}$. ETS R and Q are conjugate in H.

Now $V \mid$ some module induced ${}_R\!\uparrow^H$ and $S \mid V_Q$

So again by Mackey S is relatively $Q \cap hRh^{-1}$ projective for some $h \in H$.

But S has vertex Q so $Q \cap hRh^{-1} = Q \Rightarrow Q \le hRh^{-1}$

~~But~~ so $R \le sQs^{-1}$ so $|R| = |Q|$ and $Q = hRh^{-1}$ //