Lecture 13

Review
- We classified irreducible $\mathbb{C}\Sigma_n$ modules $\forall n \geq 1$ at once, denoted them $\{S^\lambda \mid \lambda \vdash n\}$
- $\dim S^\lambda = \#$ standard Young tableaux of shape λ
- SYT of shape $\lambda \leftrightarrow$ paths $\emptyset \to \cdots \to \lambda$ in Young lattice
  $\leftrightarrow$ GZ basis vectors
- $T \leftrightarrow$ residue sequence = weight of corresponding GZ basis vector.
- Branching graph is Young graph
- Hook formula $\dim S^\lambda = n! / \prod$ hook lengths

- $S^\lambda \otimes \mathrm{sgn} \cong S^{\lambda'}$
- Explicit formulas for matrix of s_i in terms of GZ basis, shows $\mathbb{Q}$ is a splitting field.

Problem: From beginning this approach needed us to work over $\mathbb{C}$, we used Wedderburn to define A_n etc.

"Classical" approach to representation theory, over any field

Def Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash n$. The Young subgroup $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_s}$

Def The permutation module $M^\lambda \cong \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_n} k$, $\dim M^\lambda = \dfrac{n!}{\prod(\lambda_i!)}$

Tabloids Let t be a λ-tableau. Define C_t, R_t, $t_1 \sim t_2$ and tabloids.

Rmk The set of λ-tabloids give a basis of M^λ.

Specht Modules

Def For t a tableau, $\kappa_t = \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\sigma$, $\rho_t = \sum_{\sigma \in R_t} \sigma$. The **polytabloid** is $e_t = \kappa_t\{t\} = \sum_{\sigma \in C_t} \operatorname{sgn}\sigma \{\sigma t\}$.

Ex $t = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$ $e_t = \overline{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}} - \overline{\begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}} - \overline{\begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix}} + \overline{\begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix}}$

Exc $\pi e_t = e_{\pi t}$

Def The **Specht module** $S^\lambda \leq M^\lambda$ is the submodule spanned by all the e_t as t runs over the set of λ-tableaux.

Remarks

1. e_t depends on t not just $\{t\}$
2. S^λ is cyclic, gen by any e_t
3. S^λ defined over any field, not clear that dimension is independent.

Interlude on $*$ and $\perp$

Situation M a kG-module with a nondegenerate, symmetric, G-invariant bilinear form $\langle , \rangle$

Example M^λ, set $\langle \{t_1\}, \{t_2\} \rangle = \delta_{\{t_2\}}^{\{t_1\}}$

Def For $U \leq M$, $U^\perp = \{m \in M \mid \langle u, m \rangle = 0 \ \forall u \in U\}$

Exc U a submodule $\Rightarrow U^\perp$ is a submodule

Lemma 1 Let $0 \le U \le V \le M$. Then $V/U \cong \left(U^\perp/V^\perp\right)^*$.

Proof Define $\theta: V \to \left(U^\perp/V^\perp\right)^*$ by $\theta(v) = \psi_v$ where

$$\psi_v(x + V^\perp) = \langle v, x\rangle.$$ Check:

- $\ker \theta = U$
- ψ_v is well-defined, onto (use dim)
- ψ_v is a G-module map

Cor 1 $V \cong \left(M/V^\perp\right)^*$

Pf Set $U = 0$.

Lemma 2 $(U+V)^\perp = U^\perp \cap V^\perp$

Pf By Def

Thm $V/V\cap V^\perp$ is self dual.

Proof

$$\begin{tikzcd}[column sep=small]
 & M \arrow[d, dash] & \\
 & V+V^\perp \arrow[dl, dash] \arrow[dr, dash] & \\
V \arrow[dr, dash] & & V^\perp \arrow[dl, dash] \\
 & V\cap V^\perp \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}$$

$$V/V\cap V^\perp \cong V+V^\perp/V^\perp \quad \cong \text{Thm}$$

$$\cong \left(V/(V+V^\perp)^\perp\right)^* \quad \text{Lemma 1}$$

$$= \left(V/V^\perp \cap (V^\perp)^\perp\right)^* \quad \text{Lemma 2}$$

$$= \left(V/V^\perp \cap V\right)^*$$

since $(V^\perp)^\perp = V$ (obviously $V \le (V^\perp)^\perp$, use dim. //

Def Given a basis $\langle e_i \rangle$, the Gram matrix of $\langle , \rangle$ wrt $\{e_i\}$ is

$$G_{ij} = \langle e_i, e_j \rangle$$

Thm $\dim \left(V / V \cap V^\perp \right) =$ rank of Gram matrix.

Proof Define $\theta : V \to V^*$ by $\theta(v) = \langle v, - \rangle$. Choosing a basis of V and dual basis of V^*, check that matrix of θ is the Gram matrix. Thus

$$\text{Rank} = \dim V - \dim \ker \theta = \dim V - \dim (V \cap V^\perp). \quad //$$

Back to Σ_n ... **Lemma** t a λ-tab, $u \in M^\lambda$, $\kappa_t u$ is a mult of e_t.

Thm (James' Submodule Thm)
Suppose $U \subseteq M^\lambda$. Then either $S^\lambda \subseteq U$ or $U \subseteq S^{\lambda\perp}$.

Proof Suppose $\kappa_t u = 0$ $\forall t$, $\forall u \in U$. Then $\langle e_t, u \rangle = \langle \{t\}, \kappa_t u \rangle = 0$ $\forall t, u$ so $U \subseteq (S^\lambda)^\perp$.
Otherwise $\exists u \in U, t$ s.t. $\kappa_t u = c e_t \neq 0 \Rightarrow e_t \in U \Rightarrow S^\lambda \subseteq U$.

COR Either $S^\lambda \cap S^{\lambda\perp}$ is the unique max submodule of S^λ or is all of S^λ.

Proof Any submodule of S^λ is either all of S^λ or also in $(S^\lambda)^\perp$

Thm $S^\lambda / (S^\lambda \cap S^{\lambda\perp})$ is either zero or absolutely irreducible.

If it is nonzero then $S^\lambda \cap (S^\lambda)^\perp = \operatorname{rad} S^\lambda$ and $S^\lambda / S^\lambda \cap S^{\lambda\perp}$ is self-dual!

Now use lots of combinatorics to prove over $\mathbb{C}$

1. $\mathrm{Hom}_{\Sigma_n}(S^\lambda, M^\mu) \neq 0 \Rightarrow \lambda \trianglerighteq \mu$

2. $\mathrm{Hom}_{\Sigma_n}(S^\lambda, M^\lambda) \cong \mathbb{C}$

3. $\langle , \rangle$ is inner product, $S^\lambda \cap S^{\lambda\perp} = 0$.

<u>COR</u> $\{S^\lambda\}$ are a complete set of irred $\mathbb{C}\Sigma_n$-modules.

<u>Aside</u> Inside $\mathbb{C}\Sigma_n$ Let t be a λ-tableau

$M^\lambda \cong \mathbb{C}\Sigma_n \rho_t$

$S^\lambda \cong \mathbb{C}\Sigma_n \rho_t \kappa_t$