

Lecture 12

Review • $W(\mathfrak{h}) = \{ \lambda \in \mathbb{C}^n \mid \lambda \text{ is a weight of some irreducible module } \}$

i.e. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(\mathfrak{h})$ means $L_i v = \lambda_i v$.

- \mathbb{Z} basis of an irreducible is a basis of weight vectors
- $\mathcal{B}_i = \langle s_i, L_i, L_{i+1} \rangle$ is a homomorphic image of \mathfrak{H}_2 .

Thm Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(\mathfrak{h})$, $M = s_i \lambda$ Then

1. $\lambda_i \neq \lambda_{i+1}$

2. If $\lambda_{i+1} = \lambda_i \pm 1$ then $s_i V_\lambda = \pm V_\lambda$ and M is not a weight of V .

3. If $\lambda_{i+1} \neq \lambda_i \pm 1$ then M is a weight w/ nonzero wt vector

$$W = (s_i - \frac{1}{\lambda_{i+1} - \lambda_i}) V_\lambda \text{ and } \langle V_\lambda, W \rangle \cong \mathbb{C} \langle \lambda_i, \lambda_{i+1} \rangle \text{ as a } \mathcal{B}_i\text{-module.}$$

Proof Uses class of irreducible \mathfrak{H}_2 -modules.

Remark Any time $\lambda_{i+1} \neq \lambda_i \pm 1$ in a weight, you can swap them and get another weight in same irreducible.

COR Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(\mathfrak{h})$ Then

1. $\lambda_i = 0$

2. $\{ \lambda_i - 1, \lambda_i + 1 \} \cap \{ \lambda_1, \lambda_2, \dots, \lambda_{i-1} \} \neq \emptyset$

3. If $i < j$, $\lambda_i = \lambda_j = a$ then $\{ a-1, a+1 \} \subseteq \{ \lambda_{i+1}, \dots, \lambda_{j-1} \}$

Remark Easy now to see all $\lambda_i \in \mathbb{Z}$.

Proof

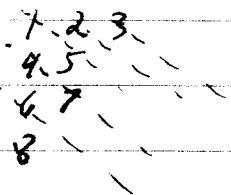
1. Clear, $L=0$
2. Suppose not. By Thm (3) can get $(\lambda_1, \lambda_2, \lambda_3, \dots)$ a weight.
If $\lambda_i = 0$ \neq , if not, \neq !
3. Suppose not, choose $i < j$ w/ $|\lambda_i - \lambda_j|$ minimal.

Young Diagrams

- Define Young diagram of a partition
- Define the Young graph.
- Standard α -tableau \leftrightarrow α -paths $\emptyset \rightarrow \dots \rightarrow \alpha$
- Residue weights, etc.

Easy Exercise #1 The set of weights in the corollary correspond exactly to residue sequences of a SYT.

Prk Suppose i and $i+1$ do not lie on adjacent diagonals in a SYT. Then swapping gives another SYT.



Exercise #2 Fix α . Can get from any SYT of shape α to any other by admissible transpositions

COR Given $\lambda, \mu \in W(n)$ corr to SYT of same shape then $\lambda \sim \mu$?

Problem Maybe more equivalences?

Answer No! The # of irred $\mathbb{C}S_n$ modules is the # of partitions.

Thm For each partition $\alpha \vdash n$ there is an irreducible $\mathbb{C}S_n$ module S^α such that

1. $\dim S^\alpha = \# \text{SYT of shape } \alpha$

2. The SYT \leftrightarrow $\mathbb{C}Z$ basis vectors

$\text{res} \leftrightarrow \text{weight}$
 seq

3. The Young graph is the branching graph!!

Ex $\text{Res}_{\mathbb{Z}/4} S^{8331} = S^{7331} \oplus S^{8321} \oplus S^{833}$

Ex $\alpha = (n)$, only $T = \boxed{1|2|3|\dots|n}$ weight = $(0, 1, 2, 3, \dots, n-1)$

$S^{(n)} \cong$ trivial module \mathbb{C}

Ex $\alpha = (1^n)$ only $T = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$ weight = $(0, -1, -2, \dots, -n+1)$

$S^{(1^n)} =$ sign representation.

Ex $\alpha = (n-1, 1)$ $n-1$ dimensional natural module

Define α'

COR Let $\alpha \vdash n$. Then $S^\alpha \otimes \text{sgn} \cong S^{\alpha'}$

Proof: weights

Young's Semidual Form

Thm Let $d \vdash n$ and let $\{V_T\}$ be an appropriately scaled GZ basis.
Let T be a SYT shape α and

1. If $\text{res } T_{i+1} = \text{res } T_i \pm 1$ then $s_i V_T = \pm V_T$

2. Else $s_i T = S$ another SYT

$$\text{Let } p = \frac{1}{\text{res } T_{i+1} - \text{res } T_i}$$

Matrix of s_i on these 2 basis elts is

$$\begin{pmatrix} p & 1-p^2 \\ 1 & -p \end{pmatrix}$$

COR \mathbb{Q} is a splitting field

Thm Let $f^\lambda = \dim S^\lambda = \# \text{ SYT shape } \lambda$. Then

$$f^\lambda = \frac{n!}{z_\lambda}$$

Ex $\lambda = 421$

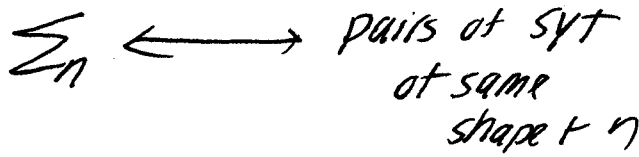
4	2	1	1
3	1		
1			

$$\dim S^\lambda = \frac{7!}{6 \cdot 4 \cdot 3 \cdot 2}$$

R-S-K

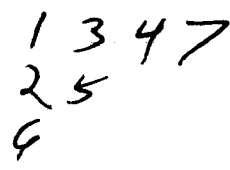
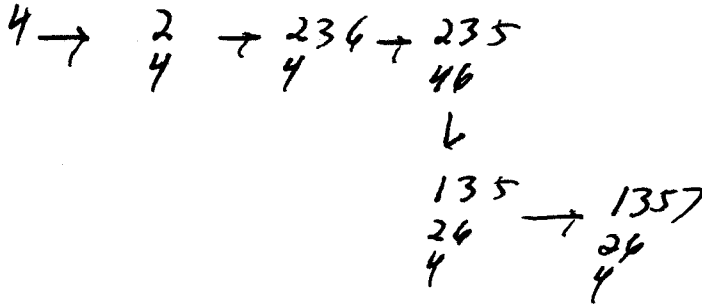
COR $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$

RSK gives bijection



Ex $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{matrix}$

Recording tableau



$\pi \rightarrow \left(\begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 \\ 4 \end{matrix}, \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 \\ 6 \end{matrix} \right)$