Lecture 12

Review • $W(n) = \{\lambda \in \mathbb{C}^n \mid \lambda$ is a weight of some irreducible module$\}$

i.e. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(n)$ means $L_i v = \lambda_i v$.

- GZ basis of an irreducible is a basis of weight vectors
- $\mathcal{B}_i = \langle s_i, L_i, L_{i+1} \rangle$ is a homomorphic image of H_2.

Thm Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(n)$, $\mu = s_i \lambda$. Then

1. $\lambda_i \neq \lambda_{i+1}$
2. If $\lambda_{i+1} = \lambda_i \pm 1$ then $s_i v_\lambda = \pm v_\lambda$ and μ is not a weight of V.
3. If $\lambda_{i+1} \neq \lambda_i \pm 1$ then μ is a weight w/ nonzero wt vector

$$w = \left(s_i - \frac{1}{\lambda_{i+1} - \lambda_i}\right) v_\lambda \quad \text{and} \quad \langle v_\lambda, w \rangle \cong L(\lambda_i, \lambda_{i+1})$$

as a $\mathcal{B}_i$-module

Proof Uses class. of irreducible H_2-modules.

Rmk Any time $\lambda_{i+1} \neq \lambda_i \pm 1$ in a weight, you can swap them and get another weight in same irreducible.

COR Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W(n)$. Then

1. $\lambda_1 = 0$
2. $\{\lambda_i - 1, \lambda_i + 1\} \cap \{\lambda_1, \lambda_2, \dots, \lambda_{i-1}\} \neq \emptyset$
3. If $i < j$, $\lambda_i = \lambda_j = a$ then $\{a-1, a+1\} \subseteq \{\lambda_{i+1}, \dots, \lambda_{j-1}\}$

Rmk Easy now to see all $\lambda_i \in \mathbb{Z}$.

Proof 1. Clear, $\lambda_1 = 0$

2. Suppose not. By Thm (3) can get $(\lambda_1, \lambda_i, \lambda_2 \ldots)$ a weight. If $\lambda_i = 0$ $\divideontimes$, if not, $\divideontimes$!

3. Suppose not, choose $i < j$ w/ $|j-i|$ minimal.

Young Diagrams

- Define Young diagram of a partition
- Define the Young graph
- Standard α-tableau $\longleftrightarrow$ α-paths $\emptyset \to \to \cdots \to \alpha$
- Residue weights, etc. . .

Easy Exercise #1 The set of weights in the corollary correspond exactly to residue sequences of a SYT.

Rmk Suppose i and $i+1$ do not lie on adjacent diagonals in a SYT. Then swapping gives another SYT.

1 2 3
4 5
6 7
8

Exercise #2 Fix α. Can get from any SYT of shape α to any other by admissible transpositions

COR Given $\lambda, \mu \in W(n)$ corr to SYT of same shape then $\lambda \sim \mu$?

Problem Maybe more equivalences?

Answer No! The # of irred $\mathbb{C}S_n$ modules is the # of partitions.

Thm For each partition $\alpha \vdash n$ there is an irreducible $\mathbb{C}\Sigma_n$ module S^α such that

1. $\dim S^\alpha = \#$ SYT of shape α
2. The SYT $\leftrightarrow$ GZ basis vectors

   res seq $\leftrightarrow$ weight

3. The Young graph is the branching graph!!

Ex $\mathrm{Res}_{\Sigma_{14}} S^{8331} = S^{7331} \oplus S^{8321} \oplus S^{833}$

Ex $\alpha = (n)$, only $T = \boxed{1\,|\,2\,|\,3\,|\,\cdots\,|\,n}$ weight $= (0,1,2,3,\dots,n-1)$

$S^{(n)} \cong$ trivial module $\mathbb{C}$

Ex $\alpha = (1^n)$ only $T = \begin{array}{|c|}\hline 1 \\ 2 \\ \vdots \\ n \\ \hline\end{array}$ weight $= (0,-1,-2,\dots,-n+1)$

$S^{(1^n)} =$ sign representation

Ex $\alpha = (n-1,1)$ $n-1$ dimensional natural module

Define α'

COR Let $\alpha \vdash n$. Then $S^\alpha \otimes \mathrm{sgn} \cong S^{\alpha'}$.

Proof: weights

Young's Seminormal Form

Thm Let $\alpha \vdash n$ and let $\{V_T\}$ be an appropriately scaled GZ basis.
Let T be a SYT shape α and

1. If $\operatorname{res} T_{i+1} = \operatorname{res} T_i \pm 1$ then $s_i V_T = \pm V_T$

2. Else $s_i T = S$ another SYT

$$\text{Let } \rho = \frac{1}{\operatorname{res} T_{i+1} - \operatorname{res} T_i}$$

Matrix of s_i on these 2 basis elts is

$$\begin{pmatrix} \rho & 1-\rho^2 \\ 1 & -\rho \end{pmatrix}$$

COR $\mathbb{Q}$ is a splitting field.

Thm Let $f^\lambda = \dim S^\lambda = \#$ SYT shape λ. Then

$$f^\lambda = \frac{n!}{\prod h_e}$$

Ex $\lambda = 421$

$$\begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \qquad \dim S^\lambda = \frac{7!}{6 \cdot 4 \cdot 3 \cdot 2}$$

R-S-K

COR $$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

RSK gives bijection

$$\Sigma_n \longleftrightarrow \text{pairs of SYT of same shape} \vdash n$$

Ex $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{matrix}$

$$4 \to \begin{matrix} 2 \\ 4 \end{matrix} \to \begin{matrix} 2\,3\,6 \\ 4 \end{matrix} \to \begin{matrix} 2\,3\,5 \\ 4\,6 \end{matrix}$$

$$\downarrow$$

$$\begin{matrix} 1\,3\,5 \\ 2\,6 \\ 4 \end{matrix} \to \begin{matrix} 1\,3\,5\,7 \\ 2\,6 \\ 4 \end{matrix}$$

Recording tableau

$$\begin{matrix} 1\,3\,4\,7 \\ 2\,5 \\ 6 \end{matrix}$$

$$\pi \to \left(\begin{matrix} 1\,3\,5\,7 \\ 2\,6 \\ 4 \end{matrix} \;,\; \begin{matrix} 1\,3\,4\,7 \\ 2\,5 \\ 6 \end{matrix} \right)$$