

Lecture 11

- Review
- We are classifying irreducible $\mathbb{C}\Sigma_n$ modules, we know the # of them is the # of partitions of n
 - Using that $\mathbb{C}(\Sigma_{n-1})$ is commutative we showed $\text{Res}_{\Sigma_{n-1}} V$ is multiplicity free for any simple Σ_n -module V .
 - Defined Jucys-Murphy elements $L_1 = 0, L_2 = (12), L_3 = (13) + (23), \dots, L_n = (1n) + \dots + (n-1n)$

Def The Branching graph \mathcal{B} has vertices the irreducible $\mathbb{C}\Sigma_n$ modules $V \uparrow n$ and edges $W \rightarrow V$ if $\text{Res}_{\Sigma_{n-1}} V \cong W \oplus \dots$

Def For V an irreducible $\mathbb{C}\Sigma_n$ module, we saw V has a Gelfand-Zetlin basis, unique up to scalars, with each basis vector corresponding to a path $W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_n = V$ in \mathcal{B}

Prop For V_T corr to this path, $\mathbb{C}\Sigma_n \cdot V_T = W_T$

GZ subalgebra

1. Write $\mathbb{C}\Sigma_n \cong \bigoplus_{V \text{ irred}} \text{End}_{\mathbb{C}}(V)$ This is canonical!
2. Choose the GZ basis of each V to get $\mathbb{C}\Sigma_n \cong \bigoplus_V M_{\dim V}(\mathbb{C})$ (*)

Def The GZ subalgebra of $\mathbb{C}\Sigma_n$ corresponds to matrices diagonal in each coordinate in (*). This is clearly a maximal commutative, semisimple algebra, call it A_n

Then $A_n = \langle L_1, L_2, \dots, L_n \rangle$

Def Suppose V is an irreducible $\mathbb{C}\Sigma_n$ -module, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n - \{0\}$.
 Say λ is a weight of V if $\exists v \neq 0, L_i v = \lambda_i v \ \forall i$. The set of such v is the λ -weight space V_λ . Note: V is a \oplus of weight spaces.

Theorem

1. Elements of a GZ basis are weight vectors
2. In an irreducible $\mathbb{C}\Sigma_n$ module, all weight spaces are 1-dim.
3. If λ is a weight of an irreducible V , it is not a weight of any other irreducible $V' \neq V$. (Write $V(\lambda) = V$)
4. There is a 1-1 correspondence between all weights for all Σ_n and paths in B from W_0 .

Proof

1. Since $L_i \in A$ then matrix of L_i is diagonal in GZ basis.
2. If two different V_j had same action of all L_i this would contradict A being all diagonal matrices.
3. Same as 2
4. We know weights $\leftrightarrow GZ$ basis vectors \leftrightarrow paths //

Remark

Suppose $(\lambda_1, \lambda_2, \dots, \lambda_n)$ a weight of V . Then $(\lambda_1, \dots, \lambda_r)$ is a weight of $\text{Res } V$. To understand B we need:

1. Which weights in \mathbb{C}^n actually occur; call this $W(n)$.
2. For $\lambda, \mu \in W(n)$, when are λ, μ in same irreducible?

Notation

$\lambda \in W(n)$ then $V(\lambda)$ is the irreducible with λ as a weight.

$V_\lambda = V_{T_\lambda}$ is the corresponding GZ basis vector.

Degenerate Affine Hecke Algebra

Def $\mathcal{H}_2 = \langle s, x, y \mid xy = yx, s^2 = 1, sx = ys - 1 \rangle$ (thus $xs = sy - 1$)

Irreducible \mathcal{H}_2 -modules : Fix $a, b \in \mathbb{C}$, construct $L(a, b)$:

Case 1 $a = b - 1$ $x \rightarrow (a)$ $y \rightarrow (b)$ $s \rightarrow (1)$

Case 2 $a = b + 1$ $x \rightarrow (a)$ $y \rightarrow (b)$ $s \rightarrow (-1)$

Case 3 $a \neq b \pm 1$ $x \rightarrow \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}$ $y \rightarrow \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ $s \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* Subcase $a = b$ then x, y do not act semisimply

* Subcase $a \neq b, b \neq 1$ New basis

$$x \rightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad y \rightarrow \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \quad s \rightarrow \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}$$

Then Every irreducible \mathcal{H}_2 -module is \cong to some $L(a, b)$.

If $a \neq b \pm 1$ then $L(a, b) \cong L(b, a)$, and no further \cong exist.

"Proof" Choose v a simultaneous eigenvector for x, y . Consider $\langle v, sv \rangle$ w/ values a, b .

$$ysv = (sx+1)v = sxv + v = a sv + v \text{ etc. } //$$

Exercise $\langle s_i, L_i, L_{i+1} \rangle$ satisfy relations of \mathcal{H}_2 . Thus \exists a surjection

$$\mathcal{H}_2 \twoheadrightarrow \mathcal{B}_i := \langle s_i, L_i, L_{i+1} \rangle$$

Classifying $W(\mathfrak{n})$

Thm Suppose $\lambda \in W(\mathfrak{n})$, $V = V(\lambda)$, $M = S_i \lambda = (\lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots, \lambda_{i-1})$

1. $\lambda_i \neq \lambda_{i+1}$
2. If $\lambda_i = \lambda_{i+1} \pm 1$ then $S_i V_\lambda = \pm V_\lambda$ and M is not a weight of V .
3. If $\lambda_i \neq \lambda_{i+1} \pm 1$ then M is a weight of V with nonzero wt vector

$$w := \left(s_i - \frac{1}{\lambda_{i+1} - \lambda_i} \right) V_\lambda \quad \langle V_\lambda, w \rangle \text{ is 2-dim } \mathcal{B}_i \text{ module} \cong \text{subspace } L(\lambda_i, \lambda_{i+1})$$

Proof We know $\mathbb{C} \sum_{i+1} V_\lambda \cong V(\lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{i+1})$. Consider

$$M = \text{Hom}_{\mathbb{Z}_{i+1}} (V(\lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{i+1}), \text{Res } V(\lambda_{i+2}, \dots, \lambda_{i+1}))$$

as a $\mathbb{Z}_{i+1,2} = \langle \mathcal{B}_i, \mathbb{Z}_{i+1} \rangle$ module, it is irreducible. \mathcal{B}_i and \mathbb{Z}_{i+1} commutes or \mathbb{Z}_{i+1} acts as scalars by Schur so $M \mathbb{Z}_{i+1}$ is an irreducible \mathcal{B}_i -module.

Check M is irreducible \mathfrak{H}_2 -module w/ e-values λ_i, λ_{i+1} .

1. \mathbb{Z}_{i+1} semisimple but \mathfrak{H}_2 not semisimple in $L(\mathfrak{g}, \mathfrak{g})$ so $\lambda_i = \lambda_{i+1}$ impossible

Rest uses \mathfrak{H}_2 -module.

COR Suppose $\lambda = (\lambda_{i+1}, \dots, \lambda_i) \in \mathbb{C}^n$. If $\lambda_i = \lambda_{i+1} = \lambda_{i+1} \pm 1$ then $\lambda \notin W(\mathfrak{n})$

Proof $S_i V_\lambda = \pm V_\lambda$, $S_{i+1} V_\lambda = \mp V_\lambda$ contradicts claim

Final Classification of $W(n)$

Thm. Let $\lambda \in \mathbb{C}^n$. Then $\lambda \in W(n) \Leftrightarrow$

- 1 $\lambda_i = 0$
- 2 $\{\lambda_i - 1, \lambda_i + 1\} \cap \{\lambda_i, \lambda_{i+1}, \lambda_{i-1}\} \neq \emptyset \quad \forall 1 < i \leq n$
- 3 If $\lambda_i = \lambda_j = a$ some $i < j$ then $\{a-1, a+1\} \subseteq \{\lambda_{i+1}, \lambda_{j-1}\}$

Proof

1 Obvious, $\lambda_i = 0$

2 Suppose 2 fails $\lambda_i, \lambda_{i+1}, \lambda_{i-1}, \lambda_j$
no $\lambda_i - 1$ or $\lambda_i + 1$

Apply 3 repeatedly to get $\lambda_i, \lambda_{i+1}, \dots$

If $\lambda_i \neq 0$ contradicts 1.

If $\lambda_i = 0$ swap again. $\neq \lambda_i = 0$

~~swap not~~
3 Pick i, j minimizing $j - i$ where it fails

$$\lambda = (\dots, a, \uparrow, \uparrow, a, \dots)$$

must be $a \pm 1$ else can swap

~~what swap at~~

so say

$$a, a \pm 1, \dots, a \pm 1, a$$

must have $(a \pm 1) \pm 1 //$