

## Lecture 2

Review  $k$  a field, usually alg. closed  
 $A$  a finite dimensional  $k$ -algebra w/ identity.

Ex  $A \cong kG$  group algebra of a finite group.

Then an  $A$ -module  $U$  is a vector space over  $k$  with an action of  $A$  by linear maps.

Exercise Given an  $n$ -dimensional  $A$ -module  $U$ , we get an induced algebra homomorphism

$$\psi: A \rightarrow M_{n \times n}(k) \cong \text{End}_k(U) \cong \text{Hom}_k(U, U)$$

Rmk Given  $R$ -modules  $M, N$ ,  $\text{Hom}_R(M, N)$  is an abelian group, the set of  $R$ -module homomorphisms  $\text{Hom}_R(M, M)$  is a ring, the endomorphism ring of  $M$ .

$\text{Hom}_A(M, M)$  is an algebra, the endomorphism algebra.

Schur's Lemma Let  $S$  be a simple  $A$ -module.

1.  $\text{End}_A(S)$  is a division algebra.

2. When  $k$  is alg. closed then  $\text{End}_A(S) = \{ \lambda \text{Id} \mid \lambda \in k \}$  ~~is a division algebra~~.

Proof 1. Image and kernel are both submodules, so nonzero maps are invertible.

2. Choose  $e$ -value  $\lambda$  for  $f: S \rightarrow S$  look at  $f - \lambda I$ .

## Free module Review

- $A \oplus A \oplus \dots \oplus A = A^{\oplus n}$  is a free  $A$ -module of rank  $n$ .  
Let  $a_i = (0, \dots, 1, \dots, 0)$ . Then

Universal Property Given any  $A$ -module  $U$  and  $u_1, u_2, \dots, u_n \in U$ , there is a unique  $A$ -module homomorphism  $\psi: A^n \rightarrow U$  mapping  $a_i \rightarrow u_i$ .

Ex  $A = \mathbb{R}$ , this is a property of basis of vector space.

## Corollary

1. Every finite dimensional  $A$ -module is the homomorphic image of a free module  $A^n$ , some  $n$ .
2. There are only finitely many simple  $A$ -modules up to  $\cong$ .

## Proof

1. Any generating set  $\{u_1, \dots, u_n\}$  of  $U$  guarantees map  $\psi$  is onto, for example a basis of  $U$ .
2. Let  $s \in S$ ,  $s \neq 0$  &  $S$  simple. Then  $As = S$  so can choose  $n=1$ , i.e.  $\exists$  a surjection  $A \rightarrow S$ .

Now  $A$  is f.d., so has a composition series, so only finitely many possible  $S$ .

Rank Modules generated by one element are called cyclic. So of course simple  $\Rightarrow$  cyclic.

## Semisimple Modules

(3)

Def An  $A$ -module is semisimple if it is a direct sum of simple modules.

EX  $A = T_2(k)$  natural module is not semisimple.

Lemma Suppose  $U \cong S_1 \oplus S_2 \oplus \dots \oplus S_n$ ,  $S_i$  are simple and suppose  $V \subseteq U$ . Then  $\exists$  a subset  $I \subseteq \{1, 2, \dots, n\}$  such that

$$U \cong V \oplus \bigoplus_{i \notin I} S_i =: V \oplus S_I$$

Proof Choose  $I$  maximal so that  $S_I \cap V = 0$ . Then

$$S_I + V \cong S_I \oplus V, \text{ want } S_I + V = U.$$

Suppose not,  $\exists S_i$  w/  $S_i \cap (S_I + V) = 0$  so

$$S_i + S_I + V \cong S_i \oplus S_I \oplus V, \neq$$

Prop Let  $U$  be an  $A$ -module. TFAE

1.  $U$  is a  $\bigoplus$  of simple modules
2. Every submodule of  $U$  is a direct summand

Exercise: Check that the Lemma implies submodules and quotients of semisimple modules are semisimple.

Proof  $1 \rightarrow 2$  by Lemma. Suppose "2" holds for  $U$ . I claim it holds for any  $U/V$ . Let  $W/V \subseteq U/V$  so  $U = W \oplus X$  by 2.

Check  $U/V = W/V \oplus X/V$ .

Now choose simple submodule  $S$ , so  $U = S \oplus T$ ,  $T \cong U/S$ .

But 2 holds for  $T$  so  $T$  is a  $\bigoplus$  of simple modules by induction on comp. length

# The Radical

Recall Let  $U$  be an  $A$ -module. The annihilator  $\text{ann } U = \{a \in A \mid aU = 0 \ \forall U \in U\}$  is a 2-sided ideal.

Def The radical,  $\text{rad } A = \bigcap_{\substack{\text{simple} \\ A\text{-modules} \\ S}} \text{ann } S$ , so this is a 2-sided ideal, the largest ideal annihilating all semisimple  $A$ -modules

Recall An ideal is nilpotent if  $I^N = 0$  for some  $N$ .

Thm The radical of  $A$  is equal to each of:

1. The smallest submodule of  ${}_A A$  with semisimple quotient.
2. The  $\bigcap$  of all maximal submodules of  ${}_A A$  (a.k.a. max left ideals of  $A$ )
3. The largest nilpotent ideal.

Def An algebra  $A$  is semisimple if  $\text{rad } A = 0$ .

COR TFAE

1.  $A$  is semisimple
2.  ${}_A A$  is semisimple
3. Every  $A$ -module is semisimple.

Exercise  $\text{rad}(A/\text{rad } A)$  is semisimple, i.e.  $A/\text{rad } A$  is a semisimple algebra.

Proof of Thm

First note  $I^n = J^n = 0 \Rightarrow (I+J)^{nm} = 0$  so sum of all nilp ideals is nilpotent, so (3) exists, call it  $N$ .

If  $S$  is simple then  $NS$  is a submodule, 0 or  $S$ . But  $N^n = 0 \Rightarrow N^n S = 0 \Rightarrow NS = 0$ . Thus  $N \in \text{rad } A$ .

Let  $0 = A_0 \subset A_1 \subset \dots \subset A_n = A$  be a comp series.

$$\begin{aligned} \text{rad } A(A_i/A_{i-1}) = 0 &\Rightarrow \text{rad } A(A_i) \subseteq A_{i-1} \\ &\Rightarrow (\text{rad } A)^n A = 0 \Rightarrow (\text{rad } A)^n = 0 \Rightarrow \text{rad } A \text{ nilp} \\ &\Rightarrow \text{rad } A \subseteq N. \end{aligned}$$

Thus  $N = \text{rad } A$ .

Suppose  $M_1, \dots, M_r$  are maximal submodules so  $A/M_i$  is simple so  $A/M_1 \oplus \dots \oplus A/M_r$  semisimple. Let  $I = M_1 \cap \dots \cap M_r$

$$A/M_1 \cap \dots \cap M_r \hookrightarrow A/M_1 \oplus \dots \oplus A/M_r \text{ so } A/M_1 \cap \dots \cap M_r \text{ is ss}$$

Thus  $A/I$  is semisimple. But if  $A/M$  is semisimple, say  $A/M = k_1/m_1 + \dots + k_s/m_s$ , build max submodule.

Thus  $M \supseteq I$ . Thus (1)  $\Rightarrow$  (2).

Finally Let  $a \in \text{rad } A$ ,  $M$  a max submodule. Then  $aA \subseteq M$  so  $a \cdot 1 \in M \Rightarrow a \in M$ . Thus  $\text{rad } A \subseteq I$ .

Suppose  $\text{rad } A \subset I$ . Choose  $S$  simple so  $IS \neq 0$ . Choose  $s$  so  $Is \neq 0$ . Thus  $Is = Ss$  if  $x \in I$  so  $xS = -s$  so  $(x+1)s = 0$  so  $x+1 \in \text{ann } S$  so  $x+1 \in \text{some max}$ , so  $x, x+1 \in I$ .  $\neq$  Ex:  $\mathbb{Z}/6\mathbb{Z}$