Math 620 Midterm Exam #1- February 27, 2012- SOLUTIONS

1. Short Answer- no work necessary (30 points)

a. Let M be an R module. State the universal property satisfied by the symmetric algebra $\mathcal{S}(M)$ in terms of an adjoint pair of functors.

Let A be a commutative R algebra. Then R-module homomorphisms from M to A correspond to R-algebra homomorphisms from $\mathcal{S}(M)$ to A. The pair of adjoint functors is the functor $\mathcal{S} : R - mod \to \text{commutative}\, R - alg$, and the forgetful functor going the other way. That is:

$$\mathrm{Hom}_{R-mod}(M, A) \cong \mathrm{Hom}_{R-alg}(\mathcal{S}(M), A).$$

b. State two equivalent conditions for an R-module Q to be *injective*.

Q is injective iff any SES $0 \to Q \to U \to N$ splits. Also iff given any injection $f : U \to V$ and $\psi : U \to Q$ there exists $\Psi : V \to Q$ such that $\Psi \circ f = \psi$.

c. Let $\mathcal{F}$ and $\mathcal{G}$ be covariant functors $C \to D$. Then $\mathcal{F}$ and $\mathcal{G}$ are *naturally isomorphic* if ...

For each $A \in C$ there is an isomorphism $\nu_A : \mathcal{F}(A) \to \mathcal{G}(A)$. And further, for any $f \in \mathrm{Hom}_C(A, B)$ we have:

$$\nu_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \nu_A.$$

2. (20 points) Let F be a field and consider the 8-dimensional $F[x]$-module:

$$F[x]/(x-1)^2(x-2)^2 \bigoplus F[x]/(x-1)^3(x-2).$$

Give the rational canonical form and Jordan canonical form for the linear map given by the action of x on this module.

Using the Chinese remainder theorem we see that the elementary divisors are $\{(x-1)^2, (x-1)^3, (x-2)^2, (x-2)\}$. Then the invariant factors must be $\{a_2(x) = (x-1)^3(x-2)^2, a_1(x) = (x-1)^2(x-2)\}$. Expanding out the invariant factors, this gives JCF and RCF respectively:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -19 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

3. (20 points) Let A be a nonzero finite abelian group. Prove that A is neither projective nor injective as a $\mathbb{Z}$-module.

We know A is a direct sum of cyclic groups, so there is an obvious surjection $\pi : \mathbb{Z}^n \to A$ which cannot split, as $\mathbb{Z}^n$ has no nonzero torsion elements. Thus A is not projective.

We can argue similarly to show A is not injective, or use Baer's criterion. Since $|A| \cdot A = 0$, we have that A is not divisible, so not an injective $\mathbb{Z}$-module.

4. (20 points) Let V be a 3-dimensional vector space with basis $\{v_1, v_2, v_3\}$ and let $T : V \to V$ be a linear transformation represented in this basis by the matrix:

$$\begin{pmatrix} -1 & 2 & 1 \\ -2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Give the matrices representing the linear transformations

$$\begin{aligned} \Lambda^2(T) : \Lambda^2(V) &\to \Lambda^2(V) \\ \Lambda^3(T) : \Lambda^3(V) &\to \Lambda^3(V), \end{aligned}$$

using appropriate bases.

Use $\{v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3\}$. Notice that

$$\begin{aligned} \Lambda^2(T)(v_1 \wedge v_2) &= T(v_1) \wedge T(v_2) \\ &= (-v_1 - 2v_2 + v_3) \wedge (2v_1 + v_3) \\ &= -v_1 \wedge v_3 - 4v_2 \wedge v_1 - 2v_2 \wedge v_3 + 2v_3 \wedge v_1 \\ &= 4v_1 \wedge v_2 - 3v_1 \wedge v_3 - 2v_2 \wedge v_3. \end{aligned}$$

Similar calculations for the other 2 basis vectors give the matrix:

$$\begin{pmatrix} 4 & -1 & 6 \\ -3 & -2 & 1 \\ -2 & -5 & -3 \end{pmatrix}$$

Recall that $\Lambda^3(V)$ is one-dimensional, spanned by $v_1 \wedge v_2 \wedge v_3$ and $\Lambda^3(T)$ is just multiplication by the determinant, so the matrix is (11).

5. (10 points) Let A be a 3×3 matrix over $\mathbb{Q}$ with $A^8 = I$. Prove that $A^4 = I$.

A satisfies $x^8 - 1$. which factors over $\mathbb{Q}$ into irreducibles as $(x^4+1)(x^2+1)(x-1)(x+1)$. So its minimal polynomial divides this. However its minimal polynomial also divides the characteristic polynomial, which is degree 3. Thus the minimal polynomial actually divides $x^4 - 1$ ($\mathbb{Q}[x]$ is a UFD), which means $A^4 - I = 0$.