Math 620 Spring 2012 Midterm 2 Solutions

1. Suppose $\deg(f(x)) = n$ and let α be a root of $f(x)$ in some extension field. Since $f(x)$ is irreducible over K we have $[K(\alpha) : K] = n$. Let $[L : K] = m$ so $\gcd(n, m) = 1$. We have $[L(\alpha) : K] = [L(\alpha) : L] \cdot m = [L(\alpha) : K(\alpha)] \cdot n$. The relatively prime condition forces $[L(\alpha) : L] = n$, which means $f(x)$ is still the minimal polynomial of α over L, so $f(x)$ is irreducible over L.

2. We have

$$[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)] \cdot [F(\alpha^2) : F].$$

Since α is a root of $x^2 - \alpha^2 \in F(\alpha^2)[x]$, the degree of the extension $F(\alpha)/F(\alpha^2)$ is at most 2. But $[F(\alpha) : F]$ is odd, so $[F(\alpha) : F(\alpha^2)] = 1$, i.e. $F(\alpha^2) = F(\alpha)$.

3. Let $\eta_6 = 1/2 + \sqrt{-3}/2$ be a primitive sixth root of unity. So $\eta_6^2 = \eta_3 = -1/2 + \sqrt{-3}/2$ is a primitive third root of unity. The four roots of $x^4 + x^2 + 1$ are $\{\pm\eta_6, \pm\eta_3\}$. One can see this directly using the substitution $y = x^2$ and the quadratic formula, or by factoring $x^4 + x^1 + 1 = (x^2 + x + 1)(x^2 - x + 1) = \Phi_3(x)\Phi_6(x)$. The the splitting field is just $\mathbb{Q}(\sqrt{-3})$, which has degree 2 over $\mathbb{Q}$.

4. Let $\alpha = \sqrt{2 + \sqrt{2}}$. Check that $(\alpha^2 - 2)^2 = 2$, i.e. α is a root of $x^4 - 4x^2 + 2$. Using the quadratic formula one finds the roots of this polynomial are $\pm\sqrt{2 \pm \sqrt{2}}$, so the polynomial is separable and the splitting field is a Galois extension. The polynomial is irreducible by Eisenstein ($p = 2$). I claim all 4 roots already lie in $F := \mathbb{Q}(\alpha)$, so this is the splitting field.

Just check that $\alpha^2 - 2 = \sqrt{2} \in F$. Also that $\sqrt{2 + \sqrt{2}}\sqrt{2 - \sqrt{2}} = \sqrt{2}$ so $\sqrt{2 - \sqrt{2}} \in F$, and we have the splitting field, and $[F : Q] = 4$. (The four roots are $\{\alpha, -\alpha, \frac{\alpha^2-2}{\alpha}, \frac{\alpha^2-2}{-\alpha}\}$. The Galois group has order 4 and since α can go to any of the four roots, we see it is cyclic or order 4, generated by $\sigma(\alpha) = \frac{\alpha^2-2}{\alpha}$. Observe that $\sigma^2(\alpha) = -\alpha$. The subgroup of order 2 generated by σ^2 fixes $\sqrt{2}$. The picture of the Galois correspondence is:

$$\begin{array}{cc}
\mathbb{Q}(\alpha) & 1 \\
| & | \\
\mathbb{Q}(\sqrt{2}) & \{1, \sigma^2\} \\
| & | \\
\mathbb{Q} & \{1, \sigma, \sigma^2, \sigma^3\}
\end{array}$$

5. This problem needs an assumption that $char K = 0$. Let $\alpha \in K$ and $\alpha \notin F$. Let $p(x)$ be the minimal polynomial of α over F, so $p(x)$ is an irreducible quadratic, and is separable. Since it's degree 2 it splits over K, i.e. K is the splitting field of a separable (since $char K = 0$) polynomial, i.e. it's Galois.