1a. Let $F = \mathbb{F}_2(t)$. Then the splitting field of the polynomial $x^2 - t$ is an inseparable extension of F.

b. The Galois group of $x^4 - 4x^2 + 2$, which is the minimal polynomial of $\sqrt{2+\sqrt{2}}$, is cyclic of order 4 (see homework).

c. $\mathbb{Q}[x_1, x_2, \ldots, x_n, \ldots]$ is not Noetherian.

d. Any R_P is a local ring, for instance $\mathbb{Z}_{(2)}$.

e. The Galois group of $x^4 - 2$ is dihedral of order 8.

f. Note that $\alpha^2 = 7 + 2\sqrt{10}$ so α satisfies $(x^2 - 7)^2 = 40$ which gives $m(x) = x^4 - 14x^2 + 9$, and since the degree of the extension is 4, this polynomial is irreducible.

2. Define $\phi : \mathbb{A}^2 \to \mathbb{A}^3$ by

$$\phi(x, y) = (x, y, xy).$$

This is a regular map (x, y, and xy are polynomials) and it is clear that the image lies in the zero set of the polynomial $xy - z$. Moreover any point in this zero set satisfies $z = xy$, so is of the form (x, y, xy). Thus ϕ is $1-1$ and onto as a map from $\mathbb{A}^2$ to V.

Now define a map $\psi : \mathbb{A}^3 \to \mathbb{A}^2$ by $\psi(x, y, z) = (x, y)$. This is again clearly a regular map. When restricted to V it is one-to one and onto and one easily checks that ψ and ϕ are mutual inverses from $\mathbb{A}^2$ to V and back. Thus the two varieties are isomorphic.

Now $k[V] = k[x, y, z]/(xy - z)$ and $k[\mathbb{A}^2] \cong k[s, t]$. The associated map $\tilde{\phi}$ is just composition with ϕ. It maps $k[V]$ to $k[\mathbb{A}^2]$ and with $\tilde{\phi}(\overline{x}) = s$ and $\tilde{\phi}(\overline{y}) = t$. Notice this is an isomorphism and is well-defined since $xy - z$ vanishes on V. Its inverse map takes s back to $\overline{x}$ and t back to $\overline{y}$.

3. Let $e^{2\pi i/3} = \zeta_3$ be a primitive cube root of unity. Since $\zeta_3 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$, we get that the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$. The extension has degree 6, thus the Galois group is the full symmetric group S_3 of permutations on the 3 roots, $\sqrt[3]{2}$, $\sqrt[3]{2}\zeta_3$, $\sqrt[3]{2}\zeta_3^2$.

Define:

$$\sigma(\sqrt[3]{2}) = \sqrt[3]{2}\zeta_3, \ \sigma(\sqrt{-3}) = \sqrt{-3}.$$
$$\tau(\sqrt[3]{2}) = \sqrt[3]{2}, \ \tau(\sqrt{-3}) = -\sqrt{-3}.$$

Then σ and τ generate the Galois group, corresponding to a 3-cycle and a transposition in S_3. Notice that τ is just the restriction of complex conjugation.

There is a subgroup of size 3, generated by σ. The corresponding fixed field is $\mathbb{Q}(\sqrt{-3})$, which is Galois (splitting field of $x^2 - 3$) since the subgroup is normal.

There are 3 subgroups of size 2, corresponding to the three transpositions $\{\tau, \tau\sigma, \tau\sigma^2.\}$ The fixed field of $< \tau >$ is just $\mathbb{Q}(\sqrt[3]{2})$. The other two correspond to $\mathbb{Q}(\sqrt[3]{2}\zeta_3)$ and $\mathbb{Q}(\sqrt[3]{2}\zeta_3^2)$.

4. By assumption $L/\mathbb{Q}$ is Galois and p divides the order of the Galois group $G = Gal(L/\mathbb{Q})$. By Cauchy's theorem G has a subgroup of size p, which corresponds to an intermediate field F with $[L : F] = p$, by the FTOGT. If $\alpha \notin F$ then $F(\alpha)$ is strictly larger than F and thus must be all of L (since $[L : F]$ is prime). Suppose then that $\alpha \in F$.

Recall that L is just the splitting field of the minimal polynomial of α. Since $L \neq \mathbb{Q}(\alpha)$, there is another root β (called a Galois conjugate of α) such that $\beta \notin F$. If σ is an element of the Galois group taking α to β, then $\sigma(F) := F'$ will give us the desired field. It is still index p since σ is an automorphism. It does not contain α so $F'(\alpha) = L$.

5. Let $V = \mathcal{Z}(J)$ be an irreducible variety and suppose it is not connected, so it is the disjoint union of two open sets. Since the topology of V is the subspace topology inherited from the Zariski topology on $\mathbb{A}^n$, we can assume there are open sets U_1 and U_2 in $\mathbb{A}^n$ such that the intersections with V are disjoint and together give V. The complements of U_i are closed sets, so are of the form $\mathcal{Z}(I_1)$ and $\mathcal{Z}(I_2)$ for radical ideals I_1 and I_2. From the definitions we have:

$$U_1 \cap V = U_2^c \cap V = \{\overline{a} \mid f(\overline{a}) = 0 \forall f \in J,\ h(\overline{a}) = 0 \forall h \in I_2.\} = \mathcal{Z}(J + I_2)$$

$$U_2 \cap V = U_1^c \cap V = \{\overline{a} \mid f(\overline{a}) = 0 \forall f \in J,\ h(\overline{a}) = 0 \forall h \in I_1.\} = \mathcal{Z}(J + I_1)$$

Thus $U_i \cap V$ are proper subvarieties whose union is V, contradicting V being irreducible. (One can check further that $J = I_1 + I_2$, although this is not necessary here).

6. By the Nullstellensatz we have a bijection between varieties and radical ideals, which reverses inclusions. Thus the descending chain of varieties corresponds to an ascending chain of ideals $I_1 \subset I_2 \subset \cdots$ in $k[\mathbb{A}^n] \cong k[x_1, x_2, \ldots, x_n]$. The latter ring is Noetherian, by the Hilbert basis theorem. Thus the ascending chain of ideals must stabilize, and thus the corresponding descending chain of varieties does as well (since the maps are bijections).