

## Review

$G$  a finite group.

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$$

- $e = z_1 + z_2 + \dots + z_r$  central primitive idempotents
- $S_1, \dots, S_r$  simple  $\mathbb{C}G$ -modules,  $S_i \cong$  col vectors for  $M_{n_i}(\mathbb{C})$   
 $\dim S_i = n_i, \quad z_j S_i = z_i S_i \quad \forall j \neq i$

- Let  $\rho_1, \dots, \rho_r$  be reps,  $\chi_1, \chi_2, \dots, \chi_r$  corr irreducible characters,  $\chi_i(1) = n_i$

Thm Let  $M$  be a  $\mathbb{C}G$ -module. Then  $\chi_M = \sum_{i=1}^r a_i \chi_i$  with  $a_i \geq 0$  unique  
so  $M \cong \sum_{i=1}^r a_i S_i$ . Further  $M \cong N$  iff  $\chi_M = \chi_N$ .

Recall A class function is  $f: \mathbb{C}G \rightarrow \mathbb{C}$  such that  $f(xgx^{-1}) = f(g) \quad \forall x, g \in G$ .

Prop The set of class functions is a vector space /  $\mathbb{C}$  with basis  $\{\chi_i\}_{i=1}^r$

## Basic Properties

Let  $\rho$  be a representation w/ character  $\chi_\rho = \chi$

1.  $\rho(g)$  is diagonalizable.
2.  $\chi(g)$  is a sum of  $n^{\text{th}}$  roots of unity, where  $n = \dim \rho$ . In particular  $\chi(g)$  is an algebraic integer.
3.  $\rho^*(g) = (\rho(g)^{-1})^T$  is a representation, called contragredient dual,  
and  $\chi_{\rho^*}(g) = \overline{\chi(g)} = \chi(g^{-1})$
4.  $|\chi(g)| \leq \chi(1)$  with equality iff  $\rho(g) = \lambda I_n$
5.  $\chi(g) = \chi(1)$  iff  $g \in \text{Ker } \rho$

Proof - - -

Regular rep...

Lemma Let  $\pi$  be the character of regular representation. Then:

$$1. \pi = \sum_{i=1}^r n_i \chi_i = \sum_{i=1}^r \chi_i(1) \chi_i \text{ so } \pi(g) = \sum_{i=1}^r \chi_i(1) \chi_i(g)$$

$$2. \pi(g) = \begin{cases} |G| & g=1 \\ 0 & \text{else} \end{cases}$$

Now we can compute C.O.B matrix from conj. class sums to  $\{z_i\}$

Prop 
$$z_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(1) \chi_i(g^{-1}) g$$

Proof

Let  $z_i = \sum_{g \in G} a_g g$ . Then:

$$\pi(z_i g^{-1}) = a_g |G| = \sum_{j=1}^r \chi_j(1) \chi_j(z_i g^{-1})$$

But  $\rho_j(z_i g^{-1}) = \rho_j(z_i) \rho_j(g^{-1}) = \delta_{ij} \rho_j(g^{-1})$ . Thus

$$a_g |G| = \chi_i(1) \chi_i(g^{-1}) \text{ so } a_g = \frac{1}{|G|} \chi_i(1) \chi_i(g^{-1}) //$$

Def The character table of  $G$  has rows indexed by irreducible characters, columns by conjugacy classes, and entries are character values

- Thus it is a square matrix
- Traditionally 1<sup>st</sup> row is triv. char, 1<sup>st</sup> col is identity

Ex

$$G = S_3$$

	$\{e\}$	$\{R\}$	$\{R^2\}$
triv = $\chi_1$	1	1	1
sgn = $\chi_2$	1	-1	1
$\chi_3$	2	0	-1

Notice 1<sup>st</sup> col is character degrees

Thm (Orthogonality Relations) Let  $g_i \in C_i$  where  $\{C_i\}_{i=1}^r$  are conj classes

1. (Row  $\perp$ )  $\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}$

2. (Col  $\perp$ )  $\sum_{k=1}^r \chi_k(g_i) \chi_k(g_j) = \delta_{ij} |C_i(g_i)| = \delta_{ij} \frac{|G|}{|C_i|}$

Prks Later:  $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$  is inner product on class functions  
 $\{\chi_i\}$  is orthonormal basis

Ex  $G = D_8$   $G' = Z(G) = \{e, r^2\}$

$$G/G' \cong V = \{eZ, sZ, rZ, srZ\}$$

so 4 1-dim reps

	$e$	$r^2$	$\{s, sr^2\}$	$\{r, r^3\}$	$\{sr, sr^3\}$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	1	-1	-1	1
$\chi_3$	1	1	-1	1	-1
$\chi_4$	1	1	1	-1	-1
$\chi_5$	2	-2	0	0	0

red entries from  $\perp$  relations

## Proof of Orthogonality Relations

We know  $z_i z_j = \delta_{ij} z_i$ . Use formula for  $z_i$ :

Row

$$\left( \frac{1}{|G|} \sum_{x \in G} \chi_i(11) \chi_i(x^{-1}) x \right) \left( \frac{1}{|G|} \sum_{y \in G} \chi_j(11) \chi_j(y^{-1}) y \right) \\ = \left( \frac{1}{|G|} \sum_{g \in G} \chi_i(11) \chi_i(g^{-1}) g \right) \delta_{ij}$$

Compare Coef of identity on each side:

LHS  $\frac{1}{|G|^2} \sum_{g \in G} \chi_i(11) \chi_i(g^{-1}) \chi_j(11) \chi_j(g)$

RHS  $\frac{1}{|G|} \delta_{ij} \chi_i(11)^2$

so  $\frac{1}{|G|} \delta_{ij} \chi_i(11)^2 = \frac{1}{|G|^2} \chi_j(11) \chi_j(11) \sum_{g \in G} \chi_i(g^{-1}) \chi_j(g)$  //

Column

Start w/ row  $\perp$  and collect terms

$$\frac{1}{|G|} \sum_{k=1}^r \chi_i(g_k) \overline{\chi_j(g_k)} \frac{|G|}{|G(g_k)|} = \delta_{ij}$$

$$\sum_{k=1}^r \chi_i(g_k) \left( \frac{1}{|G(g_k)|} \overline{\chi_j(g_k)} \right) = \delta_{ij}$$

Let  $A = r \times r$  matrix  $a_{ik} = \chi_i(g_k)$  char table

$B = r \times r$  matrix  $b_{kj} = \overline{\chi_j(g_k)} / |G(g_k)|$  so  $AB = I'$

Thus  $BA = I$  so  $\sum_{k=1}^r b_{jk} a_{ki} = \delta_{ij}$  Thus

$$\sum_{k=1}^r |G(g_k)|^{-1} \overline{\chi_k(g_j)} \chi_k(g_i) = \delta_{ij}$$

$$\sum_{k=1}^r \chi_k(g_i) \overline{\chi_k(g_j)} = |G(g_i)| \delta_{ij} //$$

Recall 1. Alg integers are roots of monic polynomials in  $\mathbb{Z}[x]$ , they form a subring of  $\mathbb{C}$ .

2.  $\{\text{Alg integers}\} \cap \mathbb{Q} = \mathbb{Z}$ .

~~Lemma~~ Suppose  $x = \sum_{g \in G} a_g g \in \mathbb{C}G$  and  $a_g \in \mathbb{Z}$ . Suppose  $u \in \mathbb{C}G$  and

$xu = \lambda u$ . Then  $\lambda$  is an alg int.

Proof Let  $G = \{g_1, \dots, g_n\}$ . Then  $xg_i = \sum_j a_{g_j} g_j$  with  $a_{g_j} \in \mathbb{Z}$ . Thus  $x$  acts by integer matrix  $A$  on  $\mathbb{C}G$  and  $\lambda$  is e-value.  $\square$

Thm. Let  $C_j$  be conj class sum and  $w_{ij} = \frac{|G|}{|C_j|} \cdot \frac{\chi_i(g_j)}{\chi_i(1)}$ .

1.  $\rho_i(C_j) = w_{ij} I$

2.  $w_{ij}$  is an alg integer.

Proof 1. Since  $C_j$  is central,  $\rho_i(C_j) = \lambda C_j$  some  $\lambda$ . But

$$\begin{aligned} \rho_i(C_j) &= \sum_{g \in C_j} \rho_i(g) \text{ so} \\ \text{tr}(\rho_i(C_j)) &= |\text{class } C_j| \chi_i(g_j) \\ &= \frac{|G|}{|C_j|} \chi_i(g_j) \end{aligned}$$

$$\text{Thus } \lambda = \frac{|G|}{|C_j|} \cdot \frac{\chi_i(g_j)}{\chi_i(1)}$$

2. We just showed  $C_j$  acts by scalar  $\omega_j$  on  $S_j \in \mathbb{C}G$ , so that scalar is an alg int, by Lemma. //

Thm Let  $\chi$  be irreducible. Then  $\chi(1) \mid |G|$ .

Proof

$$\sum_{i=1}^r \frac{|G|}{|C_G(g_i)|} \frac{\chi(g_i)}{\chi(1)} \overline{\chi(g_i)} = \sum_i \omega_{S_i} \overline{\chi(g_i)} \text{ is alg int.}$$

But this is  $\frac{|G|}{\chi(1)}$  by col orthogonality.

Thus  $\frac{|G|}{\chi(1)} \in \mathbb{Q} \cap \{\text{alg int}\} = \mathbb{Z}$ . //

### Commutator Char

Def Let  $g \in G$ . Define  $\chi^{[2]}(g) = \#\{(x, y) \mid g = xyx^{-1}y^{-1}\}$ ,

$\chi^{[2]}$  is obviously a class function.

Exercise  $\chi^{[2]} = \sum_{i=1}^r \frac{|G|}{\chi_i(1)} \chi_i$ .

Problem Find a  $\mathbb{C}G$ -module with character  $\chi^{[2]}$ , thus proving  $\chi(1) \mid |G|$  via alg integer method.

Ex Character table of  $S_3$

- $S_3$  is homomorphic image  $S_4/V$
- 2 more via  $\perp$

- $e \rightarrow e$
- $(12)(34) \rightarrow e$
- $(12) \rightarrow (12)$
- $(1234) \rightarrow (12)$
- $(123) \rightarrow (123)$

class	1	3	6	6	8
represent	$1$	$(12)(34)$	$(12)$	$(1234)$	$(123)$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	1	-1	-1	1
$\chi_3$	2	2	0	0	-1
$\chi_4$	3	$a=1$	1	-1	0
$\chi_5$	3	$b=-1$	-1	1	0

$3a+3b = -6$   
 $a^2+b^2 = 1$

Burnside  $p^a q^b$  Thm

Suppose  $|G| = p^a q^b$  with  $p, q$  prime. Then  $G$  is solvable.

Proof WLOG  $p < q$ .

Lemma 1 Suppose  $\alpha$  is a sum of  $n$  roots of unity and  $\alpha^n$  is an alg int. Then  $\alpha^n = 0$  or a root of unity.

Proof Let  $f(x)$  be minpoly  $(\alpha^n)$ , w/ constant term  $c$ . Let  $B_1, \dots, B_n$  be Galois conjugates of  $\alpha^n$ , so  $c = \prod B_i$ .

Now  $\alpha^n = (\lambda_1 + \dots + \lambda_n)^n$  each  $\lambda_i$  a root of unity, then each  $B_i$  is also of this form. Thus  $|B_i| \leq 1 \forall i$ .

Case 1 Some  $|B_i| < 1 \Rightarrow c = 0 \Rightarrow \alpha^n = 0$

Case 2 So  $|\alpha^n| = 1$  so  $\alpha^n = n\lambda$ . //

Lemma 2 Suppose  $(\chi_i(1), \frac{|G|}{|G_i(g_j)|}) = 1$  for some  $i, j$ .

Then either  $\chi_i(g_j) = 0$  or  $\chi_i(g_j) = \lambda \chi_i(1)$  some root of unity  $\lambda$ .

Proof  $\exists a, b \in \mathbb{Z}$  s.t.  $a\chi_i(1) + b \frac{|G|}{|G_i(g_j)|} = 1$  so

$$a\chi_i(g_j) + b \frac{|G|}{|G_i(g_j)|} \frac{\chi_i(g_j)}{\chi_i(1)} = \frac{\chi_i(g_j)}{\chi_i(1)}$$

Thus  $\frac{\chi_i(g_j)}{\chi_i(1)}$  is an alg integer, apply Lemma 1. //

(9)

Thm Suppose  $g \in G$  and  $|\text{Class } g| = p^a > 1$ . Then  $G$  is not simple.

Proof

Col 1 gives

$$1 + \sum_{i=2}^r \chi_i(1) \chi_i(g) = 0.$$

If  $p$  divided every  $\chi_i(1)$  such that  $\chi_i(g) \neq 0$  then we would get

$$1 + \sum p \cdot \frac{\chi_i(1)}{p} \chi_i(g) = 0$$

$$-1 = p \sum \frac{\chi_i(1)}{p} \chi_i(g) \text{ so } -1/p \text{ is alg int. //}$$

Thus  $\exists j$  with  $\chi_j(g) \neq 0$  and  $p \nmid \chi_j(1)$

Thus  $(\chi_j(1), \frac{|G|}{|C_G(g)|}) = 1$  so

$$\chi_j(g) = \lambda \chi_j(1) \text{ by Lemma 2, same nonprincipal char } \chi_j.$$

Thus  $p_j(g) \in Z(p|G)$  so  $G$  is not simple. //

Remark No known group theory proof of this thm!!

COR  $p^a \nmid a^b$  thm

Proof Let  $x \in Z(\text{Syl}_p)$ . Then  $P \leq C_G(x)$  so

$$|\text{Class } x| = a^c \neq$$