Lecture 8

p488 #4, 6, 10, 11, 14, 17
p494 #3, 11, 18, 21, 22, 24, 37, 39

Review $T: V \to V$ linear, V/F. Then $\lambda \in F$ is an eigenvalue if $T\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$. $\{\vec{v} \mid T\vec{v} = \lambda\vec{v}\}$ is called the λ-eigenspace, subspace.

Similarly define eigenvalue of a matrix.

$T v = \lambda v \Leftrightarrow (\lambda I - T)\vec{v} = 0 \Leftrightarrow \lambda I - T$ singular $\Leftrightarrow \det(\lambda I - T) = 0$.

Thus

Thm The eigenvalues are roots of characteristic polynomial $c_T(x) = \det(xI - T)$. (Similarly $c_A(x)$ for a matrix).

Obvious Fact: $c_T(x)$ is monic of degree $\dim V$.

Recall $V \cong F[x]/(a_1(x)) \oplus \cdots \oplus F[x]/(a_s(x))$ $\quad a_i(x) \mid a_{i+1}(x)$, monic

as an $F[x]$ module. Under obvious choice of basis we get

$$[T] = \begin{pmatrix} C_{a_1(x)} & & \\ & \ddots & \\ & & C_{a_s(x)} \end{pmatrix}$$

is rational canonical form, determines V up to $\cong$ (so A up to conj)

Def Let $m_T(x)$ be unique, monic polynomial so $\mathrm{Ann}(V) = (m_T(x))$, called minimal polynomial of T (or A).

Prop

1. $m_T(x) = a_s(x)$ the largest invariant factor.
2. $c_T(x) = \prod_{i=1}^{s} a_i(s)$ so $m_T(x) \mid c_T(x)$
3. $c_T(x)$ divides some power of $m_T(x)$, so $c_T(x)$ & $m_T(x)$ have same roots.

Applications / Problems

Thm 1. Rational canonical form is unique, and determines A uniquely up to conjugation.

2. Suppose $A, B \in M_{n\times n}(F)$ and $F \subseteq K$. Then A and B are similar $/F$ (i.e. $A = PBP^{-1}$, $P \in GL_n(F)$) iff similar $/K$ (i.e. $A = QBQ^{-1}$, $Q \in GL_n(K)$)

Proof 1. is just part of our fund thm.

2. Put A in RCF$/F$. Well this is still in RCF$/K$ so same inv. factors over any field.
Thus A, B similar $/K \to$ same RCF$/K \to$ same RCF$/F \to$ similar $/F$.

Rmk Motivation for rational canonical form, the RCF of a matrix always lies in smallest subfield containing it's entries

Thm (Cayley-Hamilton) A satisfies its characteristic polynomial.
Proof Since $m_A(x) \mid c_A(x)$ then $m_A(A) = 0 \Rightarrow c_A(A) = 0$.

Problem How to find invariant factors?
Ex 22-25, row & col ops on $xI - A$ to put matrix in Smith Normal Form

MAPLE: ratform(A)

Algorithm on p 481-482

Jordan Canonical Form   Assume $F = \bar{F}$ alg closed.

Then prime poly of form $(x-\lambda)$ so $V \cong F[x]/(x-\lambda_1)^{a_1} \oplus \cdots \oplus F[x]/(x-\lambda_d)^{a_d}$

Check: $F[x]/(x-\lambda)^a$, action of x in "obvious" basis is

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

Jordan block of size k $(=5)$ and eigenvalue λ. $J_k(\lambda)$

Def A matrix is in <u>Jordan canonical form</u> if it is block diagonal with Jordan blocks

<u>Thm</u>

1. Two matrices are similar iff they have same JCF, up to order of the blocks
2. Every matrix can be conjugated to its JCF.

<u>Ex</u>

$$\left(\begin{array}{ccccccccc} 3 & 1 & 0 & 0 & 0 & 0 & & & \\ 0 & 3 & 1 & 0 & 0 & 0 & & & \\ 0 & 0 & 3 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 3 & 0 & 0 & & & \\ 0 & 0 & 0 & 2 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & 2 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \end{array}\right)$$

$$\left(\begin{array}{ccc|c|cc|ccc} 3 & 1 & 0 & & & & & & \\ 0 & 3 & 1 & & & & & & \\ 0 & 0 & 3 & & & & & & \\ \hline & & & 3 & & & & & \\ \hline & & & & 2 & 1 & 0 & & \\ & & & & 0 & 2 & & & \\ \hline & & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 & 0 \end{array}\right)$$

nilpotent J.b.

How To compute? See book
or Jordan(A) in Maple

Lemma

1. $\text{charpoly}(J_k(\lambda)) = (x-\lambda)^k = \text{minpoly}(J_k(\lambda))$

2. min poly of matrix easy to compute.

$$\begin{pmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & & & \\ 0 & & & 2 & 1 & \\ & & & 0 & 2 & \\ & & & & & 3 \end{pmatrix} \qquad \begin{aligned} c.p. &= (x-2)^5(x-3) \\ m.p. &= (x-2)^3(x-3) \end{aligned}$$

3. Easy to get dimensions of eigenspace

$$\text{Geometric Mult} = \dim E_\lambda = \underset{\text{e-value } \lambda}{\#\text{ Blocks}} = \dim(\ker(\lambda I - A))$$

$$\text{Algebraic Mult} = \text{degree of } (x-\lambda) \text{ in } CP$$
$$= \dim\Big(\underbrace{\bigcup_n \ker(\lambda I - A)^n}_{\text{generalized } \lambda \text{ eigenspace}}\Big)$$

Def A is <u>diagonalizable</u> (over F) if $\exists\, P \in GL_n(F)$ with $PAP^{-1} = D$ diagonal matrix.

Thm Let $F = \overline{F}$. TFAE

1. A is diagonalizable
2. $m_A(x) = \prod$ distinct linear terms
3. Geom Mult = Alg Mult $\forall$ evalue.

COR If $C_A(x)$ has distinct roots in F, then A is diagonalizable.