Lecture 7

Review

1. $\mathbb{Z}$-modules $\xleftrightarrow[\text{cats}]{\cong \text{ of}}$ abelian groups
2. F a field,

   $F[x]$ modules $\longleftrightarrow$ Vector space V/F w/ linear $T: V \to V$

   $\cong$ corr to change of basis

3. $\mathbb{Z}$ and $F[x]$ both PID's.

<u>Goal</u> Nice thm on f.g. modules/a PID, use to classify f.g. abelian groups, linear trans

Def

1. A left R-module M is <u>Noetherian</u> if any increasing chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules stabilizes. (called ACC)

2. R is (left) Noetherian if ${}_R R$ is Noetherian, i.e. no ∞ AC of left ideals

<u>Exs</u>

1. Any PID is Noetherian. Proof
2. $F[x_1, x_2, \dots]$ not Noeth. $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \dots$
3. Hilbert Basis Thm: R Noetherian $\Rightarrow R[x]$ is Noetherian.
4. Any f.g. comm ~~ring~~ alg
5. R Noeth $\Rightarrow R/I$ Noeth.
6. R Noeth $\Leftrightarrow$ <u>Every</u> f.g. R-module is Noeth

Thm TFAE

1. M is Noeth.
2. Every nonempty set of submodules has a max elt under inc.
3. <u>Every</u> submodule is f.g.

Proof $1 \to 2$ if not build AC

$2 \to 3$ Let $\mathcal{C} = \{$f.g. submodules of $N \subseteq M.\}$ Chose $\overline{N}$ max in $\mathcal{C}$. chose $(\overline{N}, x)$

<u>$3 \to 1$</u> Suppose $M = M_1 \subseteq M_2 \subseteq \dots$ Let $N = \bigcup$, find gens

Cor PID $\to$ Noetherian

<u>Some linear Alg type Results</u>

<u>Recall</u> <u>Def</u> $Ann(M) = \{r \mid rm = 0 \ \forall m \in M\}$
$Tor(M) = \{x \mid rx = 0 \text{ some } r \neq 0\}$ Torsion submodule.

<u>Thm</u> Suppose R is an int domain, $M \cong R^n$ free of rank n. $\{m_1, \dots, m_{n+1}\} \subseteq M$.
Then $\exists\ r_1, r_2, \dots, r_{n+1}$, not all 0 with $\sum r_i m_i = 0$.

<u>Proof</u> Embed R is field of fractions K, $R^n \subseteq K^n$ vector space.

<u>Def</u> M an R-module, the <u>rank</u> of M is maxima # of lin. ind. elts in M.

Suppose M f.g. R-module, gens $\{m_1, m_2, \dots, m_n\}$. Then $R^n \xrightarrow{\varphi} M$ so
$M \cong R^n / \ker\varphi$.

<u>Thm</u> Suppose R is a PID and $N \leq R^n$ submodule. Then:

1. N is free of rank $\leq m$.

2. $\exists$ a basis $\{y_1, y_2, \dots, y_n\}$ of R^n and $a_1 | a_2 | \dots | a_m$ so
$\{a_1 y_1, a_2 y_2, \dots, a_m y_m\}$ basis of N.

<u>Ex</u> $M \cong \mathbb{Z} \oplus \mathbb{Z}$ $N = (4, 4)$ $a_1 = 4$ $y_1 = (1, 1)$ $y_2 = (1, -1)$

Def M is cyclic if $M = Rm$ some $m \in M$.

Thm (Inv Factor Form) R a PID, M a f.g. R-module. Then

1. $M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$ w/ $a_i \neq 0$, $a_1 | a_2 | a_3 | \cdots | a_m$

2. M is torsion free $\leftrightarrow$ free

3. $Tor(M) \cong R/(a_1) \oplus \cdots \oplus R/(a_m)$

Def 1. r is called the free rank (or Betti #) of M
2. $\{a_i\}$ called invariant factors, they are unique up to units.

Ex $R = \mathbb{Z}$ $\quad M \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/60\mathbb{Z}$

Proof Choose min generating set so $M \cong R^n/\ker \pi$, use previous Thm + induction

Recall

CRT $\Rightarrow$ if $a = p_1^{a_1} \cdots p_s^{a_s}$ distinct primes then
$$R/(a) \cong R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_s^{a_s})$$

Thm (Elt divisor Form)
$$M \cong R^r \oplus R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_s)^{a_s}$$

Def $\{p_1^{a_1}, \ldots, p_s^{a_s}\}$ are elementary divisors of M, note these primes may not be distinct.

Ex $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/60\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

E.D $\{2, 2, 3, 3, 4, 5\}$

Def $\bigoplus R/p^{a_i}$ fixed p called <u>p-primary component</u>.

Thm M_1, M_2 f.g. over PID R are $\cong$ iff same free rank and same invariant factors. (elt divisors)

Fact

1. Easy to go back and forth, elt divisors are prime power factors of inv factors
2. Largest inv factor is just $\prod$ largest power of each prime.

Ex $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

E.D. $(2, 2, 2^3, 3, 3^2, 5)$ Inv Factors $2^3 \cdot 3^2 \cdot 5,$ $2 \cdot 3,$ $2,$

$\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/360\mathbb{Z}$

COR Fund Thm of f.g. abelian groups

Ex $|A| = 2^4 \cdot 3^2 \cdot 5^2$

ED	$(2^4$	3^2	5^2
	or	or	or
	$2^3, 2$	$3, 3$	$5, 5$
	or		
	$2^2, 2^2$		
	or		
	$2^2, 2, 2$		
	or		
	$2, 2, 2, 2$		

so 20 possible.

Rational canonical Form

Consider $F[x]$ module V, $x: V \to V$. Then:

$$V \cong F[x]/(a_1(x)) \oplus F[x]/(a_2(x)) \oplus \cdots F[x]/(a_n(x))$$

with $a_i(x) | a_{i+1}(x)$

Let $a(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$. How does x act on $F[x]/(a(x))$?

Basis of $F[x]/(a(x))$ is $\langle \bar{1}, \bar{x}, \bar{x^2}, \ldots, \overline{x^{k-1}} \rangle$

Matrix of x in this basis is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & & 0 & -b_2 \\ 0 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & -b_{k-1} \end{pmatrix}$$

called <u>companion matrix</u> to $a(x)$.

<u>Def</u> A matrix is in <u>rational canonical form</u> if it is the direct sum of companion matrices for monic polynomials $a_1(x) | a_2(x) | \cdots | a_m(x)$.

The $a_i(x)$ are called invariant factors of the matrix.

Ex $a_1(x) = x-1 \quad a_2(x) = x^2-1 \quad a_3(x) = (x+2)(x^2-1) = x^3+2x^2-x-2$

$$a_1(x) \to (1) \qquad a_2(x) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad a_3(x) \to \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$ is in RCF

Lin Alg Review

$T: V \to V$

Define
- Eigenvalue
- Eigenspace

Thm (TFAE)
1. λ an e-value
2. $\lambda I - T$ singular
3. $\det(\lambda I - T) = 0$

$$C.P. = \det(xI - T)$$

Def Let $m(x)$ be unique monic polynomial generating $\operatorname{ann}(V)$, called minimal polynomial $m_T(x)$

similarly minimal poly of a matrix

COR minimal poly is largest invariant factor.