Lecture 6

p 435 #4
441 #6
454 1,4,7

p. 454 # 7, 12, 13
p ~~454 #11~~

Review. R comm w/ 1, M an R-module. $\mathcal{T}(M) \cong R \oplus M \oplus M \otimes M \oplus \cdots$ tensor algebra.

- graded algebra
- universal among R-algebras containing M
- $\mathcal{T}^i(M) \cong M \otimes \cdots \otimes M$ i times

Def. $\mathcal{C}(M)$ = ideal gen by all $m_1 \otimes m_2 - m_2 \otimes m_1$ graded ideal

$\mathcal{C}^k(M) = \mathcal{C}(M) \cap \mathcal{T}^k(M)$

$\mathcal{T}(M)/\mathcal{C}(M)$ is <u>symmetric algebra</u>

$\mathcal{T}^k(M)/\mathcal{C}^k(M) \cong S^k(M)$ k^{th} symmetric power

Thm 1. $S^k(M) \cong M \otimes M \otimes \cdots \otimes M \big/ (m_1 \otimes \cdots \otimes m_k - m_{\sigma 1} \otimes \cdots \otimes m_{\sigma k} \mid \sigma \in S_k)$

this is an R-module

2. U.P. $\varphi : M \times M \times \cdots \times M \to N$ any <u>symmetric</u> R multilinear map gives ! $\tilde{\varphi} : S^k(M) \to N$

3. $S(M)$ has univ. property among <u>commutative</u> R-algs containing M.

Ex. $V = \langle v_1, \ldots, v_n \rangle$ vector space / F. Then

1. $S(V) \cong F[x_1, x_2, \ldots, x_n]$

2. $\dim S^k(V) = \binom{k+n-1}{n-1}$

Exterior Algebras

Def $A(M)$ = ideal of $S(M)$ gen by all $x \otimes x$, graded ideal.

$\Lambda(M) := S(M)/A(M)$ <u>exterior algebra</u>

Notation: $\overline{m_1 \otimes \cdots \otimes m_n} = m_1 \wedge m_2 \wedge \cdots \wedge m_n$, <u>wedge product</u>

$\Lambda^k(M) = \tau^k(M)/A^k(M)$ <u>k^{th} exterior power</u>.

Rmk $m \wedge m' = -m' \wedge m$ but $a \wedge b \neq -b \wedge a$ in general.

Ex $m_1 \wedge m_2 = a$ $\quad m_3 = b$ $\quad a \wedge b = b \wedge a$

Thm

1. $\Lambda^k(M) = M \otimes M \otimes \cdots \otimes M \Big/$ sub gen by $m_1 \otimes \cdots \otimes m_k$ some $m_i = m_j$ $i \neq j$

2. Any <u>alternating</u> multilinear map $M \times M \cdots \times M \to N$ gives ! map $\Lambda^k(M) \to N$

3. $\Lambda(M)$ universal w.r.t. R-algebras such that $a^2 = 0$.

Important special case, $V = \langle v_1, v_2, \ldots, v_n \rangle$ vector space $/F$

- Recall that $v_{i_1} \wedge \cdots \wedge v_{i_k}$ is 0 if any repeat.

Thm 1. $\Lambda^k(V)$ has basis $\{ v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n \}$

2. $\dim \Lambda^k(V) = \binom{n}{k}$

3. Thus $\Lambda(V)$ is a finite dimensional algebra of dimension 2^n.

Rmk Same thm holds for M free of rank n

$\Lambda^k(M)$ free of rank $\binom{n}{k}$

Fact $\mathcal{T}^k$ and Λ^k and S^k are functors.

Given $\varphi: M \to N$ define $\mathcal{T}^k(\varphi): \mathcal{T}^k(M) \to \mathcal{T}^k(N)$

by $\mathcal{T}^k(m_1 \otimes \cdots \otimes m_k) = \varphi(m_1) \otimes \cdots \otimes \varphi(m_k)$

Ex $\mathcal{T}^k\varphi$: $\mathcal{C}^k \to \mathcal{C}^k$, $A^k \to A^k$ so get

$S^k\varphi: S^k(M) \to S^k(N)$

$\Lambda^k\varphi: \Lambda^k(M) \to \Lambda^k(N)$

<u>Example</u> $\dim V = n$ $\quad T: V \to V$ gives $\Lambda^n T: \Lambda^n(V) \to \Lambda^n(V)$

$V = \langle v_1, \ldots, v_n \rangle$ then basis of $\Lambda^n V$ is $v_1 \wedge v_2 \wedge \ldots \wedge v_n$

<u>Thm</u> Let $[T] = A$ be matrix of T w/t basis. Then

$$\Lambda^n T(v_1 \wedge \ldots \wedge v_n) = \det A \; v_1 \wedge \ldots \wedge v_n$$

<u>Proof</u> Define $D: M_{n\times n}(F) \to F$ by $D(A) = D(T)$
- check it is a det function
- Check it is 1 on id.

<u>Ex</u> $R = \mathbb{Z}[x,y]$

$I = (x,y)$

1. $\Lambda^2 R = 0$ since R free of rank 1

2. $\Lambda^2 I \neq 0$ proof

$\varphi(ax+by, cx+dy) = (ad-bc) \bmod (x,y)$

is alt R bilinear map

$I \times I \to \mathbb{Z} = R/I$

$\varphi(x,y) = 1$ so $\mathbb{Z}$ $x \wedge y \neq 0$

So $0 \to I \to R$ submodule

$0 \to \Lambda^2(I) \not\to \Lambda^2(R)$

Λ^n is not left exact

Symmetric & Alternating Tensors

Def $z \in \mathcal{T}^k(M)$ is a symmetric tensor if $\sigma z = z$ $\forall \sigma \in S_k$
alternating tensor if $\sigma z = \operatorname{sgn} z$ $\forall \sigma \in S_k$.

Ex $\cancel{m_1 \otimes m_2 \otimes m_3 \in \mathcal{T}^3}$

$m_1 \otimes m_2 + m_2 \otimes m_1 \in \mathcal{T}^2$ sym

$m_1 \otimes m_1 \otimes m_3 + m_1 \otimes m_3 \otimes m_1 + m_3 \otimes m_1 \otimes m_1$

Ex $m_1 \otimes m_2 - m_2 \otimes m_1$ alt.

Prop

1. $\mathcal{C}^k(M)$, $\mathcal{A}^k(M)$ both preserved by S_k, so S_k acts on $S^k(M)$, $\Lambda^k(M)$
2. $\sigma w = w$ $\forall w \in S^k(M)$
   $\sigma w = \operatorname{sgn}\sigma\, w$ $\forall w \in \Lambda^k(M)$

Ex $(12)\, m_1 \wedge m_2 \wedge m_3 = m_2 \wedge m_1 \wedge m_3 = -m_1 \wedge m_2 \wedge m_3$

Submodule Symmetric Tensors $\xleftrightarrow{?}$ Quotient Module $S^k(M)$

Def $z \in \mathcal{T}^k(M)$ $\operatorname{Sym}(z) = \sum_{\sigma \in S_k} \sigma z$

$\operatorname{Alt}(z) = \sum_{\sigma \in S_k} \operatorname{sgn}\sigma\, z$

Ex $\operatorname{sym}(m_1 \otimes m_1 \otimes m_3) = 2 \cdot (m_1 \otimes m_1 \otimes m_3 + \cdots)$

Thm If $k!$ is a unit in R Then $\frac{1}{k!}\operatorname{Sym}$ is $\cong$
$\frac{1}{k!}\operatorname{Alt}$ is $\cong$