Lecture 5

Define V^*, dual basis

Thm V f.d. then $\mathcal{B}^*$ is a basis.

Rmk 1. Never True if $\dim V = \infty$, Ex $k[x]$
2. linear dual vs continuous dual, reflexive spaces.

Thm $ev: V \to V^{**}$ natural injection which is $\cong$

Fact $V \to V^*$ is contravariant functor, transposes matrices.

R comm ring w/ 1, V_1, V_2, W R-modules.

Def $\varphi: V_1 \times V_2 \to W$ is R-bilinear, similarly R-multilinear

Def $V \times V \to R$ call it a bilinear form or multilinear form.
Ex dot product on $\mathbb{R}^n$

Def
1. $\varphi: V_1 \times \ldots \times V_n \to R$ is alternating if $\varphi(V_1, V_2, \ldots, V_n) = 0$ whenever $V_i = V_{i+1}$
2. Symmetric if $\varphi(V_{\sigma 1}, \ldots, V_{\sigma n}) = \varphi(V_1, \ldots, V_n)$ $\forall \sigma \in S_n$

Prop φ alt n-form
1. $\varphi(V_{\sigma(1)}, \ldots, V_{\sigma(n)}) = \operatorname{sgn}\sigma \, \varphi(V_1, V_2, \ldots, V_n)$
2. Any $V_i = V_j$, $i \neq j$ then $\varphi(V_1, \ldots, V_n) = 0$
3. Replace V_i by $V_i + \lambda V_j$, $j \neq i$, preserves φ.

Proof

Rmk $\varphi(x,y) = -\varphi(y,x) \not\Rightarrow \varphi(x,x) = 0$ in char 2

COR. Suppose

$$W_1 = a_{11}\vec{v}_1 + \cdots + a_{n1}\vec{v}_n$$
$$W_2 = a_{12}\vec{v}_1 + \cdots + a_{n2}v_n$$
$$\vdots$$
$$W_n = a_{1n}\vec{v}_1 + \cdots + a_{nn}v_n$$

Then

$$\varphi(W_1, W_2, \ldots, W_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}\, \varphi(v_1, \ldots, v_n)$$

Proof expand

Def. An $n \times n$ det function on R is $\det : M_{n\times n}(R) \to R$ s.t.

1. det is alt n-form
2. $\det I = 1$

Thm $\exists!$ $n \times n$ det function $\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$

Proof We know $\det(\vec{e}_1, \ldots, \vec{e}_n) = 1$, use cor above to expand, check this actually works

Properties

1. $\det A = \det A^t$
2. $\det(AB) = \det A \det B$
3. R int domain then $\det A = 0 \leftrightarrow$ cols lin dep
4. Cofactor formula for inverse $\Rightarrow$ $\det A \in U(R) \leftrightarrow A \in Gl_n(R)$
5. Row/Col reduction to compute determinants

Tensor Algebras

R comm w/1, usually a field.

Recall R comm w/1, an R-algebra is a ring A w/ a hom $f: R \to A$ mapping $1 \to 1$ s.t. $f(R) \subset Z(A)$

Def M an R-module $\mathcal{T}^k(M) = M^{\otimes k}$, $\mathcal{T}^0(M) = R$. k-tensors

Define $\mathcal{T}(M) = R \oplus \underset{\overset{\|}{M}}{\mathcal{T}^1(M)} \oplus \mathcal{T}^2(M) \oplus \cdots = \bigoplus_{i \geq 0} \mathcal{T}^i(M)$

tensor alg. of M

Thm

1. $\mathcal{T}(M)$ is an R-alg containing M, which is a graded algebra
2. Given any R-alg A and $\varphi: M \to A$ R-mod thm $\exists!$ Φ $T(M) \to A$ ..

Proof..

Special Case V f.d. vs basis $\mathcal{B} = \{v_1, \ldots, v_n\}$

$\mathcal{T}^k(V)$ basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}$

Rmk Like poly alg, noncommutative, variables $v_1, v_2, \ldots, v_n$

Def Let $R = \bigoplus_{i \geq 0} R_i$ be graded ring $I \subset R$ is a graded ideal

if $I = \bigoplus_{i \geq 0} I \cap R_i$ S, R graded rings

$\varphi: S \to R$ is hom of graded rings if $\varphi(S_k) \subset R_k$.

Rmk R_0 is subalgebra

Ex $\mathbb{Z}[x]$ graded, $(1+x)$ not graded

(x^2) is

Thm Graded ring / graded ideal is graded

Def. The symmetric algebra $S(M) = \mathcal{T}(M) \big/ (m_1 \otimes m_2 - m_2 \otimes m_1 \ \ \forall m \in M)$

a. $\mathcal{C}(M)$ graded ideal, $S^k(M) = \mathcal{T}^k(M) / \mathcal{C}^k(M) \quad \overset{''}{\mathcal{C}}(M)$

Ex 1. This makes order not matter

2. $\mathcal{C}(M) \cap \mathcal{T}_1 = 0$ so $M \cong S^1(M)$

Thm 1. $S^k(M) \cong M \otimes M \cdots \otimes M \big/ (m_1 \otimes \cdots \otimes m_k - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)})$

2. $\varphi: M \times \cdots \times M \to N$ symmetric, multilinear then $\exists!$ R-module homo $\Phi: S^k(M) \to N$ w/ $\varphi = \Phi \circ \iota$

3. A any commutative R-alg, $\varphi: M \to A$ then $\exists!$ $\Phi: S(M) \to A$ ...

Ex V n-dimention, 1. $S(V) \cong F[x_1, x_2, \ldots, x_n]$

2. $\dim S^k(V) = \binom{k+n-1}{n-1}$

$\overset{''}{\#}$ monomials of degree k

Repeat for alternating maps...

Def. The exterior algebra of M is $\mathcal{T}(M) / A(M)$

$A(M)$ = ideal gen by all $m \otimes m$

$\Lambda(M) = \mathcal{T}(M) / A(M)$

write $m_1 \wedge m_2 \wedge \cdots \wedge m_k$ instead of $\overline{m_1 \otimes \cdots \otimes m_k}$.

Prop

1. $A(M)$ is graded ideal, so define $\Lambda^k(M) = \mathcal{T}^k(M)/A^k(M)$
   - $\Lambda^0(M) = R$ k^{th} exterior power.
     $\Lambda^1(M) = M$

2. $(m_1 \wedge \ldots \wedge m_i) \wedge (m_{j+1} \wedge \ldots \wedge m_3) = m_1 \wedge \ldots \wedge m_3$ called wedge product.

   Facts
   1. $m \wedge m' = -m' \wedge m$
   2. $m_1 \wedge \ldots \wedge m_k = 0$ if any repeats
   3. ✱ Not true that $\Lambda(M)$ is antic.
      $(m_1 \wedge m_2) \wedge m_3 = m_3 \wedge (m_1 \wedge m_2)$

Thm

1. $\Lambda^k(M) = M \otimes \ldots \otimes M \Big/ \Big($submodule gen by $m_1 \otimes \ldots \otimes m_k \mid m_i = m_j$ so

2. $\varphi: M^k \to N$ alt k-linear $\exists!$ $\Phi: \Lambda^k(M) \to N$..

Vector Space Example

$V = \langle v_1, \ldots, v_n \rangle$

$\Lambda^k(V)$ has basis $\{v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \mid 1 \le i_1 < \ldots < i_k \le n\}$

$\dim_F(\Lambda^k(V)) = \binom{n}{k}$

Rmk $\Lambda^n(V)$ is 1-dimensional, $\det \in \Lambda^n(V)$