Lecture 3

Universal Objects

Def $I \in \mathrm{ob}\,\mathcal{C}$ is <u>universal</u> (aka <u>initial</u>) if $\exists!\ I \to C \quad \forall C \in \mathcal{C}$.
$T \in \mathrm{ob}\,\mathcal{C}$ is <u>couniversal</u> (<u>terminal</u>) if $\exists!\ C \to T \quad \forall T \in \mathcal{C}$

<u>Ex/Rmks:</u>

1. $\{e\}$ is initial & terminal in Grp (called <u>zero object</u>)
2. Any two initial (terminal) objects are $\cong$.
3. $\mathbb{Z}$ is initial in Ring.

4. Suppose $\{A_i \mid i \in I\} \subset \mathrm{ob}\,\mathcal{C}$. Define new category $\mathcal{E}$ with
$\mathrm{ob}\,\mathcal{E}$ = pairs $(B, \{f_i\})$ $B \in \mathcal{C}$, $f_i : B \to A_i$

morphism $(B, \{f_i\}) \to (D, \{g_i\})$ is $h : B \to D$
with $f_i = g_i \circ h$.

Then $(\prod A_i, \{\pi_i\})$ is a terminal object in $\mathcal{E}$ if it exists

Cor $\prod A_i$ is unique up to $\cong$.

Similarly coproduct is initial obj.

i.e. given any $B \xrightarrow{f_i} A_i$ $\exists$ $h : B \to \prod A_i$ . . .

5. $f : A \to B$ hom of abelian groups. Let $\mathcal{C}$ be category of
$\mathrm{ob}\,\mathcal{C} = \{(X, \varphi) : \varphi : X \to A,\ f\varphi = 0\}$
mor $(X, \varphi) \to (Y, \psi)$ is map $r : X \to Y$ so $\psi \circ r = \varphi$

Ker f is a terminal object in this category.

<u>Moral</u>: Many many things can be described as universals in appropriate category.

<u>Def</u> Let $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ a functor. Let $X \in \mathcal{D}$. Consider the category of <u>morphisms from X to $\mathcal{F}$</u>

objects: pairs (A, φ) $A \in \mathcal{C}$, $\varphi: X \to \mathcal{F}(A)$

morphisms: $(A,\varphi) \to (B,\psi)$ are maps $A \xrightarrow{g} B$ so

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(B) \\ X \arrow[u, "\varphi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes.

<u>Def</u> A <u>universal arrow from X to $\mathcal{F}$</u> is an initial object in the category above, i.e.

a pair $(U(X), i)$ with $U(X) \in A$, $i: X \to \mathcal{F}(U(X))$ such that for any $(A, X \xrightarrow{\varphi} F(A))$ $\exists\,!\ \Phi: U(X) \to A$ so $\mathcal{F}(\Phi) i = \varphi$.

<u>Rmk</u> "Universal properties" can be stated in terms of initial or terminal objects in appropriate categories

<u>Ex1</u> Recall R a ring, free R-module on set A was

$\mathcal{F}(A)$ so any map $A \to M$ extends to

unique $\mathcal{F}(A) \to M$.

Let's Restate This...

$\mathcal{F}: R\text{-mod} \to Set$ forgetful, A any set, $i: A \to \mathcal{F}(F(A))$.
Then $(F(A), i)$ is a universal arrow from A to $\mathcal{F}$.

Thm Suppose $(U_1(X), i_1)$ and $(U_2(X), i_2)$ are universal arrows from X to $\mathcal{F}$. Then there exists a unique isomorphisms $h: U_1(X) \to U_2(X)$ such that $i_2 = \mathcal{F}(h) i_1$

Rmk Can dualize to get universal from $\mathcal{F}$ to an object in $\mathcal{D}$.

Many more examples: products, coproducts, tensor products, kernel, cokernel, inverse & direct limits etc.

Representable Functors

Let $A \in ob\, \mathcal{C}$. Then $Hom_{\mathcal{C}}(A, -)$ is covariant functor $\mathcal{C} \to Set$. These functors "come with $\mathcal{C}$"

Def A cov. functor $\mathcal{F}: \mathcal{C} \to Set$ is representable if $\exists$ a natural $\cong$ of $\mathcal{F}$ with $Hom_{\mathcal{C}}(A, -)$ for some $A \in ob\, \mathcal{C}$.

Similarly a contravariant functor is representable if naturally $\cong$ to some $Hom_{\mathcal{C}}(-, B)$, $B \in ob\, \mathcal{C}$.

Rmk If $Hom_{\mathcal{C}}(A, B)$ has extra structure (ab gp, vect space, R-module), can say $\mathcal{F}: \mathcal{C} \to \begin{matrix} Ab \\ R\text{-mod} \\ VS \end{matrix}$ is representable.

Yoneda's Lemma

Informally: Natural transf from repr. functors are representable.

Suppose $\mathcal{F}: \mathcal{C} \to \mathcal{S}et$ is covariant. Set $h_A = Hom_{\mathcal{C}}(A, -)$. Then natural transformations from h_A to $\mathcal{F}$ are in natural bijection with $\mathcal{F}(A)$, via the map:

$$Nat(h_A, \mathcal{F}) \xrightarrow{\cong} \mathcal{F}(A)$$
$$\eta \longrightarrow \eta_A(1_A)$$

Recall: $\eta_A : h_A(A) \to \mathcal{F}(A)$

Similar for Contravariant case.

Proof We build an inverse. Let $a \in \mathcal{F}(A)$. Need

$$\eta_B : Hom_{\mathcal{C}}(A,B) \to \mathcal{F}(B)$$

Well $f \in Hom_{\mathcal{C}}(A,B)$ then $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$

so $\eta_B(f) = \mathcal{F}(f)(a)$ //

Application Choose $\mathcal{F} = h_B$. Then

$$Nat(h_A, h_B) \xleftrightarrow{\text{nat bij}} Hom_{\mathcal{C}}(B, A).$$

Functor category $\mathcal{S}et^{\mathcal{C}}$: objects: Functors: $\mathcal{C} \to \mathcal{S}et$
morph: Nat tran. Thus

$A \to h_A$ is a full, faithful embedding of $\mathcal{C}^{op}$ into $\mathcal{S}et^{\mathcal{C}}$, by Yoneda. ~~Entries not representable~~

An abstract excursion...

Suppose $\mathcal{F} : \mathcal{C} \to Set$ is representable, so $\eta_e : h_A \to \mathcal{F}(A)$.
By Yoneda η_e corresponds to element $a \in \mathcal{F}(A)$, think of as an arrow $a : * \to \mathcal{F}(A)$.

Claim Universal Arrows $\longleftrightarrow$ repr. of functors

Adjoint Functors

Def. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{L} : \mathcal{D} \to \mathcal{C}$. An adjunction between $\mathcal{F}$ and $\mathcal{L}$ consists of bijections:

$$Hom_{\mathcal{D}}(\mathcal{F}X, Y) \to Hom_{\mathcal{C}}(X, \mathcal{L}Y)$$

which are natural in each variable $X \in \mathcal{C}$, $Y \in \mathcal{D}$.

Say $\mathcal{F}$ is the left adjoint of $\mathcal{L}$.
$\mathcal{L}$ is the right adjoint of $\mathcal{F}$.

Thm If adjoints exist, they are unique up to natural $\cong$.

Basic Example $X \times -$ and $Hom(X, -)$ are adjoints on Set.

$$Hom(\underbrace{X \times Y}_{\mathcal{F}Y}, Z) \cong Hom(\underbrace{\cancel{X}Y}_{Y}, \underbrace{Hom(X, Z)}_{\mathcal{L}(Z)})$$

<u>Ex 2</u> inc $Ab \to Gp$ has left adjoint abelianization

<u>Ex 3</u> $\mathcal{F} = M_R \otimes - : R\text{-mod} \to Ab$

$\mathcal{G} : Ab \to R\text{-mod} \quad \mathcal{G}(A) = Hom_{\mathbb{Z}}(M, A)$

<u>Ex 4</u> Suspension and loop space $[\Sigma X, Y] = [X, \Omega Y]$
etc...

<u>Rmk</u> Definition can be stated in terms of initial and terminal objects