Review

R a comm ring w/1, $D \subseteq R$ mult closed (For example $D = R - P$, P prime)

Def Localization of R at D: $D^{-1}R = \{ \frac{r}{d} \mid r \in R, d \in D \} / \left(\frac{r_1}{d_1} = \frac{r_2}{d_2} \text{ if } (r_1 d_2 - r_2 d_1)x = 0 \text{ some } x \in D \right)$

- $D^{-1}R = 0 \leftrightarrow 0 \in D$
- no zero divisors in D, $0 \notin D \Rightarrow D^{-1}R$ is integral domain containing R
- R an int domain, $D = R^* \Rightarrow D^{-1}R$ is field of fractions of R
- $\pi: R \to D^{-1}R$ $\pi(r) = \frac{rd}{d}$ is well-defined, Kernel $= \{ r \mid rd = 0 \text{ some } d \}$

- π allows us to consider correspondences between certain ideals in R and $D^{-1}R$.

Def D, R as above, $I \subset R$ an ideal. Let $D^{-1}I = \{ \frac{a}{d} \mid a \in I, d \in D \}$, notice $D^{-1}I = I \cdot D^{-1}R$.

Prop

1. $D^{-1}I$ is an ideal in $D^{-1}R$ called extension of I in $D^{-1}R$. (in no zero divisor case $I \subseteq D^{-1}I$)

2. $D^{-1}(I+J) = D^{-1}I + D^{-1}J$, etc.. for $\cap$, $\cancel{\cup}$, pro !

Proof easy exercise

Prop 2 $I \subset R$, D as above. Then $D^{-1}I = D^{-1}R$ iff $D \cap I \neq \emptyset$.

Proof ~~Summary~~.

Suppose $d \in D \cap I$, then $\frac{d}{d} = 1 \in D^{-1}I \Rightarrow D^{-1}I = D^{-1}R$.

Conversely suppose $D^{-1}I = D^{-1}R$, let $1 = \frac{a}{d}$ $a \in I, d \in D$.

Thus

$$\frac{d}{d} = \frac{a}{d} \Rightarrow (d^2 - ad)x = 0 \qquad x \in D$$

$$d^2x = adx \qquad d^2x \in D \quad adx \in I,$$

neither 0 unless $0 \in D$.

Def Now let J be an ideal in $D^{-1}R$. Then $\pi^{-1}(J)$ is an ideal in R, called the <u>contraction</u> of J.

Rmk $\pi^{-1}(J)$ is the set of numerators which appear in elts of J.

Ex $R = \mathbb{Z}$ $\quad D^{-1}R = \mathbb{Z}_{(3)}$ has ideal $\underset{J}{\underset{\parallel}{5\cdot\mathbb{Z}_{(3)}}} = \{ \frac{5a}{b} \mid 3 \nmid b \}$

contraction of J is $5\mathbb{Z}$

Prop 3
1. $I \subseteq \pi^{-1}(D^{-1}I)$
2. Every ideal in $D^{-1}R$ is of form $D^{-1}I$, some $I \subset R$.
3. If P is prime in R and $D \cap P = \emptyset$ then $D^{-1}P$ is prime in $D^{-1}R$ and $\pi^{-1}(D^{-1}P) = P$.

Proof Exc.

COR $\exists$ 1-1 corr between prime ideals of R disjoint from D and prime ideals of $D^{-1}R$ given by $P \to D^{-1}P$.

<u>Special Case</u>: $D = R - P$, Q prime, $\underset{D}{\text{disjoint from}} \longleftrightarrow \subseteq P$

Thm Let $P \leq R$ be prime.

1. $\exists$ 1-1 corr between prime ideals of R contained in P and prime ideals of R_P given by

$$Q \to D^{-1}Q = Q_P.$$

2. Thus $P_P = P \cdot R_P$ is unique maximal ideal in R_P.

<u>Proof</u> #1 follows from Cor, special case.

2. M maximal $\Rightarrow$ M prime $\Rightarrow M = Q_P$ some $Q_P \subseteq P_P$ (strict as

However $P_P \subset R_P$ strict by Prop 2. Thus M maximal $\Rightarrow Q = P$ and $M = P_P$.

DEF A commutative ring w/ 1 and a unique maximal ideal is called a <u>local ring</u>

<u>Rmk</u> 1. Thus every ideal lies in M, lattice 

2. R_P is a local ring

Thm R comm w/ 1. TFAE

1. R is local w/ maximal ideal M.
2. {nonunits} forms an ideal.
3. $\exists$ maximal ideal such that $1+m$ is a unit $\forall m \in M$.

Proof $\underline{1 \to 2}$ Suppose a nonunit, (a) proper $\Rightarrow (a) \subseteq M$ so M has all nonunits so M = all nonunits

$\underline{2 \to 1}$ Clear, any proper ideal is $\subseteq$ {nonunits}

$\underline{3 \to 1}$ Suppose $a \notin M$ so $(a)+M = R$ so $1 = ar+m$ so $ar = 1-m$ is a unit so a is a unit so M = {nonunits}

$\underline{1 \to 3}$ $1+m \notin M \Rightarrow$ unit. //

Local Rings in Algebraic Geometry

Setting: $k=\bar{k}$, V a irreducible variety, note $k[V]$ is an int domain (since $\mathcal{I}(V)$ prime)

Def $k(V)$ = field of fractions of $k[V]$, think rational functions on V.

Def f/g is regular at $v \in V$ (aka defined at v) if $\exists f_1, g_1 \in k[V]$ so

$f/g = f_1/g_1$ and $g_1(v) \neq 0$.

Remarks 1. Unless $k[V]$ were a UFD, no "lowest terms" for f/g

2. If f_1/g_1 regular at v then regular on open neighborhood $\{g_1 \neq 0\}$ of v, which is dense in V.

<u>Def</u>. Let $\mathcal{O}_{v,V} = \{ f/g \in k(V) \mid f/g$ is regular at $v \}$ the <u>local ring of V at v</u>

<u>Rmk</u> 1. $\mathcal{O}_{v,V}$ is a local ring, it is $= k[V]_{\chi(v)}$ (This is intrinsic to the variety)

2. Let $m_{v,V}$ be the unique maximal ideal, so:

$$m_{v,V} = \chi(v)_{\chi(v)} = \{ f/g \mid f/g = f_1/g_1 \text{ with } f_1(v)=0,\ g_1(v) \neq 0 \}$$

<u>Prop</u> $k[V] = \{ f/g \in k(V) \mid$ regular everywhere $\}$

<u>Prop</u> Suppose $\varphi : V \to W$ regular map and $\varphi(v) = w$, with $\tilde{\varphi} : k[W] \to k[V]$. Then this induces

$$\hat{\varphi} : \mathcal{O}_{w,W} \to \mathcal{O}_{v,V} \qquad \hat{\varphi}(h/k) = \tilde{\varphi}(h)/\tilde{\varphi}(k)$$

~~mapping~~ such that $\hat{\varphi}^{-1}(m_{v,V}) = m_{w,W}$ (local homomorphism)

Local Ring can be used to give alg def of smoothness, dimension tangent planes, etc..

Tangent Space
"old way" $V = Z(f)$ hypersurface, $v \in V$.

$D_v(f)(x_1, \dots, x_n) = \frac{\partial f}{\partial x_1}(v)\, x_1 + \dots + \frac{\partial f}{\partial x_n}(v)\, x_n$. Then

$D_v f\,(x_1 - v_1, \dots, x_n - v_n)$ is linear approx to $f(x_1, \dots, x_n)$ at v.

Tangent space $T = Z(D_v(f))$, $\vec{T} + \vec{v}$ is tangent plane.

Ex $f(x,y) = x^2 + y^2 - 4$ circle $= Z(f)$ at $(\sqrt{2}, \sqrt{2})$

$\underset{2\sqrt{2}}{\frac{\partial f}{\partial x} = 2x} \quad \underset{2\sqrt{2}}{\frac{\partial f}{\partial y} = 2y} \qquad D_{(\sqrt{2},\sqrt{2})} f = \sqrt{2}\,x + \sqrt{2}\,y$

$\sqrt{2}\,x + \sqrt{2}\,y = 0$ shift up to tangent line

Def $T_{v,V} = Z(\{D_v(f)(x_1, \dots, x_n) \mid f \in \mathcal{I}(V)\})$ tangent space at v.
($= \cap$ tangent spaces of hypersurfaces)

Thm $V, v, \mathcal{O}_{v,V}, m_{v,V}$ as above. Then

$$(T_{v,V})^* \cong m_{v,V}/m_{v,V}^2 \quad \text{as } k\text{-vector spaces}$$

DEF dimension of V = Trans degree $k(V)$ over k,
determined by any local ring since $k(V) = $ f. fractions $\mathcal{O}_v, V$.

DEF V is nonsingular at v if dim tang space = dim V.
Else singular.

V is smooth if nonsing everywhere.