Review

A commutative ring with 1, $D \subseteq R$ multiplicative closed. (For example $D = R - P$, $P$ prime)

Def. Localization of $R$ at $D$: $D^{-1}R = \left\{ \frac{f}{g} \mid f, g \in R, g \in D \right\}$

- $D^{-1}R = 0 \iff 0 \in D$
- No zero divisors in $D$, $0 \in D \implies D^{-1}R$ is integral domain containing $R$
- $R$ an integral domain, $D = R^*$ $\implies D^{-1}R$ is field of fractions of $R$
- $\eta: R \to D^{-1}R$, $\eta(r) = \frac{r}{1}$ is well-defined, Kernel = $\{ r \mid r \cdot 0 \text{ some} \}

- It allows us to consider correspondences between certain ideals in $R$ and $D^{-1}R$

Def. $D, R$ as above, $I \subseteq R$ an ideal. Let $D^{-1}I = \left\{ \frac{a}{g} \mid a \in I, g \in D \right\}$.

Notice $D^{-1}I = I \cdot D^{-1}R$

Proof

1. $D^{-1}I$ is an ideal in $D^{-1}R$ called extension of $I$ in $D^{-1}R$.
   (In no zero divisor case $I \subseteq D^{-1}I$)


Proof easy exercise
Prop 2. \( I \subseteq R, D \) as above. Then \( D^I = D^R \) iff \( D \cap I \neq \emptyset \).

Proof. Suppose \( D \cap I \neq \emptyset \), then \( \frac{d}{d} = 1 \in D^I \Rightarrow D^I = D^R \).

Conversely suppose \( D^I = D^R \), let \( I = \frac{a}{d} \in I, d \in D \).

Thus \( \frac{d}{d} = \frac{a}{d} \Rightarrow (d^2 - ad)x = 0 \quad x \in D \)
\( d^2x = adx \quad d^2x \in D \quad ad \in I \)
neither 0 unless \( 0 \in D \).

Def. Now let \( J \) be an ideal in \( D^R \). Then \( \Pi^{-1}(J) \) is an ideal in \( R \), called the contraction of \( J \).

Rem. \( \Pi^{-1}(J) \) is the set of numerators which appear in \( \Pi \) of \( J \).

Ex. \( R = \mathbb{Z} \quad D^R = \mathbb{Z}_3 \) has ideal \( \mathbb{Z}/(3) = \{ \frac{5a}{b} | 3 \not| b \} \)
contraction of \( J \) is \( 5\mathbb{Z} \)

Prop 3. 1. \( I \subseteq \Pi^{-1}(D^I) \)
2. Every ideal in \( D^R \) is of form \( D^I \), same \( I \subseteq R \).
3. If \( P \) is prime in \( R \) and \( DAP = 0 \) then \( D^P \) is prime in \( D^R \)
and \( \Pi(D^P) = P \).

Proof. Exc.
2. A commutative ring is a local ring if and only if a unique maximal ideal exists.

3. If $P$ is a maximal ideal, then $P = P_P$.

4. If $Q$ is not maximal in $R$, then $Q = Q_Q$.

**Theorem:** Let $P < P$ be prime.

**Proof:**

1. In 1-1 correspondence between prime ideals of $R$ minus $R$, and prime ideals of $R$ minus $P$ (since $P$ is prime).

**Special Case:** $P = P_P$, $P$ prime, disjoint from $P$. 

**Corollary:** Every prime ideal of $R$ minus $R$ is disjoint from $P$.
Then $R$ comm w/ I. TFAE:

1. $R$ is local w/ maximal ideal $M$.
   a. \{nonunits\} forms an ideal.
   3. \exists maximal ideal such that $1 + M$ is a unit \forall € M.

Part 1 $\rightarrow$ 2: Suppose a nonunit \( a \), proper $\Rightarrow (a) \cap M = a + M$ so $M$ has all nonunits so $M = \text{all nonunits}$.

2. $\rightarrow$ 1: Clear any proper ideal is $\subseteq \{\text{nonunits}\}$.

3. $\rightarrow$ 1: Suppose $a \notin M$ so $(a) + M = R$ so $1 = ar + m$ so $ar = 1 - m$
   is a unit so $a$ is a unit so $M = \{\text{nonunits}\}$.

1 $\rightarrow$ 3: $1 + M = \text{unit}$.

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Local Rings in Algebraic Geometry

Setting: $k = \mathbb{R}$, $V$ a variety, note $k[V]$ is an int domain (since $k[V]$ prime).

Def: $k(V)$ = field of fractions of $k[V]$, think rational functions on $V$.

Def: $f/g$ is regular at $v \in V$ (aka defined) if $\exists f, g \in k[V]$ so

$f/g = f/vg$ and $g(v) \neq 0$.

Remarks

1. Unless $k[V]$ were a UFD, no "lowest terms" for $f/g$.

2. If $f/g$ regular at $V$ then regular on open neighborhood $3g \neq 0$ of $v$, which is dense in $V$. 
**Def.** Let \( \mathcal{O}_V = \{ f/y \in k(V) \mid f/y \text{ is regular at } V \} \) the local ring at \( V \).

**Rmk.** \( \mathcal{O}_V \) is a local ring, it is \( = k[V]_{x(V)} \). (This is intrinsic)

\[ M_{x(V)} = \mathfrak{m}_{x(V)} = \{ f/y \in k[V]_{x(V)} \mid f(V) = 0 \}, \]

**Prop.** \( k[V] = \{ f/y \in k(V) \mid \text{regular everywhere} \} \).

**Prop.** Suppose \( \varphi : V \rightarrow W \) regular map and \( \varphi(V) = W \) with \( \varphi : k[V] \rightarrow k[W] \). Then this induces

\[ \varphi : \mathcal{O}_V \rightarrow \mathcal{O}_W \quad (\varphi)(f/y) = (\varphi(f))/\varphi(y) \]

making such that \( \varphi^{-1}(M_{y(W)}) = M_{x(V)} \) (local homomorphism).

Local Ring can be used to give alg def of smoothness, dimension, tangent planes, etc.
Tangent Space

- Old way: \( V = Z(f) \) hypersurface, \( v \in V \).

\[ D_v(f)(x_1, \ldots, x_n) = \frac{df}{dx_1}(v_x) x_1 + \ldots + \frac{df}{dx_n}(v_x) x_n. \]

Then

\[ D_{v_1}(x_1 - v_1, \ldots, x_n - v_n) \] is linear approx to \( f(x_1, \ldots, x_n) \) at \( v \).

Tangent space \( \tilde{T} = Z(D_v(f)) \) at \( v \), \( \tilde{T} + v \) is tangent plane.

\[ f(x, y) = x^2 + y^2 - v \] circle = \( Z(f) \) at \( (\sqrt{v}, \sqrt{v}) \)

\[ \frac{df}{dx} = 2x, \quad \frac{df}{dy} = 2y \]

\[ D_{(\sqrt{v}, \sqrt{v})} f = \sqrt{v} x + \sqrt{v} y \]

\( \sqrt{v} x + \sqrt{v} y = 0 \) shift up to tangent line.

Def: \( T_{\gamma, v} = Z\{D_v(f)(x_1, \ldots, x_n) | f \in X(V) \} \) tangent space at \( v \).

\( (\gamma, v) \) tangent space of hypersurface.

Then \( V, v, \alpha_v, m_v \) as above. Then

\[ (T_{\gamma, v})^* = m_v/v \] as \( k \)-vector spaces

Def: dimension of \( V = \text{Trans degree } R(V) \) over \( R \),
determined by any local ring since \( R(V) = \text{fractions } \alpha_v, v \).

Def: \( V \) is nonsingular at \( v \) if \( \dim \text{ tangent space} = \dim V \)
Else singular.

\( V \) is smooth if nonsing everywhere.