<u>Lecture 2</u>

<u>Review</u> Functor $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$ can be covariant or contravariant.

<u>Def</u> $\mathcal{F}$ is <u>full</u> if $\mathcal{F}: Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ is surjective $\forall A,B$

full subcategory means inclusion functor is full

$\mathcal{F}$ is <u>faithful</u> if $\mathcal{F}: Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ is 1-1 $\forall A,B$

<u>Rmk</u> Opposite category reduces study to either contravariant or covariant, although no reason to do this!

<u>Def</u> Suppose $\mathcal{F}, \mathcal{G}: \mathcal{C} \to \mathcal{D}$ are covariant functors.
A <u>natural transformation</u> from $\mathcal{F}$ to $\mathcal{G}$ is a map η that assigns to each object $A \in \mathcal{C}$ a morphism $\eta_A \in Hom_{\mathcal{D}}(\mathcal{F}A, \mathcal{G}A)$ such that $\forall A,B \in \mathcal{C}$, $\forall f \in Hom_{\mathcal{C}}(A,B)$,

the following diagram commutes

$$\begin{tikzcd} \mathcal{F}A \arrow[r, "\eta_A"] \arrow[d, "\mathcal{F}f"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}f"] \\ \mathcal{F}B \arrow[r, "\eta_B"'] & \mathcal{G}B \end{tikzcd}$$

<u>Rmk</u> 1. Obvious alteration for natural transf between contravariant functors.

2. Easy exercise that composition of two nat trans is natural trans

3. If all η_A are $\cong$'s then called a natural $\cong$.
"naturally isomorphic" is an equiv. relation on functors $\mathcal{C} \to \mathcal{D}$.

4. Suppose M is a left R-module. Then $R \otimes_R M$ and M are "naturally isomorphic".
This is shorthand for
$R \otimes_R -$ and Id are naturally $\cong$ functors.

Example 1
V a fd vector space $/k$, $ev_v \in V^{**}$ defined by $ev_v(f) = f(v)$ $\forall f \in V^*$.

Define
D^2: fd k-vec space $\to$ fd k-vec space
$D^2 V = V^{**}$, $T: V \to W$
$D^2 T$: ~~$W^{**} \to V^{**}$~~ $V^{**} \to W^{**}$
$ev_v \to ev_{T(v)}$

Thm Let $\eta_V: V \to V^{**}$ be given by $v \to ev_v$.

Then η gives a natural isomorphism from the identity functor to the functor D^2.

Proof

1. η_V is an $\cong$ from $Id\,V \to D^2 V$.

2. Suppose

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "ev"'] & W \arrow[d, "ev"] \\ V^{**} \arrow[r, "D^2 T"] & W^{**} \end{tikzcd}$$

$$\begin{tikzcd} v \arrow[r] \arrow[d] & Tv \arrow[d] \\ ev_v \arrow[r] & ev_{Tv} \end{tikzcd}$$

✓ commutes!

Example 2

Define a functor $\mathcal{GL}_n$: CRing $\to$ Grp by $R \to GL_n(R)$

~~$R \to U(R)$~~ f.

and $f: R \to S$ goes to map $GL_n(R) \longrightarrow GL_n(S)$
act on elements

Define a functor $\mathcal{U}$: CRing $\to$ Grp by $R \to U(R)$

Thm det: $GL_n(R) \to U(R)$ gives a natural trans from $\mathcal{GL}_n$ to $\mathcal{U}$.

Isomorphisms & Equivalences

Def Two categories $\mathcal{C}, \mathcal{D}$ are $\cong$ if $\exists$ $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{C}$ with $\mathcal{F}\mathcal{G} = Id_{\mathcal{D}}$, $\mathcal{G}\mathcal{F} = Id_{\mathcal{C}}$

Ex

1. $\mathbb{Z}$-mod $\cong$ Ab
2. Elt abelian p-groups $\cong$ VS over F_p

Rmk Not useful, requires same # of objects in each $\cong$ class

Def. $\mathcal{C}$ and $\mathcal{D}$ are <u>equivalent</u> if $\exists$ functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G}\mathcal{F}$ is naturally $\cong$ to $I_{\mathcal{C}}$
$\mathcal{F}\mathcal{G}$ " " " " $I_{\mathcal{D}}$.

$\mathcal{F}$ (and $\mathcal{G}$) are called equivalences of categories

<u>Rmks</u>

1. Equivalence of categories is an equiv. relation
2. Contravariantly equiv vs covariantly equiv.
3. Many important conjectures / thms in rep are about equivalences of categories

** 4. $\mathcal{F}$ is an equivalence of categories iff it is full, faithful and "essentially surjective," meaning its image contains an element of every $\cong$ class in $\mathcal{D}$.

5. All important cat theory notions are preserved

Example 1

$$\begin{tikzcd} c \arrow[loop right, "1_c"] \end{tikzcd} \qquad\qquad \begin{tikzcd} d_1 \arrow[loop left, "1_{d_1}"] \arrow[r, bend left] & d_2 \arrow[l, bend left] \arrow[loop right, "1_{d_2}"] \end{tikzcd}$$

$\mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$

$\mathcal{F}(c) = d_1$

$\mathcal{F}(1_c) = 1_{d_1}$ is an equivalence,

but $\mathcal{C} \not\cong \mathcal{D}$

Ex 2 Modules for R and for $M_n(R)$ are equivalent categories

$M \in R$-mod, $\Longrightarrow$ colvectors of length n from M

Def Two rings R, S are Morita equivalent if mod-R and mod-S are (additively) equivalent categories

Ex 3 $\mathcal{C}$Ring and affine schemes are equiv categories

$R \longrightarrow$ Prime ideal Spectrum

Products, Coproducts, universals

Def on HW for products coproducts.

Thm They are ! up to $\cong$ if they exist.
Pf Exerc using univ property.

Def $I \in \mathcal{C}$ is universal (or initial) if for each object $C \in \mathcal{C}$ there is a unique morphism $I \to C$.
$T \in \mathcal{C}$ is couniversal (or terminal) if for each $C \in \mathcal{C}$ there is unique morphism $C \to T$.

Ex 1. $\{e\}$ initial & terminal in Grp

2. Any two universals are $\cong$ (Proof)

3 Let $\{A_i \mid i \in I\} \subseteq ob\, \mathcal{C}$. Define a new category $\mathcal{E}$ with

$ob\,\mathcal{E}$ = pairs $(B, \{f_i \mid i \in I\})$ where $B \in \mathcal{C}$ and $f_i : B \to A_i$

morphism $(B, \{f_i\}) \to (D, \{g_i\})$ is a morphism $h: B \to D$ so $f_i = g_i \circ h$.

Exercise

If $(\prod A_i, \{p_i\})$ exists it is a terminal object in category $\mathcal{E}$.

Similarly coproduct is a universal in approp. cat.